On Hyper Pseudo BCK-algebras

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ABSTRACT. In this paper, we introduce the notion of hyper pseudo BCK-algebras, which is a generalization of pseudo BCK-algebras and hyper BCK-algebras and we investigates some related properties. In follow, we define some kinds of hyper pseudo BCK-ideals of a hyper pseudo BCK-algebra and we find the relations among them. Finally, we characterize the hyper pseudo BCK-ideals of type 4 generated by a nonempty subset.

Keywords: Hyper pseudo BCK-algebras, Hyper pseudo BCK-ideals, Generated hyper pseudo BCK-ideals.

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1. INTRODUCTION AND PRELIMINARIES

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [8] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. In order to extend BCK-algebras in a noncommutative form, Georgescu and Iorgulescu [4] introduced the notion of pseudo BCK-algebras and studied their properties. The pseudo BCK-algebras as generalization

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of BCK-algebras in order to give a structure corresponding to pseudo MV-algebras, as the bounded commutative BCK-algebras corresponds to MV-algebra. Hyperstructure theory (called also multialgebras) was introduced in 1934 by F. Marty [11] at the 8th Congress of Scandinavian Mathematiciens. Since then many researchers have worked on algebraic hyperstructures and developed it. A recent book [3] contains a wealth of applications. Via this book, Corsini and Leoreanu presented some of the numerous applications of algebraic hyperstructures, especially those from the last fifteen years, to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [1, 10, 12, 13], R. A. Borzooei, M.M. Zahedi et al. applied the hyperstructures to BCK-algebras and introduced the notion of a hyper BCK-algebra (hyper K-algebra) which is a generalization of BCK-algebra and investigated some related properties. Now, in this paper we define the notions of hyper pseudo BCK-algebra, hyper pseudo BCK-ideals of hyper pseudo BCK-algebra and we obtain some related results which have been mentioned in the abstract.

Definition 1.1. [10] By a hyper BCK-algebra we mean a nonempty set H endowed with a hyperoperation "∘" and a constant 0 satisfy the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x \circ H \ll \{x\}$,
(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the hyperorder in H.

Definition 1.2. [10] Let I be a nonempty subset of hyper BCK-algebra H and $0 \in I$. Then I is said to be a

(i) *hyper BCK-ideal* of H if for all $x, y \in H$, $y \in I$ and $x \circ y \ll I$, imply $x \in I$.

(ii) *weak hyper BCK-ideal* of H if for all $x, y \in H$, $y \in I$ and $x \circ y \subseteq I$, imply $x \in I$.

Definition 1.3. [4] A pseudo BCK-algebra is a structure $\mathfrak{X} = (X, *, \diamond, 0)$, where "$*$" and "$\diamond$" are binary operations on X and "0" is a constant element of X, that satisfies the following:

(a1) $(x * y) \diamond (x * z) \preceq z * y$, $(x \diamond y) * (x \diamond z) \preceq z \diamond y$,
(a2) $x * (x \diamond y) \preceq y$, $x \diamond (x * y) \preceq y$,
(a3) $x \preceq x$,

(a4) $0 \preceq x$,

(a5) $x \preceq y$, $y \preceq x$ implies $x = y$,

(a6) $x \preceq y \Leftrightarrow x * y = 0 \Leftrightarrow x \diamond y = 0$.

for all $x, y, z \in X$.

Let X be a pseudo BCK-algebra, we denote $*(y, I) = \{x \in X \mid x * y \in I\}$ and $\diamond(y, I) = \{x \in X \mid x \diamond y \in I\}$.

Note that, $*(y, I) \cap \diamond(y, I) = \{x \in X \mid x * y \in I, x \circ y \in I\}$.

Definition 1.4. [9] Let I be a nonempty subset of pseudo BCK-algebra X and $0 \in I$. Then I is said to be a

(i) *pseudo-ideal* of X if for any $y \in I$, $*(y, I) \subseteq I$ and $\diamond(y, I) \subseteq I$.

(ii) *weak pseudo-ideal* of X if for any $y \in I$, $*(y, I) \cap \diamond(y, I) \subseteq I$.

2. Hyper pseudo BCK-algebra

In 2001, G. Georgescu and A. Iorgulescu [3], extended the BCK-algebras in a noncommutative form and defined the notion of pseudo BCK-algebras. They gave a structure corresponding between pseudo BCK-algebras and pseudo MV-algebras. Moreover, in 2000, R. A. Borzooei and et al. [1,10], defined the notion of hyper BCK-algebras and hyper K-algebras as a generalization of BCK-algebras and in [5], Sh. Ghorbani and et al. defined the notion of hyper MV-algebra and they found a structure corresponding between hyper MV-algebras and hyper K-algebras, by some conditions. Now, in this section we generalize the notion of pseudo BCK-algebras and hyper BCK-algebras and we define the concept of *hyper pseudo BCK-algebra*. One of our motivation is to find a corresponding between this structure and hyper pseudo MV-algebras, in the future.

Definition 2.1. A *hyper pseudo BCK-algebra* is a structure $(H, \circ, *, 0)$, where "$*$" and "$\circ$" are hyper operations on H and "0" is a constant element, that satisfies the following:

(PHK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$, $(x * z) * (y * z) \ll x * y$.

(PHK2) $(x \circ y) * z = (x * z) \circ y$.

(PHK3) $x \circ H \ll \{x\}$, $x * H \ll \{x\}$.

(PHK4) $x \ll y$ and $y \ll x$ imply $x = y$.

for all $x, y, z \in H$, where $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$.

Theorem 2.2. *Let H be a hyper pseudo BCK-algebra. If $x * y = x \circ y$, for all $x, y \in H$, then H is a hyper BCK-algebra and if "$*$" and "$\circ$" are singleton, then H is a pseudo BCK-algebra.*

Proof. Let H be a hyper pseudo BCK-algebra and $x * y = x \circ y$, for all $x, y \in H$. It is easy to see that H is a hyper BCK-algebra. Now, assume that "$*$" and "$\circ$"

are singleton. Then $x \ll y \iff x*y = 0 \iff x \circ y = 0$. Clearly (a6) and (a5) are hold. By (PHK1), $(x \circ z) \circ (y \circ z) \ll x \circ y$ and so $((x \circ z) \circ (y \circ z)) * (x \circ y) = 0$. By (PHK2), $((x \circ z) * (x \circ y)) \circ (y \circ z) = 0$, i.e. $(x \circ z) * (x \circ y) \ll (y \circ z)$. By the similar way, we can prove that if $(x*z)*(y*z) \ll x*y$, then $(x*z) \circ (x*y) \ll y*z$. Therefore, (a1) is hold. Now, we need to show that $0 * 0 = 0$. By (PHK3), $0 * 0 \ll 0$. So, it is enough to show that $0 \ll 0 * 0$. If $0 \circ (0 * 0) = b$, we will prove that $b = 0$. Since by (PHK3), $0 \circ (0 * 0) \ll 0$, then $b \ll 0$, i.e. $b * 0 = 0$. We know that $b * 0 \ll b$ and so $0 \ll b$. Hence $b = 0$ by (PHK4) and so $0 * 0 = 0$. By (PHK3), $0 \circ x \ll 0$ and so by (PHK2), $0 = (0 \circ x) * 0 = (0 * 0) \circ x = 0 \circ x$, i.e. $0 \ll x$ and so (a4) is hold. By (PHK3), $x * 0 \ll x$ and using (PHK2), we obtain $0 = (x * 0) \circ x = (x \circ x) * 0$ i.e. $x \circ x \ll 0$. Since by (a4) $0 \ll x \circ x$, hence by (PHK4), $x \circ x = 0$, i.e. $x \ll x$ and (a3) is hold. By (PHK2) and (a3), we have $(x * (x \circ y)) \circ y = (x \circ y) * (x \circ y) = 0$, i.e. $x * (x \circ y) \ll y$. By the similar way, we can prove that $x \circ (x * y) \ll y$, hence (a2) is hold. Therefore, H is a pseudo BCK-algebra. $\square$

Example 2.3. (i) Let $H = \{0, 1, 2, \ldots\}$. The hyperoperations " $*$ " and " $\circ$ " on H be defined as follows:

$$x * y = \begin{cases} \{0\} & , \text{ if } x < y, \\ \{0, x\} & , \text{ if } x = y, \\ \{x\} & , \text{ if } x > y. \end{cases} \qquad x \circ y = \begin{cases} \{0, x\} & , \text{ if } x \le y, \\ \{x\} & , \text{ if } x > y. \end{cases}$$

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra.

(ii) Let $H = \{0, 1, 2, 3\}$. The hyperoperations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0}	{0}	{0}
2	{2}	{1}	{0}	{1}
3	{3}	{3}	{0,3}	{0,3}

$*$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0}	{0}	{0}
2	{2}	{1}	{0,1}	{1}
3	{3}	{3}	{0,3}	{0,3}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra.

Proposition 2.4. *In any hyper pseudo BCK-algebra H, the following hold:*

(i) $0 \circ 0 = \{0\}$, $0 * 0 = \{0\}$.

(ii) $0 \ll x$.

(iii) $x \ll x$.

(iv) $A \ll A$.

(v) $A \subseteq B$ *implies* $A \ll B$.

(vi) $0 \circ x = \{0\}$, $0 * x = \{0\}$.

(vii) $0 \circ A = \{0\}$, $0 * A = \{0\}$.

(viii) $A \ll \{0\}$ *implies* $A = \{0\}$.

(ix) $x \circ 0 = \{x\}$, $x * 0 = \{x\}$.

(x) $x \circ 0 \ll \{y\}$ *implies* $x \ll y$ *and* $x * 0 \ll \{y\}$ *implies* $x \ll y$.

(xi) $y \ll z$ *implies* $x \circ z \ll x \circ y$ *and* $x * z \ll x * y$.

(xii) $x \circ y = \{0\}$ *implies* $(x \circ z) \circ (y \circ z) = \{0\}$ *and* $x \circ z \ll y \circ z$, $x * y = \{0\}$
   implies $(x * z) * (y * z) = \{0\}$ *and* $x * z \ll y * z$.

(xiii) $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$ *and* $A * \{0\} = \{0\}$ *implies* $A = \{0\}$

(xiv) $(A \circ c) \circ (B \circ c) \ll A \circ B$, $(A * c) * (B * c) \ll A * B$.

for all $x, y, z \in H$ *and for all nonempty subsets A and B of H.*

Proof. (i) Let $a \in 0 * 0$. Since by (PHK3), $0 * 0 \ll \{0\}$, then $a \ll 0$. Now, we show that $0 \ll a$. Let $b \in 0 * a$. Since by (PHK3) $0 * a \ll \{0\}$, then $b \ll \{0\}$ and so $0 \in b * 0$. But, since $b * 0 \ll \{b\}$, then $0 \ll \{b\}$ and so by (PHK4), $b = 0$. Hence $0 \in 0 * a$ and so $0 \ll a$. Therefore, by (PHK4), $a = 0$ and so $0 * 0 = \{0\}$. By the similar way, we can prove that $0 \circ 0 = \{0\}$.

(ii) By (PHK3), $0 \circ x \ll \{0\}$ and so by (PHK2) and (i), $0 \in (0 \circ x) * 0 = (0 * 0) \circ x = 0 \circ x$. Hence $0 \ll x$, for all $x \in H$.

(iii) By (PHK3), $x \circ 0 \ll \{x\}$ and by using (PHK2), we obtain $0 \in (x \circ 0) * x = (x * x) \circ 0 = \bigcup_{a \in x * x} a \circ 0$. Thus there exists $a_0 \in x * x$ such that $0 \in a_0 \circ 0$ i.e. $a_0 \ll 0$. Using (ii) we get $a_0 = 0$ and so $0 \in x * x$, i.e. $x \ll x$.

(iv) By (iii), the proof is clear.

(v) The proof is easy.

(vi) Let $a \in 0 * x$ for $x \in H$. Since by (PHK3), $0 * x \ll \{0\}$, then $a \ll 0$ and so by (ii) and (PHK4), $a = 0$. Hence $0 * x = \{0\}$. By the similar way, we can proof that $0 \circ x = \{0\}$.

(vii) By (vi), the proof is clear.

(viii) Let $A \ll \{0\}$ and $a \in A$. Then $a \ll 0$ and so $a = 0$. Therefore $A = \{0\}$.

(ix) By (iv) and (PHK2), $0 \in (x \circ 0) * (x \circ 0) = (x * (x \circ 0)) \circ 0$. Hence there exists $a \in x * (x \circ 0)$ such that $0 \in a \circ 0$ and so $a \ll 0$. Now, by (ii) and (PHK4), $a = 0$ and so $0 \in x * (x \circ 0)$. Hence there exists $t \in x \circ 0$ such that $0 \in x * t$ and so $x \ll t$. Now, since by (PHK3), $x \circ 0 \ll \{x\}$, then $t \ll x$ and so $t = x$. Thus, $x \in x \circ 0$. Now, let $y \in x \circ 0$. Then by (iii) and (PHK2), $0 \in y * y \subseteq (x \circ 0) * y = (x * y) \circ 0$. Hence there exists $u \in x * y$ such that $0 \in u \circ 0$ and so $u \ll 0$. Now, by (ii) and (PHK4), $u = 0$ and so $0 \in x * y$. Hence $x \ll y$. Now, since by (PHK3), $x \circ 0 \ll \{x\}$, then $y \ll x$ and so $y = x$. Thus, $x \circ 0 = \{x\}$. By the similar way, we can prove that $x * 0 = \{x\}$.

(x) By (ix), the proof is clear.

(xi) Let $y \ll z$, for $y, z \in H$. Since $0 \in y \circ z$, then by (PHK1) for any $x \in H$, $(x \circ z) \circ 0 \subseteq (x \circ z) \circ (y \circ z) \ll x \circ y$ and so $(x \circ z) \circ 0 \ll x \circ y$. Hence for any $a \in x \circ z$, there exists $b \in x \circ y$ such that $a \circ 0 \ll b$. Now, since by (ix), $a \in a \circ 0$, then $a \ll b$ and this means that $x \circ z \ll x \circ y$. By the similar way, we can prove that $x * z \ll x * y$.

(xii) Let $x * y = \{0\}$, for $x, y \in H$. Then by (PHK1), $(x * z) * (y * z) \ll x * y = \{0\}$ and so by (viii), $(x * z) * (y * z) = \{0\}$ and this means that $x * z \ll y * z$. By the similar way, we can prove the other case.

(xiii) Let $\emptyset \neq A \subseteq H$, $A * \{0\} = \{0\}$ and $a \in A$. Then by (ix), $a \in a * \{0\} \subseteq A * \{0\} = \{0\}$ and so $a = 0$. Hence $A = \{0\}$. The proof of other case is similar.

(xiv) Let A and B are two nonempty subsets of H and $x \in (A \circ c) \circ (B \circ c)$. Then there exist $a \in A$ and $b \in B$ such that $x \in (a \circ c) \circ (b \circ c)$. Since by (PHK1), $(a \circ c) \circ (b \circ c) \ll a \circ b$ then there exists $y \in a \circ b$ such that $x \ll y$. Now, since $y \in a \circ b \ll A \circ B$, then $(A \circ c) \circ (B \circ c) \ll A \circ B$. By the similar way, we can prove that $(A * c) * (B * c) \ll A * B$. $\qquad \square$

Note: (i) From now on, in this paper H is a hyper pseudo BCK-algebra, unless otherwise state.

(ii) For any nonempty subset I of H and any element y of H, we denote

$$*(y, I)^{\ll} = \{x \in H \mid x * y \ll I\}, \; \circ(y, I)^{\ll} = \{x \in H \mid x \circ y \ll I\}$$

$$*(y, I)^{\subseteq} = \{x \in H \mid x * y \subseteq I\}, \; \circ(y, I)^{\subseteq} = \{x \in H \mid x \circ y \subseteq I\}$$

3. Hyper pseudo BCK-ideals

In this section we generalize the notion of pseudo and weak pseudo BCK-ideals in pseudo BCK-algebras and (weak) hyper BCK-ideals in hyper BCK-algebras to the hyper pseudo BCK-algebras and we get the following cases for hyper pseudo BCK-ideals. In fact, these following 12 cases are the natural generalization of ideals in the hyper pseudo BCK-algebras.

define the notions of hyper pseudo BCK-ideals of type 1, 2, ..., 12. Then we state and prove some theorems which determine the relationships between these notions.

Definition 3.1. Let I be a nonempty subset of H and $0 \in I$. Then I is said to be a *hyper pseudo BCK-ideal* of

(i1)   type (1),   if for any $y \in I$, $*(y, I)^{\ll} \subseteq I$ and $\circ(y, I)^{\ll} \subseteq I$.
(i2)   type (2),   if for any $y \in I$, $*(y, I)^{\subseteq} \subseteq I$ and $\circ(y, I)^{\ll} \subseteq I$.
(i3)   type (3),   if for any $y \in I$, $*(y, I)^{\ll} \subseteq I$ and $\circ(y, I)^{\subseteq} \subseteq I$.
(i4)   type (4),   if for any $y \in I$, $*(y, I)^{\subseteq} \subseteq I$ and $\circ(y, I)^{\subseteq} \subseteq I$.
(i5)   type (5),   if for any $y \in I$, $*(y, I)^{\ll} \subseteq I$ or $\circ(y, I)^{\ll} \subseteq I$.
(i6)   type (6),   if for any $y \in I$, $*(y, I)^{\subseteq} \subseteq I$ or $\circ(y, I)^{\ll} \subseteq I$.
(i7)   type (7),   if for any $y \in I$, $*(y, I)^{\ll} \subseteq I$ or $\circ(y, I)^{\subseteq} \subseteq I$.
(i8)   type (8),   if for any $y \in I$, $*(y, I)^{\subseteq} \subseteq I$ or $\circ(y, I)^{\subseteq} \subseteq I$.
(i9)   type (9),   if for any $y \in I$, $*(y, I)^{\ll} \cap \circ(y, I)^{\ll} \subseteq I$.
(i10)   type (10),   if for any $y \in I$, $*(y, I)^{\subseteq} \cap \circ(y, I)^{\ll} \subseteq I$.
(i11)   type (11),   if for any $y \in I$, $*(y, I)^{\ll} \cap \circ(y, I)^{\subseteq} \subseteq I$.

(i12)   type (12),   if for any $y \in I$, $*(y, I)^{\subseteq} \cap \circ(y, I)^{\subseteq} \subseteq I$.

Theorem 3.2. *(i) If $x * y = x \circ y$, for all $x, y \in H$, then any hyper pseudo BCK-ideal of types 1, 2, 3, 5 and 9 in H is a hyper BCK-ideal in H and any hyper pseudo BCK-ideal of types 1, 2, 3, 4, 5, 8, 9 and 12 in H is a weak hyper BCK-ideal in H.*

(ii) If " $*$ " and " $\circ$ " are singleton, then any hyper pseudo BCK-ideal of type 4 in H is a pseudo-ideal in H and any hyper pseudo BCK-ideal of types 2, 3, 4, 8 and 12 in H is a weak pseudo-ideal in H.

Proof. (i) Assume that $x * y = x \circ y$, for all $x, y \in H$ and I be a hyper pseudo BCK-ideal of type 1. First, we prove that I is a hyper BCK-ideal in H. So, let $y \in I$ and $x \circ y \ll I$. Then $x \in \circ(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 1, then $\circ(y, I)^{\ll} \subseteq I$ and so $x \in I$. Hence I is a hyper BCK-ideal in H. Now, we prove that I is a weak hyper BCK-ideal in H. So, let $y \in I$ and $x \circ y \subseteq I$. Then $x \circ y \ll I$ and so $x \in \circ(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 1, then $\circ(y, I)^{\ll} \subseteq I$ and so $x \in I$. Hence I is a weak hyper BCK-ideal in H. By the similar way, we can prove the other cases.

(ii) Assume that " $*$ " and " $\circ$ " are singleton. Then

$$*(y, I)^{\subseteq} = \{x \in H \mid x * y \subseteq I\} = \{x \in H \mid x * y \in I\} = *(y, I)$$

$$\circ(y, I)^{\subseteq} = \{x \in H \mid x \circ y \subseteq I\} = \{x \in H \mid x \circ y \in I\} = \circ(y, I)$$

Now, let I be a hyper pseudo BCK-ideal of type 4. First, we prove that I is a pseudo-ideal in H. So, let $y \in I$, $x \in *(y, I)$ and $z \in \circ(y, I)$. Since I is a hyper pseudo BCK-ideal of type 4, then $*(y, I) = *(y, I)^{\subseteq} \subseteq I$ and $\circ(y, I) = \circ(y, I)^{\subseteq} \subseteq I$ and so $x \in I$ and $z \in I$. Hence I is a pseudo-ideal in H. Now, we prove that I is a weak pseudo-ideal in H. So, let $y \in I$ and $x \in *(y, I) \cap \circ(y, I)$. Then $x \in *(y, I)$ and $x \in \circ(y, I)$. Since I is a hyper pseudo BCK-ideal of type 4, then $*(y, I) = *(y, I)^{\subseteq} \subseteq I$ and $\circ(y, I) = \circ(y, I)^{\subseteq} \subseteq I$ and so $x \in I$. Hence I is a weak pseudo-ideal in H. The proof of other cases are the similar. $\square$

Theorem 3.3. *(i) Any hyper pseudo BCK-ideal of types 1, 2 and 3 in H are equivalent.*

(ii) *Every hyper pseudo BCK-ideal of type 1 in H is a hyper pseudo BCK-ideal of types 4 and 5.*

(iii) *Every hyper pseudo BCK-ideal of type 4 in H is a hyper pseudo BCK-ideal of type 8.*

Proof. (i) Let I be a hyper pseudo BCK-ideal of type 1. We will prove that I is a hyper pseudo BCK-ideal of type 2. It is enough to prove that for any $y \in I$, $*(y, I)^{\subseteq} \subseteq I$. Let $y \in I$ and $x \in *(y, I)^{\subseteq}$. Then $x * y \subseteq I$ and so $x * y \ll I$. Hence $x \in *(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 1, then $*(y, I)^{\ll} \subseteq I$ and so $x \in I$. Therefore, $*(y, I)^{\subseteq} \subseteq I$.

Now, let I be a hyper pseudo BCK-ideal of type 2. We will prove that I is a hyper pseudo BCK-ideal of type 3. Let $y \in I$ and $x \in \circ(y, I)^{\subseteq}$. Then $x \circ y \subseteq I$ and so $x \circ y \ll I$. Hence $x \in \circ(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 2, then $\circ(y, I)^{\ll} \subseteq I$ and so $x \in I$. Hence $\circ(y, I)^{\subseteq} \subseteq I$. Now, let $y \in I$ and $z \in *(y, I)^{\ll}$. Then $z * y \ll I$. We should prove that $z \in I$. Let $z \notin I$, by contrary. First, we claim that for any $w \in H$, if $w \notin I$ then $w \not\ll I$. Now, let $w \in H$ and $w \notin I$. Since I is a hyper pseudo BCK-ideal of type 2 and $0 \in I$, then $\circ(0, I)^{\ll} \subseteq I$ and so $w \notin \circ(0, I)^{\ll}$. Hence $w \circ 0 \not\ll I$ and so by Theorem 2.4 (ix), $w \not\ll I$. Now, since by hypothesis $*(y, I)^{\subseteq} \subseteq I$, then $z \notin *(y, I)^{\subseteq}$ and so $z * y \not\subseteq I$. Then there exists $w \in z * y$ such that $w \notin I$ and so by above, $w \not\ll I$, which is impossible. Because $z * y \ll I$ and $w \in z * y$ then $w \ll I$. Hence $z \in I$ and so $*(y, I)^{\ll} \subseteq I$.

Now, let I be a hyper pseudo BCK-ideal of type 3. We will prove that I is a hyper pseudo BCK-ideal of type 1. It is enough to prove that for any $y \in I$, $\circ(y, I)^{\ll} \subseteq I$. Similar to above, we can proof that if $x \in \circ(y, I)^{\ll}$ then $x \in I$.

(ii) Let I be a hyper pseudo-ideal of type 1. We will prove that I is a hyper pseudo BCK-ideal of type 4. Let $y \in I$, $x \in *(y, I)^{\subseteq}$ and $z \in \circ(y, I)^{\subseteq}$. Then $x * y \subseteq I$ and $z \circ y \subseteq I$ and so $x * y \ll I$ and $z \circ y \ll I$. Hence $x \in *(y, I)^{\ll}$ and $z \in \circ(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 1, then $*(y, I)^{\ll} \subseteq I$ and $\circ(y, I)^{\ll} \subseteq I$ and so $x \in I$ and $z \in I$. Hence $*(y, I)^{\subseteq} \subseteq I$ and $\circ(y, I)^{\subseteq} \subseteq I$.

Now, let I be a hyper pseudo BCK-ideal of type 1. We will prove that I is a hyper pseudo BCK-ideal of type 5. Let $y \in I$ and $x \in *(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 1, then $*(y, I)^{\ll} \subseteq I$ and so $x \in I$. Hence $*(y, I)^{\ll} \subseteq I$. If $x \in \circ(y, I)^{\ll}$ by the similar way, we can proof that $x \in I$. Hence $\circ(y, I)^{\ll} \subseteq I$.

(iii) The proof is similar to the proof of case (ii), by some modification. $\square$

The following examples show that the converse of Theorem 3.3 (ii) and (iii) are not correct in general.

Example 3.4. (i) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b		$*$	0	a	b
0	{0}	{0}	{0}		0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}		a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}		b	{b}	{b}	{0,a}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 4 but it is not a hyper pseudo BCK-ideal of type 1. Because $a \in *(b, I)^{\ll}$ but $a \notin I$.

(ii) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, a\}$ is a hyper pseudo BCK-ideal of type 5 but it is not a hyper pseudo BCK-ideal of type 1. Because $b \in *(a, I)^{\ll}$ but $b \notin I$.

(iii) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 8 but it is not a hyper pseudo BCK-ideal of type 4. Because $a \in \circ(b, I)^{\subseteq}$ but $a \notin I$.

Theorem 3.5. *Every hyper pseudo BCK-ideal of type 5 in H is a hyper pseudo BCK-ideal of types 6, 7, 8 and 9.*

Proof. Let I be a hyper pseudo BCK-ideal of type 5. We will prove that I is a hyper pseudo BCK-ideal of type 6. If $\circ(y, I)^{\ll} \subseteq I$, then the proof is straightforward. Now, let $\circ(y, I)^{\ll} \nsubseteq I$ and $x \in *(y, I)^{\subseteq}$ for $y \in I$. Then $x * y \subseteq I$ and so $x * y \ll I$. Hence $x \in *(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 5 and $\circ(y, I)^{\ll} \nsubseteq I$, then $*(y, I)^{\ll} \subseteq I$ and so $x \in I$. Hence $*(y, I)^{\subseteq} \subseteq I$. The proof of type 7 is the similar. Now, let I be a hyper pseudo BCK-ideal of type 5. We will prove that I is a hyper pseudo BCK-ideal of type 8. If $\circ(y, I)^{\subseteq} \subseteq I$, then the proof is straightforward. Now, let $\circ(y, I)^{\subseteq} \nsubseteq I$ and $x \in *(y, I)^{\subseteq}$ for $y \in I$. Then $x * y \subseteq I$ and so $x * y \ll I$. Hence $x \in *(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 5, then $\circ(y, I)^{\ll} \subseteq I$ or $*(y, I)^{\ll} \subseteq I$. It is enough to prove that $\circ(y, I)^{\ll} \nsubseteq I$. If we assume that $\circ(y, I)^{\ll} \subseteq I$, since by the hypothesis $\circ(y, I)^{\subseteq} \nsubseteq I$ there exists $a \in \circ(y, I)^{\subseteq}$ such that $a \notin I$. Then $a \circ y \subseteq I$ and so $a \circ y \ll I$. Hence $a \in \circ(y, I)^{\ll}$ and so $a \in I$. That is contrast. Therefore, $\circ(y, I)^{\ll} \nsubseteq I$. Hence $*(y, I)^{\ll} \subseteq I$ and so $x \in I$. Thus $*(y, I)^{\subseteq} \subseteq I$. Now, let I be a hyper pseudo BCK-ideal of type 5. We will prove that I is a hyper pseudo BCK-ideal of type 9. Let $y \in I$ and $x \in *(y, I)^{\ll} \cap \circ(y, I)^{\ll}$. Then $x \in *(y, I)^{\ll}$ and $x \in \circ(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 5, then $*(y, I)^{\ll} \subseteq I$ or $\circ(y, I)^{\ll} \subseteq I$ and so $x \in I$. Hence $*(y, I)^{\ll} \cap \circ(y, I)^{\ll} \subseteq I$. $\square$

Corollary 3.6. *Every hyper pseudo BCK-ideal of type 2 or 3 in H is a hyper pseudo BCK-ideal of types 6, 7, 8 and 9.*

The following examples show that the converse of Theorem 3.5 is not correct in general.

Example 3.7. (i) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 6 but it is not a hyper pseudo BCK-ideal of type 5. Because $a \in *(b, I)^{\ll}$ but $a \notin I$ and $a \in \circ(b, I)^{\ll}$ but $a \notin I$.

(ii) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 7 but it is not a hyper pseudo BCK-ideal of type 5. Because $a \in \circ(b, I)^{\ll}$ but $a \notin I$ and $a \in *(b, I)^{\ll}$ but $a \notin I$.

(iii) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 8 but it is not a hyper pseudo BCK-ideal of type 5. Because $a \in *(0, I)^{\ll}$ but $a \notin I$ and $a \in \circ(b, I)^{\ll}$ but $a \notin I$.

(iv) Let $H = \{0, a, b, c\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b	c
0	{0}	{0}	{0}	{0}
a	{a}	{0}	{a}	{0}
b	{b}	{b}	{0}	{0}
c	{c}	{b}	{c}	{0}

$*$	0	a	b	c
0	{0}	{0}	{0}	{0}
a	{a}	{0}	{a}	{0}
b	{b}	{b}	{0}	{0}
c	{c}	{c}	{a}	{0}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, a, b\}$ is a hyper pseudo BCK-ideal of type 9 but it is not a hyper pseudo BCK-ideal of type 5. Because $c \in *(b, I)^{\ll}$ but $c \notin I$ and $c \in \circ(a, I)^{\ll}$ but $c \notin I$.

Theorem 3.8. *(i) Every hyper pseudo BCK-ideal of type 6 in H is a hyper pseudo BCK-ideal of types 8 and 10.*

(ii) *Every hyper pseudo BCK-ideal of type 7 in H is a hyper pseudo BCK-ideal of types 8 and 11.*

(iii) *Every hyper pseudo BCK-ideal of type 8 in H is a hyper pseudo BCK-ideal of type 12.*

Proof. (i) Let I be a hyper pseudo BCK-ideal of type 6. We will prove that I is a hyper pseudo BCK-ideal of type 8. If $*(y, I)^{\subseteq} \subseteq I$ the proof is straightforward. Let $*(y, I)^{\subseteq} \not\subseteq I$ and $x \in \circ(y, I)^{\subseteq}$ for $y \in I$. Then $x \circ y \subseteq I$ and so $x \circ y \ll I$. Hence $x \in \circ(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 6 and $*(y, I)^{\subseteq} \not\subseteq I$, then $\circ(y, I)^{\ll} \subseteq I$ and so $x \in I$. Thus $\circ(y, I)^{\subseteq} \subseteq I$. Now, let I be a hyper pseudo BCK-ideal of type 6. We will prove that I is a hyper pseudo BCK-ideal of type 10. Let $y \in I$ and $x \in *(y, I)^{\subseteq} \cap \circ(y, I)^{\ll}$. Then $x \in *(y, I)^{\subseteq}$ and $x \in \circ(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 6, then $*(y, I)^{\subseteq} \subseteq I$ or $\circ(y, I)^{\ll} \subseteq I$ and so $x \in I$. Hence $*(y, I)^{\subseteq} \cap \circ(y, I)^{\ll} \subseteq I$.

The proof of cases (ii) and (iii) are the similar to the proof of case (i), by the some modification. □

The following examples show that the converse of Theorem 3.8 is not correct in general.

Example 3.9. (i) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 8 but it is not a hyper pseudo BCK-ideal of type 6. Because $a \in *(b, I)^{\subseteq}$ but $a \notin I$ and $a \in \circ(b, I)^{\ll}$ but $a \notin I$.

(ii) Let $H = \{0, a, b, c\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$*$	0	a	b	c
0	{0}	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}	{0,a}
b	{b}	{b}	{0}	{0}
c	{c}	{c}	{b}	{0}

$\circ$	0	a	b	c
0	{0}	{0}	{0}	{0}
a	{a}	{0,a}	{0}	{0}
b	{b}	{b}	{0,b}	{0,b}
c	{c}	{c}	{b,c}	{0,c}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 10 but it is not a hyper pseudo BCK-ideal of type 6. Because $c \in *(b, I)^{\subseteq}$ but $c \notin I$ and $a \in \circ(b, I)^{\ll}$ but $a \notin I$.

(iii) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

∘	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

*	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 8 but it is not a hyper pseudo BCK-ideal of type 7. Because $a \in \circ(b, I)^{\subseteq}$ but $a \notin I$ and $a \in *(b, I)^{\ll}$ but $a \notin I$.

(iv) Let $H = \{0, a, b, c\}$ and operations " * " and " ∘ " on H be defined as follows:

∘	0	a	b	c
0	{0}	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}	{0,a}
b	{b}	{b}	{0}	{0}
c	{c}	{c}	{b}	{0}

*	0	a	b	c
0	{0}	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}	{0,b}
c	{c}	{c}	{b,c}	{0,c}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 11 but it is not a hyper pseudo BCK-ideal of type 7. Because $a \in *(b, I)^{\ll}$ but $a \notin I$ and $c \in \circ(b, I)^{\subseteq}$ but $c \notin I$.

(v) Let $H = \{0, a, b, c\}$ and operations " * " and " ∘ " on H be defined as follows:

*	0	a	b	c
0	{0}	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}	{0,a}
b	{b}	{b}	{0}	{0}
c	{c}	{c}	{c}	{0,c}

∘	0	a	b	c
0	{0}	{0}	{0}	{0}
a	{a}	{0}	{0}	{0}
b	{b}	{b}	{0,b}	{0,b}
c	{c}	{c}	{c}	{0,c}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, c\}$ is a hyper pseudo BCK-ideal of type 12 but it is not a hyper pseudo BCK-ideal of type 8. Because $b \in *(c, I)^{\subseteq}$ but $b \notin I$ and $a \in \circ(c, I)^{\subseteq}$ but $a \notin I$.

Theorem 3.10. *(i) Every hyper pseudo BCK-ideal of type 9 in H is a hyper pseudo BCK-ideal of types 10, 11 and 12.*

  (ii) *Every hyper pseudo BCK-ideal of type 10 in H is a hyper pseudo BCK-ideal of type 12.*

  (iii) *Every hyper pseudo BCK-ideal of type 11 in H is a hyper pseudo BCK-ideal of type 12.*

Proof. (i) Let I be a hyper pseudo BCK-ideal of type 9. We will prove that I is a hyper pseudo BCK-ideal of type 10. Let $y \in I$ and $x \in *(y, I)^{\subseteq} \cap \circ(y, I)^{\ll}$. Then $x \in *(y, I)^{\subseteq}$ and $x \in \circ(y, I)^{\ll}$. Thus $x * y \subseteq I$ and $x \circ y \ll I$ and so $x * y \ll I$. Hence $x \in *(y, I)^{\ll} \cap \circ(y, I)^{\ll}$. Since I is a hyper pseudo BCK-ideal of type 9, then $*(y, I)^{\ll} \cap \circ(y, I)^{\ll} \subseteq I$ and so $x \in I$. Hence $*(y, I)^{\subseteq} \cap \circ(y, I)^{\ll} \subseteq I$. The proof of the other cases are similar.

The proof of cases (ii) and (iii) are the similar to the proof of case (i), by the some modifications.   □

The following examples show that the converse of Theorem 3.10 is not correct in general.

Example 3.11. (i) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 10 but it is not a hyper pseudo BCK-ideal of type 9. Because $a \in *(b, I)^{\ll} \cap \circ(b, I)^{\ll}$ but $a \notin I$.

(ii) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

| $*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 11 but it is not a hyper pseudo BCK-ideal of type 9. Because $a \in *(b, I)^{\ll} \cap \circ(b, I)^{\ll}$ but $a \notin I$.

(iii) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 12 but it is not a hyper pseudo BCK-ideal of type 9. Because $a \in *(0, I)^{\ll} \cap \circ(0, I)^{\ll}$ but $a \notin I$.

(iv) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 12 but it is not a hyper pseudo BCK-ideal of type 10. Because $a \in *(b, I)^{\subseteq} \cap \circ(b, I)^{\ll}$ but $a \notin I$.

(v) Let $H = \{0, a, b\}$ and operations " $*$ " and " $\circ$ " on H be defined as follows:

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

Then $(H, \circ, *, 0)$ is a hyper pseudo BCK-algebra. We can see that $I = \{0, b\}$ is a hyper pseudo BCK-ideal of type 12 but it is not a hyper pseudo BCK-ideal of type 11. Because $a \in *(b, I)^{\ll} \cap \circ(b, I)^{\subseteq}$ but $a \notin I$.

In the following diagram, we can see the relationship among all of types of hyper pseudo BCK-ideals.
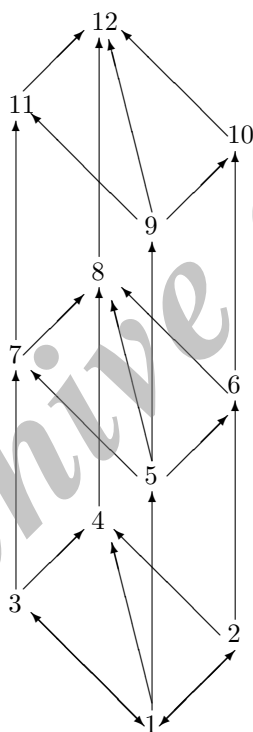
4. Characterization of generated hyper pseudo BCK-ideals

Now, in this section we characterize the hyper pseudo BCK-ideals generated by a nonempty subset.

Proposition 4.1. *Let $\{I_\lambda | \lambda \in \Lambda\}$ be a family of hyper pseudo BCK-ideals of type i, for $1 \le i \le 12$, in H. Then $\bigcap_{\lambda \in \Lambda} I_\lambda$ is a hyper pseudo BCK-ideal of type i, for $1 \le i \le 12$, in H, too.*

Proof. Let $I = \bigcap_{\lambda \in \Lambda} I_\lambda$ and for any $\lambda \in \Lambda$, I_λ be a hyper pseudo BCK-ideal of type 1. We will prove that I is a hyper pseudo BCK-ideal of type 1. Clearly $0 \in I$. Now, let $y \in I$, $x \in *(y, I)^{\ll}$ and $z \in \circ(y, I)^{\ll}$. Then $x * y \ll I$ and $z \circ y \ll I$. Since $x * y \ll I$, for any $u \in x * y$ there exists $v \in I$ such that $u \ll v$. Since $v \in I_\lambda$, for any $\lambda \in \Lambda$, then $x * y \ll I_\lambda$, for any $\lambda \in \Lambda$. Hence $x \in *(y, I_\lambda)^{\ll}$ and $y \in I_\lambda$ for any $\lambda \in \Lambda$. Since I_λ is a hyper pseudo BCK-ideal of type 1, then $*(y, I_\lambda)^{\ll} \subseteq I_\lambda$ and so $x \in I_\lambda$ for any $\lambda \in \Lambda$. Hence $x \in I$ and so $*(y, I)^{\ll} \subseteq I$. By the similar way, we can prove that $z \in I$ and so $\circ(y, I)^{\ll} \subseteq I$. The proof of other cases are the similar. $\qquad\square$

Theorem 4.2. *Let A be a nonempty subset of H. By the hyper pseudo BCK-ideal of type i, for $1 \le i \le 4$, generated by A, written $[A]_i$, we mean that intersection of all hyper pseudo BCK-ideals of type i which contain A. Then*

$$[A]_i \supseteq \{x \in H : (...((x \circ a_1) \circ a_2) \circ ...) \circ a_n = \{0\}, \ for \ some \ a_1, a_2, ..., a_n \in A\}.$$

Proof. We prove the theorem for type 1, but the proof of other cases are the similar. Assume that $x \in H$ satisfies the identity

$$(...((x \circ a_1) \circ a_2) \circ ...) \circ a_n = \{0\}$$

for some $a_1, a_2, ..., a_n \in A$. Since $0 \in [A]_1$, we have

$$(...((x \circ a_1) \circ a_2) \circ ...) \circ a_n = \{0\} \subseteq [A]_1$$

and so $(...((x \circ a_1) \circ a_2) \circ ...) \circ a_n \ll [A]_1$. Thus for each

$$a \in (...((x \circ a_1) \circ a_2) \circ ...) \circ a_{n-1}$$

we get $a \circ a_n \ll [A]_1$ i.e. $a \in \circ(a_n, [A]_1)^{\ll}$. Now, since $[A]_1$ is a hyper pseudo BCK-ideal of type 1, then $\circ(a_n, [A]_1)^{\ll} \subseteq [A]_1$ and so $a \in [A]_1$. Hence

$$(...((x \circ a_1) \circ a_2) \circ ...) \circ a_{n-1} \subseteq [A]_1$$

and so

$$(...((x \circ a_1) \circ a_2) \circ ...) \circ a_{n-1} \ll [A]_1$$

Continuing this process, we conclude that $\{x\} \subseteq [A]_1$ and so $x \in [A]_1$. Therefore

$$[A]_1 \supseteq \{x \in H : (...((x \circ a_1) \circ a_2) \circ ...) \circ a_n = \{0\}, \ for \ some \ a_1, a_2, ..., a_n \in A\}$$

. $\qquad\square$

Theorem 4.3. *Let $|x * y| < \infty$ and $|x \circ y| < \infty$ for all $x, y \in H$ and H satisfies the following condition*

$$(...((x \circ y_1) \circ y_2) \circ ...) \circ y_m = \{0\} \quad iff \quad (...((x * y_m) * y_{m-1}) * ...) * y_1 = \{0\}$$

for all $x, y_1, y_2, ..., y_m \in H$. If A is a nonempty subset of H which for all $a \in A, a \circ a = \{0\}$, then

$$[A]_4 = \{x \in H : (...((x \circ a_1) \circ a_2) \circ ...) \circ a_n = \{0\}, \quad for \ some \ a_1, a_2, ..., a_n \in A\}.$$

Proof. Let $B = \{x \in H : (...((x \circ a_1) \circ a_2) \circ ...) \circ a_n = \{0\}, \ for \ some \ a_1, a_2, ..., a_n \in A\}$. According to the Theorem 4.2, we only need to prove that $[A]_4 \subseteq B$. To do this, it is sufficient to show that B is a hyper pseudo-ideal of type 4 containing A. For each $a \in A$, we have $a \circ a = \{0\}$ and so $a \in B$. Therefore, $A \subseteq B$. Since $A \neq \emptyset$, we can take $a \in A$. By Proposition 2.4 (vi), $0 \circ a = \{0\}$ and so $0 \in B$. Now, let $y \in B$ and $x \in *(y, B)^{\subseteq}$. Then $x * y \subseteq B$. Since $|x * y| < \infty$, then there exist $a_1, ..., a_n \in A$ such that

$$(...(((x * y) \circ a_1) \circ a_2) \circ ...) \circ a_n = \{0\}.$$

By using $(PHK2)$, we see that

$$((...((x \circ a_1) \circ a_2) \circ ...) \circ a_n) * y = \{0\}.$$

Thus for each $u \in (...((x \circ a_1) \circ a_2) \circ ...) \circ a_n$, we have $u * y = \{0\}$. Since $y \in B$, then there exist $b_1, ..., b_m \in A$ such that $(...((y \circ b_1) \circ b_2) \circ ...) \circ b_m = \{0\}$ and so by hypothesis $(...((y * b_m) * b_{m-1}) * ...) * b_1 = \{0\}$. Hence by Proposition 2.4 (xiv),

$$
\begin{aligned}
&((...((u * b_m) * b_{m-1}) * ...) * b_1) * \{0\} \\
=\ &(((...((u * b_m) * b_{m-1}) * ...) * b_2) * b_1) \\
&*(((...((y * b_m) * b_{m-1}) * ...) * b_2) * b_1) \\
\ll\ &((...((u * b_m) * b_{m-1}) * ...) * b_2) * ((...((y * b_m) * b_{m-1}) * ...) * b_2) \\
\ll\ &((...((u * b_m) * b_{m-1}) * ...) * b_3) * ((...((y * b_m) * b_{m-1}) * ...) * b_3) \\
&\vdots \\
\ll\ &u * y \\
=\ &\{0\}
\end{aligned}
$$

Thus by Proposition 2.4 (xiii), $(...((u * b_m) * b_{m-1}) * ...) * b_1 = \{0\}$ and so by hypothesis we obtain $(...((u \circ b_1) \circ b_2) \circ ...) \circ b_m = \{0\}$. Hence

$$(...(((...((x \circ a_1) \circ a_2) \circ ...) \circ a_n) \circ b_1) \circ b_2) \circ ...) \circ b_m = \{0\}.$$

Since $a_i, b_j \in A$, for $i = 1, 2, ..., n; j = 1, 2, ..., m$, then $x \in B$. Now, let $y \in B$ and $z \in \circ(y, B)^{\subseteq}$. Then $z \circ y \subseteq B$. Since $|z \circ y| < \infty$, then by the similar way,

we get that $z \in B$. Hence $*(y, B)^{\subseteq} \subseteq B$ and $\circ(y, B)^{\subseteq} \subseteq B$ i.e. B is a hyper pseudo-ideal of type 4 in H containing A. □

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