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Binary Multiquasigroups with Medial-Like Equations

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Abstract. In this paper paramedial, co-medial and co-paramedial binary multiquasigroups are considered and a characterization of the corresponding component operations of these multiquasigroups is given.

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1. Introduction

One way to define a binary quasigroup is that it is a groupoid (A, f) in which for any $a, b \in A$ there are unique solutions x, y to the equations $f(a, x) = b$, $f(y, a) = b$. A loop is a quasigroup with unit (e) such that

$$f(e, x) = f(x, e) = x.$$

Groups are associative quasigroups, i.e., they satisfy:

$$f(f(x, y), z) = f(x, f(y, z)).$$

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There are various generalization of a group (see, [2, 3]). Most of the notions defined for binary quasigroups can be easily generalized to n-ary operations which are called n-quasigroups. An n-quasigroup is an n-groupoid (A, f) ($f : A^n \to A, n > 0$) in which for every n-sequence $a_1, \ldots, a_n$ of elements from A, every $a \in A$ and every i ($1 \leq i \leq n$), there is a unique solution x of the equation

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = a.$$

For example, 1-quasigroups are just bijections.

Let A be a nonempty set, n and m be positive integers and $f : A^n \to A^m$ be an arbitrary function. Then (A, f) is called $[n, m]$-groupoid. The n-ary operations, $f_1, \ldots, f_m$, are defined by the following:

$$f(x_1, \ldots, x_n) = (y_1, \ldots, y_m) \Leftrightarrow y_i = f_i(x_1, \ldots, x_n),$$

for every $1 \leq i \leq m$, are called the component operations of f and they are denoted by $f = (f_1, \ldots, f_m)$ [22, 23, 26]. The $[n, m]$-groupoid is proper iff $n, m, |Q| \geq 2$.

The $[n, m]$-groupoid (A, f) is called $[n, m]$-quasigroup (or multiquasigroup [9, 10, 27]) iff for every injection, $\phi : N_n \to N_{n+m}$, where $N_n = \{1, \ldots, n\}$, and every $(a_1, \ldots, a_n) \in Q^n$ there exists a unique $(b_1, \ldots, b_{n+m}) \in Q^{n+m}$ such that:

$$f(b_1, \ldots, b_n) = (b_{n+1}, \ldots, b_{n+m}) \quad \text{and} \quad b_{\phi(i)} = a_i,$$

for $i = 1, \ldots, n$.

It is clear that $Q(f)$ is an $[n, 1]$-quasigroup iff $Q(f)$ is an n-quasigroup [6]. $Q(f)$ is a $[1, m]$-quasigroup iff there exist permutations, $f_1, \ldots, f_m$, of Q such that $f(x) = (f_1(x), \ldots, f_m(x))$. It is also clear that all components of a multiquasigroup are quasigroup operations.

If the component operations of the $[n, m]$-quasigroup are binary operations, i.e. $n = 2$, then we say that the $[n, m]$-quasigroup is a binary multiquasigroup.

Let us consider the following hyperidentities [17, 18, 19]:

$$g(f(x, y), f(u, v)) = f(g(x, u), g(y, v)), \quad \text{(Mediality)} \quad (1.1)$$

$$g(f(x, y), f(u, v)) = f(g(v, y), g(u, x)), \quad \text{(Paramediality)} \quad (1.2)$$

$$g(f(x, y), f(u, v)) = g(f(x, u), f(y, v)), \quad \text{(Co-mediality)} \quad (1.3)$$

$$g(f(x, y), f(u, v)) = g(f(v, y), f(u, x)), \quad \text{(Co-paramediality)} \quad (1.4)$$

$$f(x, x) = x. \quad \text{(Idempotency)} \quad (1.5)$$

The binary algebra, (A, F), is called:

- medial, if it satisfies the identity (1.1),
- paramedial, if it satisfies the identity (1.2),
- co-medial, if it satisfies the identity (1.3),
- co-paramedial, if it satisfies the identity (1.4),
- idempotent, if it satisfies the identity (1.5),

for every $f, g \in F$. The binary algebra, (A, F), is called mode, if it is medial and idempotent.

Medial groupoids, medial algebras and medial idempotent algebras (modes) were studied in [12, 13, 24]. Paramedial groupoids and paramedial quasigroups were studied in [7, 21, 25]. In general, the properties of mediality, paramediality, co-mediality and co-paramediality are the second order properties of the algebras in the sense of [8, 15, 19, 17].

Definition 1.1. The binary multiquasigroup (A, f) with $f = (f_1, \ldots, f_m)$ is called:

- medial, if the binary algebra, $(A, f_1, \ldots, f_m)$, is medial,
- paramedial, if the binary algebra, $(A, f_1, \ldots, f_m)$, is paramedial,
- co-medial, if the binary algebra, $(A, f_1, \ldots, f_m)$, is co-medial,
- co-paramedial, if the binary algebra, $(A, f_1, \ldots, f_m)$, is co-paramedial,
- idempotent, if the binary algebra, $(A, f_1, \ldots, f_m)$, is idempotent,
- mode, if the binary algebra, $(A, f_1, \ldots, f_m)$, is a mode.

The next characterization of binary medial multiquasigroups follows from [4, 16, 20].

Theorem 1.2. *Let (Q, f) be a binary multiquasigroup, where $f = (f_1, \ldots, f_m)$. If (Q, f) is a binary medial multiquasigroup, then there exists an abelian group, $(Q, +)$, such that:*

$$f_i(x, y) = \alpha_i x + \beta_i y + c_i,$$

where α_i, β_i are automorphisms of the group $(Q, +)$, and $c_i \in Q$ is a fixed element and: $\alpha_i\beta_j = \beta_j\alpha_i, \alpha_i\alpha_j = \alpha_j\alpha_i, \beta_i\beta_j = \beta_j\beta_i$, for $i, j = 1, \ldots, m$. The group, $(Q, +)$, is unique up to isomorphisms. Moreover, if (Q, f) is a mode, then

$$f_i(x, y) = \alpha_i x + \beta_i y,$$

where α_i, β_i are automorphisms of both the group, $(Q, +)$, and of the algebra, $(Q, f_1, \ldots, f_m)$.

2. Main Results

To characterize the paramedial, co-medial and co-paramedial multiquasigroups we need the concept of holomorphism for groups [14, 19].

Definition 2.1. If $(Q, \cdot)$ is a group, then the bijection, $\alpha : Q \to Q$, is called a holomorphism of $(Q, \cdot)$ if

$$\alpha(x \cdot y^{-1} \cdot z) = \alpha x \cdot (\alpha y)^{-1} \cdot \alpha z,$$

for every $x, y, z \in Q$. Note that this concept is equivalent to the concept of quasiautomorphism of groups [5].

The set of all holomorphisms of $(Q,\cdot)$ is denoted by $Hol(Q,\cdot)$ and it is a group under the superposition of the mappings: $(\alpha\cdot\beta)x=\beta(\alpha x)$, for every $x\in Q$.

Lemma 2.2. [19] *Let for bijections $\alpha_1,\alpha_2,\alpha_3$ on the group, $(Q,\cdot)$, the following identity be satisfied:*

$$\alpha_1(x\cdot y)=\alpha_2(x)\cdot\alpha_3(y),$$

then $\alpha_1,\alpha_2,\alpha_3\in Hol(Q,\cdot)$.

Lemma 2.3. [19] *Every holomorphism, α, of the group, $(Q,\cdot)$, has the following form:*

$$\alpha x=\varphi x\cdot k,$$

where $\varphi\in Aut(Q,\cdot)$ and $k\in Q$.

The triple, (α,β,γ), of the bijections from the set, G, onto the set, H, is called an isotopism of the groupoid, $(G,\cdot)$, onto the groupoid, $(H,\circ)$, provided: $\gamma(x\cdot y)=\alpha x\circ\beta y$, for all $x,y\in G$. $(H,\circ)$ is called an isotope of $(G,\cdot)$, and the groupoids, $(G,\cdot)$ and $(H,\circ)$, are called isotopic to each other. The isotopism of $(G,\cdot)$ onto $(G,\cdot)$ is called the autotopism of $(G,\cdot)$.

Let α and β be the permutations of G and ι denoting the identity map on G. Then (α,β,ι) is the principal isotopism of the groupoid, $(G,\cdot)$, onto the groupoid, $(G,\circ)$, meaning that (α,β,ι) is an isotopism of $(G,\cdot)$ onto $(G,\circ)$.

Theorem 2.4. *Let (Q,f) be a binary multiquasigroup, where $f=(f_1,\dots,f_m)$. If (Q,f) is a binary paramedial multiquasigroup, then there exists an abelian group, $(Q,+)$, such that:*

$$f_i(x,y)=\alpha_i x+\beta_i y+c_i,$$

where α_i,β_i are automorphisms of the group, $(Q,+)$, and $c_i\in Q$ is a fixed element and: $\alpha_i\beta_j=\alpha_j\beta_i,\alpha_i\alpha_j=\beta_j\beta_i,\beta_i\alpha_j=\beta_j\alpha_i$, for $i,j=1,\dots,m$. The group, $(Q,+)$, is unique up to isomorphisms.

Proof. If f_1 is a fixed component operation of the binary multiquasigroup, (Q,f), then by [21], f_1 is principally isotopic to the abelian group operation, $*$, on Q. Now, if f_i is any component operation, then the pair of operations, (f_1,f_i), is paramedial.

First, we use the main result of [1] (also see [4]). If the set, Q, forms a quasigroup under 6 operations, $A_i(x,y)$ (for $i=1,\dots,6$), and if these operations satisfy the equation:

$$A_1(A_2(x,y),A_3(u,v))=A_4(A_5(x,u),A_6(y,v)),\tag{2.1}$$

for all elements, x,y,u,v, of the set, Q, then there exists an operation, '+', under which Q forms an abelian group on which all these 6 quasigroups are isotopic. And there exist 8 one-to-one mappings, $\alpha, \beta, \gamma, \delta, \epsilon, \psi, \varphi, \chi$, of Q onto itself such that:

$$\begin{aligned} A_1(x,y) &= \delta x + \varphi y, & A_2(x,y) &= \delta^{-1}(\alpha x + \beta y), \\ A_3(x,y) &= \varphi^{-1}(\chi x + \gamma y), & A_4(x,y) &= \psi x + \epsilon y, \\ A_5(x,y) &= \psi^{-1}(\alpha x + \chi y), & A_6(x,y) &= \epsilon^{-1}(\beta x + \gamma y). \end{aligned}$$

Now, let $A_i^*(x,y) = A_i(y,x)$; then, putting it in (2.1), we have:

$$A_1(A_2(x,y), A_3(u,v)) = A_4^*(A_6^*(v,y), A_5^*(u,x)), \tag{2.2}$$

and

$$\begin{aligned} A_4^*(x,y) &= A_4(y,x) = \psi y + \epsilon x = \epsilon x + \psi y, \\ A_5^*(x,y) &= A_5(y,x) = \psi^{-1}(\alpha y + \chi x) = \psi^{-1}(\chi x + \alpha y), \\ A_6^*(x,y) &= A_6(y,x) = \epsilon^{-1}(\beta y + \gamma x) = \epsilon^{-1}(\gamma x + \beta y), \end{aligned}$$

since, $(Q,+)$ is an abelian group. But, by the definition of paramedial pair operations, (f_1, f_i), we know:

$$f_i(f_1(x,y), f_1(u,v)) = f_1(f_i(v,y), f_i(u,x)). \tag{2.3}$$

So, let $A_1 = A_5^* = A_6^* = f_i$ and $A_2 = A_3 = A_4^* = f_1$. With this assumption, we reach the equation (2.3), from the equation (2.2). Therefore, since $A_1 = A_5^*$, we have:

$$\begin{aligned} &\delta x + \varphi y = \psi^{-1}(\chi x + \alpha y) \\ &\Rightarrow \psi(\delta x + \varphi y) = \chi x + \alpha y \\ &\Rightarrow \psi(x + y) = \chi(\delta^{-1} x) + \alpha(\varphi^{-1} y) \\ &\Rightarrow \psi \in Hol(Q,+), \end{aligned}$$

by Lemma 2.2.

Similarly, since $A_1 = A_6^*$, we have: $\epsilon \in Hol(Q,+)$. Therefore, by Lemma 2.3, there exist $\varphi_1, \psi_1 \in Aut(Q,+)$ such that:

$$\begin{aligned} \psi x &= \varphi_1 x + a, \\ \epsilon x &= b + \psi_1 x, \end{aligned}$$

where a, b are fixed elements in Q. Hence,

$$\begin{aligned} &f_1(x,y) = A_4^*(x,y) = \psi x + \epsilon y = \\ &\varphi_1 x + a + b + \psi_1 x = \varphi_1 x + c_1 + \psi_1 x, \end{aligned}$$

where $c_1 = a + b$ is a fixed element in Q.

By the same manner, we can show that: $\delta, \varphi \in Hol(Q,+)$, since $A_2 = A_4^*$ and $A_3 = A_4^*$. So, there exist $\varphi_2, \psi_2 \in Aut(Q,+)$ such that:

$$\begin{aligned} \delta x &= \varphi_2 x + d, \\ \varphi x &= e + \psi_2 x, \end{aligned}$$

where d, e are fixed elements in Q. Hence,

$$f_i(x, y) = A_1(x, y) = \delta x + \varphi y = \varphi_2 x + c_2 + \psi_2 y,$$

where $c_2 = d + e$ is a fixed element in Q.

Now, put

$$f_1(x, y) = \varphi_1(x) + \psi_1(y) + c_1,$$
$$f_i(x, y) = \varphi_2(x) + \psi_2(y) + c_2,$$

in equation (2.3), if $x = 0$; then we obtain $\varphi_1\varphi_2 = \psi_2\psi_1$, if $y = 0$; then $\varphi_1\psi_2 = \varphi_2\psi_1$, if $u = 0$; then $\psi_1\varphi_2 = \psi_2\varphi_1$; and if $v = 0$, then $\varphi_2\varphi_1 = \psi_1\psi_2$.

Therefore, f_1 and f_i are principally isotopic to the group operation, $+$, on Q. Thus, by transitivity of isotopy, any component operation, f_i, is principally isotopic to the same abelian group operation, $+$.

The uniqueness of the group, $(Q, +)$, follows from the Albert's theorem [5, 13, 19]: if every two groups are isotopic, then they are isomorphic.

□

Lemma 2.5. *Let for bijections $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ on the group, $(Q, \cdot)$, the following identity be satisfied:*

$$\alpha_1(\alpha_2(x \cdot y) \cdot z) = \alpha_3 x \cdot \alpha_4(\alpha_5 y \cdot \alpha_6 z),$$

then $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in Hol(Q, \cdot)$ (see [19] *p. 36, for Moufang loops).*

Lemma 2.6. *Let $\alpha_0 \in Hol(Q, \cdot)$ and $k \in Q$, then the mapping,*

$$\alpha x = \alpha_0 x \cdot k,$$

$x \in Q$, is a holomorphism of the group, $(Q, \cdot)$ (see [19] *p. 36, for Moufang loops).*

Theorem 2.7. *Let (Q, f) be a binary multiquasigroup, where $f = (f_1, \ldots, f_m)$. If (Q, f) is a binary co-medial multiquasigroup, then there exists an abelian group, $(Q, +)$, such that*

$$f_i(x, y) = \alpha_i x + \beta_i y + c_i,$$

where α_i, β_i are automorphisms of the group, $(Q, +)$, and $c_i \in Q$ is a fixed element and: $\alpha_i\beta_j = \beta_i\alpha_j$, for $i, j = 1, \ldots, m$. The group, $(Q, +)$, is unique up to isomorphisms.

Proof. Let f_1, f_2 be fixed component operations; then by the definition of comediality:

$$f_1(f_2(x, y), f_2(u, v)) = f_1(f_2(x, u), f_2(y, v)).$$

Also, for every component operation, f_i, we have:

$$f_i(f_2(x, y), f_2(u, v)) = f_i(f_2(x, u), f_2(y, v)). \quad (2.4)$$

So, by the main result of [1], the algebras, (Q, f_1) and (Q, f_2), are isotopic to the abelian group, $(Q, \circ)$; and the algebras, (Q, f_1) and (Q, f_i), are isotopic to the abelian group, $(Q, \cdot)$. Thus, by transitivity of isotopy, the algebra, (Q, f_i), is isotopic to $(Q, \circ)$ and we have:

$$f_i(x, y) = \eta_i^{-1}(\alpha_i x \circ \beta_i y),$$

where $\eta_i, \alpha_i, \beta_i$ are bijections of Q.

Let $u = a \in Q$, then:

$$f_i(f_2(x, y), f_2(a, v)) = f_i(f_2(x, a), f_2(y, v)).$$

Put $f_2(a, v) = pv$ and $f_2(x, a) = qx$; then

$$\begin{aligned} f_i(f_2(x, y), pv) &= f_i(qx, f_2(y, v)), \\ f_i(f_2(x, y), v) &= f_i(qx, f_2(y, p^{-1}v)), \\ f_i(f_2(x, y), v) &= g_i(x, g_2(y, v)), \end{aligned} \tag{2.5}$$

where $g_i(x, y) = f_i(qx, y)$ and $g_2(x, y) = f_2(x, p^{-1}y)$.

Now, we use another theorem of [1, 4]: If the set, Q, forms quasigroups under all 4 operations, $A_i(x, y)$ $(i = 1, 2, 3, 4)$, and if these operations satisfy the equation:

$$A_1(A_2(x, y), z) = A_3(x, A_4(y, z)),$$

then there exists an operation, $*$, under which Q forms a group with which these 4 quasigroups are isotopic to the group $(Q, *)$.

So, by transitivity of isotopy we have:

$$\begin{aligned} g_i(x, y) &= \tau_i^{-1}(\gamma_i x \circ \epsilon_i y), \\ g_2(x, y) &= \lambda_2^{-1}(\delta_2 x \circ \mu_2 y), \end{aligned}$$

where, $\gamma_i, \tau_i, \epsilon_i, \lambda_2, \mu_2, \delta_2$ are bijections of Q. Putting it in equation (2.5), we have:

$$\begin{aligned} &\eta_i^{-1}(\alpha_i(\eta_2^{-1}(\alpha_2 x \circ \beta_2 y)) \circ \beta_i v) = \tau_i^{-1}(\gamma_i x \circ \epsilon_i(\lambda_2^{-1}(\delta_2 y \circ \mu_2 v))), \\ &(\tau_i \eta_i^{-1})(\alpha_i(\eta_2^{-1}(\alpha_2 x \circ \beta_2 y)) \circ \beta_i v) = \gamma_i x \circ \epsilon_i(\lambda_2^{-1}(\delta_2 y \circ \mu_2 v)), \\ &(\tau_i \eta_i^{-1})(\alpha_i(\eta_2^{-1}(x \circ y)) \circ v) = \gamma_i(\alpha_2^{-1}x) \circ \epsilon_i(\lambda_2^{-1}(\delta_2(\beta_2^{-1}y) \circ \mu_2(\beta_i^{-1}v))), \\ &(\tau_i \eta_i^{-1})(\alpha_i \eta_2^{-1}(x \circ y) \circ v) = \gamma_i(\alpha_2^{-1}x) \circ \epsilon_i \lambda_2^{-1}(\delta_2(\beta_2^{-1}y) \circ \mu_2(\beta_i^{-1}v)), \end{aligned}$$

Therefore, by Lemma 2.5, $\theta = \eta_2^{-1}\alpha_i \in Hol(Q, \circ)$.

If $f_i = f_2$, then $\theta_2 = \eta_2^{-1}\alpha_2$ and if $f_i = f_0$, then $\alpha_i = \alpha_0$.

Hence,

$$\eta_2 = \alpha_0 \theta^{-1},$$

$$\alpha_2 = \eta_2 \theta_2 = \alpha_0 \theta^{-1} \theta_2.$$

Thus, for every component operation, $f_* = f_2 \in F$, we have:

$$f_*(x,y) = f_2(x,y) = \eta_2^{-1}(\alpha_2 x \circ \beta_2 y) =$$
$$(\alpha_0\theta^{-1})^{-1}((\alpha_0\theta^{-1}\theta_2)x \circ \beta_2 y) = (\theta\alpha_0^{-1})((\alpha_0\theta^{-1}\theta_2)x \circ \beta_2 y) =$$
$$(\theta\alpha_0^{-1})((\theta_2(\theta^{-1}(\alpha_0 x))) \circ \beta_2 y) = \alpha_0^{-1}(\theta(\theta_2(\theta^{-1}(\alpha_0 x))) \circ \theta(\beta_2 y)) =$$
$$\alpha_0^{-1}((\theta^{-1}\theta_2\theta)(\alpha_0 x) \circ \theta(\beta_2 y)) =$$
$$\alpha_0^{-1}((\theta^{-1}\theta_2\theta)(\alpha_0 x) \circ ((\theta^{-1}\theta_2\theta)e)^{-1} \circ ((\theta^{-1}\theta_2\theta)e) \circ \theta(\beta_2 y)) =$$
$$\alpha_0^{-1}(\mu(\alpha_0 x) \circ \tau y),$$

where,

$$\mu x = (\theta^{-1}\theta_2\theta)x \circ ((\theta^{-1}\theta_2\theta)e)^{-1},$$
$$\tau x = ((\theta^{-1}\theta_2\theta)e) \circ \theta(\beta_2 x).$$

Since, $\theta^{-1}\theta_2\theta \in Hol(Q,\circ)$, by Lemma 2.6, $\mu \in Hol(Q,\circ)$.

Now, we define the new operation, $+$, by the following rule:

$$x + y = \alpha_0^{-1}(\alpha_0 x \circ \alpha_0 y),$$

then,

$$f_*(x,y) = \alpha_0^{-1}(\mu(\alpha_0 x) \circ \tau y) =$$
$$\alpha_0^{-1}(\alpha_0(\alpha_0^{-1}(\mu(\alpha_0 x))) \circ \alpha_0(\alpha_0^{-1}(\tau y))) =$$
$$\alpha_0^{-1}(\mu(\alpha_0 x)) + \alpha_0^{-1}(\tau y) = (\alpha_0\mu\alpha_0^{-1})x + (\tau\alpha_0^{-1})y =$$
$$\varphi x + \sigma y,$$

where, $\varphi = \alpha_0\mu\alpha_0^{-1}$ and $\sigma = \tau\alpha_0^{-1}$, and $\varphi \in Aut(Q,+)$ because:

$$\varphi(x+y) = (\alpha_0\mu\alpha_0^{-1})(x+y) = (\eta\alpha_0^{-1})\alpha_0(x+y) =$$
$$(\mu\alpha_0^{-1})(\alpha_0 x \circ \alpha_0 y) = \alpha_0^{-1}(\mu(\alpha_0 x \circ \alpha_0 y)) =$$
$$\alpha_0^{-1}(\mu(\alpha_0 x) \circ \mu(\alpha_0 y)) = \alpha_0^{-1}((\alpha_0^{-1}\varphi\alpha_0)(\alpha_0 x) \circ (\alpha_0^{-1}\varphi\alpha_0)(\alpha_0 y)) =$$
$$\alpha_0^{-1}(\alpha_0(\varphi x) \circ \alpha_0(\varphi y)) = \varphi x + \varphi y.$$

Hence, by insertion equation (2.4), we have:

$$\varphi_i(\varphi_2 x + \sigma_2 y) + \sigma_i(\varphi_2 u + \sigma_2 v) = \varphi_i(\varphi_2 x + \sigma_2 u) + \sigma_i(\varphi_2 y + \sigma_2 v).$$

Put $\varphi_2 x = \sigma_2 y = 0, \quad \varphi_2 u = u, \quad \sigma_2 v = v$; then:

$$\sigma_i(u+v) = \varphi_i(\sigma_2\varphi_2^{-1}u) + \sigma_i(\varphi_2\sigma_2^{-1}0 + v).$$

So, by Lemma 2.2, $\sigma_i \in Hol(Q,+)$. Thus, by Lemma 2.3, there exists $\psi_i \in Aut(Q,+)$ such that:

$$\sigma_i(x) = \psi_i(x) + c_i,$$

where $c_i \in Q$.

Hence, every component operation, f_i, is represented by the following rule:

$$f_i(x,y) = \varphi_i(x) + \psi_i(y) + c_i,$$

where $c_i \in Q$ and $\varphi_i, \psi_i \in Aut(Q,+)$. □

Theorem 2.8. *Let (Q, f) be a binary multiquasigroup, where $f = (f_1, \ldots, f_m)$. If (Q, f) is a binary co-paramedial multiquasigroup, then there exists an abelian group, $(Q,+)$, such that:*

$$f_i(x, y) = \alpha_i x + \beta_i y + c_i,$$

where α_i, β_i are automorphisms of the group, $(Q,+)$, and $c_i \in Q$ is a fixed element and $\alpha_i\alpha_j = \beta_i\beta_j$, for $i, j = 1, \ldots, m$. The group, $(Q,+)$, is unique up to isomorphisms.

Proof. The proof is similar to that of Theorem 2.7. □

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References

1. J. Aczél, V. D. Belousov, M. Hosszú, Generalized Associativity and Bisymmetry on Quasigroups, *Acta Math. Sci. Hung.*, **11**, (1960), 127-136.
2. S. M. Anvariyeh, S. Mirvakili, Canonical (m,n)-ary hypermodules over Krasner (m,n)-ary hyperrings, *Iranian Journal of Mathematical Sciences and Informatics*, **7**(2), (2012), 17-34.
3. H. Babaei, M. Jafarpour, S. Sh. Mousavi, R-parts in hyperrings, *Iranian Journal of Mathematical Sciences and Informatics*, **7**(1), (2012), 59-71.
4. V. D. Belousov, Systems of quasigroups with generalized identities, *Uspekhi. Math. Nauk.*, **20**, (1965), 75-146. [English transl. in *Russian Math. Surveys.* **20**, (1965), 73-143.]
5. V. D. Belousov, *Foundations of the theory of quasigroups and loops.* Nauka. Moscow, 1967.
6. V. D. Belousov, *n-ary quasigroups.* Shtiinca, Kishinev, 1972.
7. J. Cho, J. Ježek, T. Kepka, Paramedial groupoids, *Czechoslavak Math. J.*, **49**(2), (1999), 227-240.
8. A. Church, *Introduction to Mathematical logic.* Princeton University Press. 1: Princeton, 1956.
9. G. Čupona, J. Ušan, Z. Stojaković, Multiquasigroups and some related structures, *Prilosi MANU.* **I**, (1980), No. 2, 5-12.
10. G. Čupona, Z. Stojaković, J. *U*šan, On finite multiquasigroups, *Publ. Inst. Math.*, **29**(43), (1981), 53-59.
11. W. A. Dudek, , V. S. Trokhimenko, *Algebras of multiplace functions.* Kremenchug, 2010.
12. J. Ježek, T. Kepka, *Medial groupoids.* Monograph of Acad. Praha, 93/2, 1983.
13. A. G. Kurosh, *General algebra.* M. Nauka, 1974, (Russian).
14. S. MacLane, *Homology.* Springer-Verlag, Fourth Printing, 1994.
15. A. I. Maltsev, *The metamathematics of algebraic systems.* collected papers 1936-1967 (Translated and edited by B. F. Wells III). North-Holland, Amsterdam, 1970.
16. Yu. M. Movsisyan, *Generalization of Toyoda theoram.* Proceeding of the Loops'99, Prague, 1999.

17. Yu. M. Movsisyan, *Hyperidentities and hypervarieties in algebras.* Yerevan State University Press. Yerevan, 1990 (Russian).
18. Yu. M. Movsisyan, Hyperidentities in algebras and varieties, *Uspekhi Math. Nauk.*, **53**, (1998), 61-114. English translation in: *Russian Math. Surveys*, **53**, (1998), No. 1, 57-108.
19. Yu. M. Movsisyan, *Introduction to the theory of algebras with hyperidentities.* Yerevan State University Press. Yerevan, 1986 (Russian).
20. Yu. M. Movsisyan, E. Nazari, Transitive Modes, *Demonstratio Mathematica.*, **XLIV**(3), (2011), 511-522.
21. P. Němec, T. Kepka, T-quasigroups I, *Acta Univ. Carolin. Math. Phys.*, **12**(1), (1971), 39-49.
22. M. Polonijo, Structure theorem for C^{n+1}-systems, *Glasnik Mat.*, **16**(38), (1981), 211-217.
23. M. Polonijo, *On C^{n+1}-systems and $[n,m]$-groupoids,* Algebraic Conference, Beograd, 1982.
24. A. Romanowska, J. D. H. Smith, *Modes.* World Scientific, 2002.
25. V. A. Shcherbakov, D. I. Pushkashu, On the structure of finite paramedial quasigroups, *Comment. Math. Univ. Carolin.*, **51**(2), (2010), 357-370.
26. Z. Stojaković, On bisymetric $[n,m]$-groupoids, *Review of research Faculty of Science-University of Novi Sad.*, **12**, (1982), 399-405.
27. Z. Stojaković, Dj Paunić, *Identities on multiquasigroups.* Proc. Symp. On n-ary structures, 195-200, 1982.