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Binary Multiquasigroups with Medial-Like Equations

Amir Ehsani $^{a\ast}~$ and Yuri Movsisyan b

^a Department of Mathematics, Mahshahr Branch, Islamic Azad University, Mahshahr, Iran a.ehsani@mahshahriau.ac.ir

^b Department of Mathematics and Mechanics, Yerevan State University, Alex Manoogian 1, Yerevan 0025, Armenia yurimovsisyan@yahoo.com

> ABSTRACT. In this paper paramedial, co-medial and co-paramedial binary multiquasigroups are considered and a characterization of the corresponding component operations of these multiquasigroups is given.

Keywords: Medial, Paramedial, Co-medial, Co-paramedial, Multiquasigroup, Mode.

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1. INTRODUCTION

One way to define a binary quasigroup is that it is a groupoid (A, f) in which for any $a, b \in A$ there are unique solutions x, y to the equations f(a, x) = b, f(y, a) = b. A loop is a quasigroup with unit (e) such that

$$f(e, x) = f(x, e) = x.$$

Groups are associative quasigroups, i.e., they satisfy:

$$f(f(x,y),z) = f(x,f(y,z))$$

*Corresponding Author

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There are various generalization of a group (see, [2, 3]). Most of the notions defined for binary quasigroups can be easily generalized to n-ary operations which are called n-quasigroups. An n-quasigroup is an n-groupoid (A, f) (f : $A^n \to A, n > 0$ in which for every n-sequence a_1, \ldots, a_n of elements from A, every $a \in A$ and every $i \ (1 \le i \le n)$, there is a unique solution x of the equation

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = a.$$

For example, 1-quasigroups are just bijections.

Let A be a nonempty set, n and m be positive integers and $f: A^n \to A^m$ be an arbitrary function. Then (A, f) is called [n, m]-groupoid. The n-ary operations, f_1, \ldots, f_m , are defined by the following:

$$f(x_1,\ldots,x_n) = (y_1,\ldots,y_m) \Leftrightarrow y_i = f_i(x_1,\ldots,x_n)$$

for every $1 \leq i \leq m$, are called the component operations of f and they are denoted by $f = (f_1, \ldots, f_m)$ [22, 23, 26]. The [n, m]-groupoid is proper iff $n, m, |Q| \ge 2.$

The [n, m]-groupoid (A, f) is called [n, m]-quasigroup (or multiquasigroup [9, 10, 27] iff for every injection, $\phi : N_n \to N_{n+m}$, where $N_n = \{1, \ldots, n\}$, and every $(a_1, \ldots, a_n) \in Q^n$ there exists a unique $(b_1, \ldots, b_{n+m}) \in Q^{n+m}$ such that:

$$f(b_1, \dots, b_n) = (b_{n+1}, \dots, b_{n+m})$$
 and $b_{\phi(i)} = a_i$,

for i = 1, ..., n.

It is clear that Q(f) is an [n, 1]-quasigroup iff Q(f) is an *n*-quasigroup [6]. Q(f) is a [1,m]-quasigroup iff there exist permutations, f_1, \ldots, f_m , of Q such that $f(x) = (f_1(x), \ldots, f_m(x))$. It is also clear that all components of a multiquasigroup are quasigroup operations.

If the component operations of the [n, m]-quasigroup are binary operations, i.e. n = 2, then we say that the [n, m]-quasigroup is a binary multiquasigroup. Let us consider the following hyperidentities [17, 18, 19]:

$$g(f(x,y), f(u,v)) = f(g(x,u), g(y,v)), \quad (\text{Mediality})$$
(1.1)

$$g(f(x,y), f(u,v)) = f(g(v,y), g(u,x)), \quad (\text{Paramediality})$$
(1.2)

$$g(f(x,y), f(u,v)) = g(f(x,u), f(y,v)), \quad \text{(Co-mediality)}$$
(1.3)

$$g(f(x,y), f(u,v)) = g(f(v,y), f(u,x)), \quad \text{(Co-paramediality)}$$
(1.4)

$$f(x,x) = x.$$
 (Idempotency) (1.5)

The binary algebra, (A, F), is called:

- medial, if it satisfies the identity (1.1),
- paramedial, if it satisfies the identity (1.2),
- co-medial, if it satisfies the identity (1.3),
- co-paramedial, if it satisfies the identity (1.4),
- idempotent, if it satisfies the identity (1.5),

for every $f, g \in F$. The binary algebra, (A, F), is called mode, if it is medial and idempotent.

Medial groupoids, medial algebras and medial idempotent algebras (modes) were studied in [12, 13, 24]. Paramedial groupoids and paramedial quasigroups were studied in [7, 21, 25]. In general, the properties of mediality, paramediality, co-mediality and co-paramediality are the second order properties of the algebras in the sense of [8, 15, 19, 17].

Definition 1.1. The binary multiquasigroup (A, f) with $f = (f_1, \ldots, f_m)$ is called:

- medial, if the binary algebra, (A, f_1, \ldots, f_m) , is medial,
- paramedial, if the binary algebra, (A, f_1, \ldots, f_m) , is paramedial,
- co-medial, if the binary algebra, (A, f_1, \ldots, f_m) , is co-medial,
- co-paramedial, if the binary algebra, (A, f_1, \ldots, f_m) , is co-paramedial,
- idempotent, if the binary algebra, (A, f_1, \ldots, f_m) , is idempotent,
- mode, if the binary algebra, (A, f_1, \ldots, f_m) , is a mode.

The next characterization of binary medial multiquasigroups follows from [4, 16, 20].

Theorem 1.2. Let (Q, f) be a binary multiquasigroup, where $f = (f_1, \ldots, f_m)$. If (Q, f) is a binary medial multiquasigroup, then there exists an abelian group, (Q, +), such that:

$$f_i(x,y) = \alpha_i x + \beta_i y + c_i,$$

where α_i, β_i are automorphisms of the group (Q, +), and $c_i \in Q$ is a fixed element and: $\alpha_i\beta_j = \beta_j\alpha_i, \alpha_i\alpha_j = \alpha_j\alpha_i, \beta_i\beta_j = \beta_j\beta_i$, for i, j = 1, ..., m. The group, (Q, +), is unique up to isomorphisms. Moreover, if (Q, f) is a mode, then

$$f_i(x,y) = \alpha_i x + \beta_i y,$$

where α_i, β_i are automorphisms of both the group, (Q, +), and of the algebra, (Q, f_1, \ldots, f_m) .

2. Main Results

To characterize the paramedial, co-medial and co-paramedial multiquasigroups we need the concept of holomorphism for groups [14, 19].

Definition 2.1. If (Q, \cdot) is a group, then the bijection, $\alpha : Q \to Q$, is called a holomorphism of (Q, \cdot) if

$$\alpha(x \cdot y^{-1} \cdot z) = \alpha x \cdot (\alpha y)^{-1} \cdot \alpha z,$$

for every $x, y, z \in Q$. Note that this concept is equivalent to the concept of quasiautomorphism of groups [5].

The set of all holomorphisms of (Q, \cdot) is denoted by $Hol(Q, \cdot)$ and it is a group under the superposition of the mappings: $(\alpha \cdot \beta)x = \beta(\alpha x)$, for every $x \in Q$.

Lemma 2.2. [19] Let for bijections $\alpha_1, \alpha_2, \alpha_3$ on the group, (Q, \cdot) , the following identity be satisfied:

$$\alpha_1(x \cdot y) = \alpha_2(x) \cdot \alpha_3(y),$$

then $\alpha_1, \alpha_2, \alpha_3 \in Hol(Q, \cdot)$.

Lemma 2.3. [19] Every holomorphism, α , of the group, (Q, \cdot) , has the following form:

$$\alpha x = \varphi x \cdot k,$$

where $\varphi \in Aut(Q, \cdot)$ and $k \in Q$.

The triple, (α, β, γ) , of the bijections from the set, G, onto the set, H, is called an isotopism of the groupoid, (G, \cdot) , onto the groupoid, (H, \circ) , provided: $\gamma(x \cdot y) = \alpha x \circ \beta y$, for all $x, y \in G$. (H, \circ) is called an isotope of (G, \cdot) , and the groupoids, (G, \cdot) and (H, \circ) , are called isotopic to each other. The isotopism of (G, \cdot) onto (G, \cdot) is called the autotopism of (G, \cdot) .

Let α and β be the permutations of G and ι denoting the identity map on G. Then (α, β, ι) is the principal isotopism of the groupoid, (G, \cdot) , onto the groupoid, (G, \circ) , meaning that (α, β, ι) is an isotopism of (G, \cdot) onto (G, \circ) .

Theorem 2.4. Let (Q, f) be a binary multiquasigroup, where $f = (f_1, \ldots, f_m)$. If (Q, f) is a binary paramedial multiquasigroup, then there exists an abelian group, (Q, +), such that:

$$f_i(x, y) = \alpha_i x + \beta_i y + c_i,$$

where α_i, β_i are automorphisms of the group, (Q, +), and $c_i \in Q$ is a fixed element and: $\alpha_i\beta_j = \alpha_j\beta_i, \alpha_i\alpha_j = \beta_j\beta_i, \beta_i\alpha_j = \beta_j\alpha_i$, for i, j = 1, ..., m. The group, (Q, +), is unique up to isomorphisms.

Proof. If f_1 is a fixed component operation of the binary multiquasigroup, (Q, f), then by [21], f_1 is principally isotopic to the abelian group operation, *, on Q. Now, if f_i is any component operation, then the pair of operations, (f_1, f_i) , is paramedial.

First, we use the main result of [1] (also see [4]). If the set, Q, forms a quasigroup under 6 operations, $A_i(x, y)$ (for i = 1, ..., 6), and if these operations satisfy the equation:

$$A_1(A_2(x,y), A_3(u,v)) = A_4(A_5(x,u), A_6(y,v)),$$
(2.1)

for all elements, x, y, u, v, of the set, Q, then there exists an operation, '+', under which Q forms an abelian group on which all these 6 quasigroups are

isotopic. And there exist 8 one-to-one mappings, $\alpha, \beta, \gamma, \delta, \epsilon, \psi, \varphi, \chi$, of Q onto itself such that:

$$A_1(x,y) = \delta x + \varphi y, \qquad A_2(x,y) = \delta^{-1}(\alpha x + \beta y),$$

$$A_3(x,y) = \varphi^{-1}(\chi x + \gamma y), \qquad A_4(x,y) = \psi x + \epsilon y,$$

$$A_5(x,y) = \psi^{-1}(\alpha x + \chi y), \qquad A_6(x,y) = \epsilon^{-1}(\beta x + \gamma y).$$

Now, let $A_i^*(x, y) = A_i(y, x)$; then, putting it in (2.1), we have:

$$A_1(A_2(x,y), A_3(u,v)) = A_4^*(A_6^*(v,y), A_5^*(u,x)),$$
(2.2)

and

$$\begin{aligned} A_4^*(x,y) &= A_4(y,x) = \psi y + \epsilon x = \epsilon x + \psi y, \\ A_5^*(x,y) &= A_5(y,x) = \psi^{-1}(\alpha y + \chi x) = \psi^{-1}(\chi x + \alpha y), \\ A_6^*(x,y) &= A_6(y,x) = \epsilon^{-1}(\beta y + \gamma x) = \epsilon^{-1}(\gamma x + \beta y), \end{aligned}$$

since, (Q, +) is an abelian group. But, by the definition of paramedial pair operations, (f_1, f_i) , we know:

$$f_i(f_1(x,y), f_1(u,v)) = f_1(f_i(v,y), f_i(u,x)).$$
(2.3)

So, let $A_1 = A_5^* = A_6^* = f_i$ and $A_2 = A_3 = A_4^* = f_1$. With this assumption, we reach the equation (2.3), from the equation (2.2). Therefore, since $A_1 = A_5^*$, we have:

$$\begin{split} \delta x + \varphi y &= \psi^{-1}(\chi x + \alpha y) \\ \Rightarrow \psi(\delta x + \varphi y) &= \chi x + \alpha y \\ \Rightarrow \psi(x + y) &= \chi(\delta^{-1}x) + \alpha(\varphi^{-1}y) \\ \Rightarrow \psi \in Hol(Q, +), \end{split}$$

by Lemma 2.2.

Similarly, since $A_1 = A_6^*$, we have: $\epsilon \in Hol(Q, +)$. Therefore, by Lemma 2.3, there exist $\varphi_1, \psi_1 \in Aut(Q, +)$ such that:

$$\psi x = \varphi_1 x + a,$$

$$\epsilon x = b + \psi_1 x,$$

where a, b are fixed elements in Q. Hence,

$$f_1(x,y) = A_4^*(x,y) = \psi x + \epsilon y =$$

$$\varphi_1 x + a + b + \psi_1 x = \varphi_1 x + c_1 + \psi_1 x,$$

where $c_1 = a + b$ is a fixed element in Q.

By the same manner, we can show that: $\delta, \varphi \in Hol(Q, +)$, since $A_2 = A_4^*$ and $A_3 = A_4^*$. So, there exist $\varphi_2, \psi_2 \in Aut(Q, +)$ such that:

$$\delta x = \varphi_2 x + d,$$

$$\varphi x = e + \psi_2 x,$$

where d, e are fixed elements in Q. Hence,

$$f_i(x,y) = A_1(x,y) = \delta x + \varphi y = \varphi_2 x + c_2 + \psi_2 y,$$

where $c_2 = d + e$ is a fixed element in Q.

Now, put

$$f_1(x, y) = \varphi_1(x) + \psi_1(y) + c_1,$$

$$f_i(x, y) = \varphi_2(x) + \psi_2(y) + c_2,$$

in equation (2.3), if x = 0; then we obtain $\varphi_1 \varphi_2 = \psi_2 \psi_1$, if y = 0; then $\varphi_1\psi_2 = \varphi_2\psi_1$, if u = 0; then $\psi_1\varphi_2 = \psi_2\varphi_1$; and if v = 0, then $\varphi_2\varphi_1 = \psi_1\psi_2$.

Therefore, f_1 and f_i are principally isotopic to the group operation, +, on Q. Thus, by transitivity of isotopy, any component operation, f_i , is principally isotopic to the same abelian group operation, +.

The uniqueness of the group, (Q, +), follows from the Albert's theorem [5, 13, 19]: if every two groups are isotopic, then they are isomorphic.

Lemma 2.5. Let for bijections $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ on the group, (Q, \cdot) , the following identity be satisfied:

$$\alpha_1(\alpha_2(x \cdot y) \cdot z) = \alpha_3 x \cdot \alpha_4(\alpha_5 y \cdot \alpha_6 z),$$

then $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in Hol(Q, \cdot)$ (see [19] p. 36, for Moufang loops).

Lemma 2.6. Let $\alpha_0 \in Hol(Q, \cdot)$ and $k \in Q$, then the mapping,

$$\alpha x = \alpha_0 x \cdot k,$$

 $x \in Q$, is a holomorphism of the group, (Q, \cdot) (see [19] p. 36, for Moufang loops).

Theorem 2.7. Let (Q, f) be a binary multiquasigroup, where $f = (f_1, \ldots, f_m)$. If (Q, f) is a binary co-medial multiquasigroup, then there exists an abelian group, (Q, +), such that

$$f_i(x,y) = \alpha_i x + \beta_i y + c_i,$$

where α_i, β_i are automorphisms of the group, (Q, +), and $c_i \in Q$ is a fixed element and: $\alpha_i\beta_j = \beta_i\alpha_j$, for i, j = 1, ..., m. The group, (Q, +), is unique up to isomorphisms.

Proof. Let f_1, f_2 be fixed component operations; then by the definition of comediality:

 $f_1(f_2(x,y), f_2(u,v)) = f_1(f_2(x,u), f_2(y,v)).$

Also, for every component operation, f_i , we have:

$$f_i(f_2(x,y), f_2(u,v)) = f_i(f_2(x,u), f_2(y,v)).$$
(2.4)

So, by the main result of [1], the algebras, (Q, f_1) and (Q, f_2) , are isotopic to the abelian group, (Q, \circ) ; and the algebras, (Q, f_1) and (Q, f_i) , are isotopic to

the abelian group, (Q, \cdot) . Thus, by transitivity of isotopy, the algebra, (Q, f_i) , is isotopic to (Q, \circ) and we have:

$$f_i(x,y) = \eta_i^{-1}(\alpha_i x \circ \beta_i y),$$

where $\eta_i, \alpha_i, \beta_i$ are bijections of Q.

Let $u = a \in Q$, then:

$$f_i(f_2(x,y), f_2(a,v)) = f_i(f_2(x,a), f_2(y,v)).$$

Put $f_2(a, v) = pv$ and $f_2(x, a) = qx$; then

$$f_i(f_2(x,y), pv) = f_i(qx, f_2(y,v)),$$

$$f_i(f_2(x,y), v) = f_i(qx, f_2(y, p^{-1}v)),$$

$$f_i(f_2(x,y), v) = g_i(x, g_2(y,v)),$$
(2.5)

where $g_i(x, y) = f_i(qx, y)$ and $g_2(x, y) = f_2(x, p^{-1}y)$. Now, we use another theorem of [1, 4]: If the set, Q, forms quasigroups under all 4 operations, $A_i(x, y)$ (i = 1, 2, 3, 4), and if these operations satisfy the equation:

$$A_1(A_2(x, y), z) = A_3(x, A_4(y, z))$$

then there exists an operation, *, under which Q forms a group with which these 4 quasigroups are isotopic to the group (Q, *).

So, by transitivity of isotopy we have:

$$g_i(x,y) = \tau_i^{-1}(\gamma_i x \circ \epsilon_i y),$$

$$g_2(x,y) = \lambda_2^{-1}(\delta_2 x \circ \mu_2 y),$$

 $\gamma_i, \tau_i, \epsilon_i, \lambda_2, \mu_2, \delta_2$ are bijections of Q. Putting it in equation (2.5), we where, have:

$$\begin{split} &\eta_{i}^{-1}(\alpha_{i}(\eta_{2}^{-1}(\alpha_{2}x\circ\beta_{2}y))\circ\beta_{i}v)=\tau_{i}^{-1}(\gamma_{i}x\circ\epsilon_{i}(\lambda_{2}^{-1}(\delta_{2}y\circ\mu_{2}v))),\\ &(\tau_{i}\eta_{i}^{-1})(\alpha_{i}(\eta_{2}^{-1}(\alpha_{2}x\circ\beta_{2}y))\circ\beta_{i}v)=\gamma_{i}x\circ\epsilon_{i}(\lambda_{2}^{-1}(\delta_{2}y\circ\mu_{2}v)),\\ &(\tau_{i}\eta_{i}^{-1})(\alpha_{i}(\eta_{2}^{-1}(x\circ y))\circ v)=\gamma_{i}(\alpha_{2}^{-1}x)\circ\epsilon_{i}(\lambda_{2}^{-1}(\delta_{2}(\beta_{2}^{-1}y)\circ\mu_{2}(\beta_{i}^{-1}v)))\\ &(\tau_{i}\eta_{i}^{-1})(\alpha_{i}\eta_{2}^{-1}(x\circ y)\circ v)=\gamma_{i}(\alpha_{2}^{-1}x)\circ\epsilon_{i}\lambda_{2}^{-1}(\delta_{2}(\beta_{2}^{-1}y)\circ\mu_{2}(\beta_{i}^{-1}v)), \end{split}$$

Therefore, by Lemma 2.5, $\theta = \eta_2^{-1} \alpha_i \in Hol(Q, \circ)$. If $f_i = f_2$, then $\theta_2 = \eta_2^{-1} \alpha_2$ and if $f_i = f_0$, then $\alpha_i = \alpha_0$. Hence,

$$\eta_2 = \alpha_0 \theta^{-1},$$
$$\alpha_2 = \eta_2 \theta_2 = \alpha_0 \theta^{-1} \theta_2.$$

Thus, for every component operation, $f_* = f_2 \in F$, we have:

$$\begin{aligned} f_*(x,y) &= f_2(x,y) = \eta_2^{-1}(\alpha_2 x \circ \beta_2 y) = \\ (\alpha_0 \theta^{-1})^{-1}((\alpha_0 \theta^{-1} \theta_2) x \circ \beta_2 y) = (\theta \alpha_0^{-1})((\alpha_0 \theta^{-1} \theta_2) x \circ \beta_2 y) = \\ (\theta \alpha_0^{-1})((\theta_2(\theta^{-1}(\alpha_0 x))) \circ \beta_2 y) = \alpha_0^{-1}(\theta(\theta_2(\theta^{-1}(\alpha_0 x))) \circ \theta(\beta_2 y)) = \\ \alpha_0^{-1}((\theta^{-1} \theta_2 \theta)(\alpha_0 x) \circ \theta(\beta_2 y)) = \\ \alpha_0^{-1}((\theta^{-1} \theta_2 \theta)(\alpha_0 x) \circ ((\theta^{-1} \theta_2 \theta) e)^{-1} \circ ((\theta^{-1} \theta_2 \theta) e) \circ \theta(\beta_2 y)) = \\ \alpha_0^{-1}(\mu(\alpha_0 x) \circ \tau y), \end{aligned}$$

where,

$$\mu x = (\theta^{-1}\theta_2\theta)x \circ ((\theta^{-1}\theta_2\theta)e)^{-1},$$

$$\tau x = ((\theta^{-1}\theta_2\theta)e) \circ \theta(\beta_2 x).$$

Since, $\theta^{-1}\theta_2\theta \in Hol(Q, \circ)$, by Lemma 2.6, $\mu \in Hol(Q, \circ)$. Now, we define the new operation, +, by the following rule:

$$x + y = \alpha_0^{-1}(\alpha_0 x \circ \alpha_0 y),$$

then,

$$f_*(x,y) = \alpha_0^{-1}(\mu(\alpha_0 x) \circ \tau y) = \\ \alpha_0^{-1}(\alpha_0(\alpha_0^{-1}(\mu(\alpha_0 x))) \circ \alpha_0(\alpha_0^{-1}(\tau y))) = \\ \alpha_0^{-1}(\mu(\alpha_0 x)) + \alpha_0^{-1}(\tau y) = (\alpha_0 \mu \alpha_0^{-1})x + (\tau \alpha_0^{-1})y = \\ \varphi x + \sigma y,$$

where, $\varphi = \alpha_0 \mu \alpha_0^{-1}$ and $\sigma = \tau \alpha_0^{-1}$, and $\varphi \in Aut(Q, +)$ because: $\varphi(x+y) = (\alpha_0 \mu \alpha_0^{-1})(x+y) = (\eta \alpha_0^{-1})\alpha_0(x+y) =$ $(\mu \alpha_0^{-1})(\alpha_0 x \circ \alpha_0 y) = \alpha_0^{-1}(\mu(\alpha_0 x \circ \alpha_0 y)) =$ $\alpha_0^{-1}(\mu(\alpha_0 x) \circ \mu(\alpha_0 y)) = \alpha_0^{-1}((\alpha_0^{-1}\varphi\alpha_0)(\alpha_0 x) \circ (\alpha_0^{-1}\varphi\alpha_0)(\alpha_0 y)) =$ $\alpha_0^{-1}(\alpha_0(\varphi x) \circ \alpha_0(\varphi y)) = \varphi x + \varphi y.$

Hence, by insertion equation (2.4), we have:

$$\begin{aligned} \varphi_i(\varphi_2 x + \sigma_2 y) + \sigma_i(\varphi_2 u + \sigma_2 v) &= \varphi_i(\varphi_2 x + \sigma_2 u) + \sigma_i(\varphi_2 y + \sigma_2 v) \\ \text{Put } \varphi_2 x &= \sigma_2 y = 0, \quad \varphi_2 u = u, \quad \sigma_2 v = v; \text{ then:} \\ \sigma_i(u+v) &= \varphi_i(\sigma_2 \varphi_2^{-1} u) + \sigma_i(\varphi_2 \sigma_2^{-1} 0 + v). \end{aligned}$$

So, by Lemma 2.2, $\sigma_i \in Hol(Q, +)$. Thus, by Lemma 2.3, there exists $\psi_i \in Aut(Q, +)$ such that:

$$\sigma_i(x) = \psi_i(x) + c_i,$$

where $c_i \in Q$.

Hence, every component operation, f_i , is represented by the following rule:

$$f_i(x,y) = \varphi_i(x) + \psi_i(y) + c_i,$$

where $c_i \in Q$ and $\varphi_i, \psi_i \in Aut(Q, +)$.

Theorem 2.8. Let (Q, f) be a binary multiquasigroup, where $f = (f_1, \ldots, f_m)$. If (Q, f) is a binary co-paramedial multiquasigroup, then there exists an abelian group, (Q, +), such that:

$$f_i(x, y) = \alpha_i x + \beta_i y + c_i,$$

where α_i, β_i are automorphisms of the group, (Q, +), and $c_i \in Q$ is a fixed element and $\alpha_i \alpha_j = \beta_i \beta_j$, for i, j = 1, ..., m. The group, (Q, +), is unique up to isomorphisms.

Proof. The proof is similar to that of Theorem 2.7.

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