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On Some Fractional Systems of Difference Equations

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Abstract. This paper deals with the solutions of the systems of difference equations

$$x_{n+1}=\frac{y_{n-3}y_nx_{n-2}}{y_{n-3}x_{n-2}\pm y_{n-3}y_n\pm y_nx_{n-2}},\ y_{n+1}=\frac{y_{n-2}x_{n-1}}{2y_{n-2}\pm x_{n-1}},\ n\in\mathbb{N}_0,$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and initial values x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0 are non-zero real numbers.

Keywords: System of difference equations, Form of the solutions, Periodicity.

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1. Introduction

Difference equations enter as approximations of continuous problems and as models describing life situations in many directions. Recently there has been a great interest in studying difference equations, see, for instance [1]-[12], [14], [17]-[19] and references cited therein, as well as in studying systems of difference equations (see, e.g. [13], [15], [16]).

In this paper we will investigate the solutions of the systems of difference equations

$$x_{n+1}=\frac{y_{n-3}y_nx_{n-2}}{y_{n-3}x_{n-2}\pm y_{n-3}y_n\pm y_nx_{n-2}},\ y_{n+1}=\frac{y_{n-2}x_{n-1}}{2y_{n-2}\pm x_{n-1}},\ n\in\mathbb{N}_0,$$

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where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and initial values x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0 are non-zero real numbers.

Definition 1.1. A sequence $\{x_n\}_{n \geq -k}$ is said to be eventually periodic with period $p \in \mathbb{N}$ if there is an $n_0 \geq -k$ such that $x_{n+p} = x_n$, for all $n \geq n_0$. If $n_0 = -k$, we say that the sequence $\{x_n\}_{n \geq -k}$ is periodic with period p.

Definition 1.2. Let $\{F_n\}_{n \geq 0} = \{0, 1, 1, 2, 3, 5, 8, ...\}$ be the Fibonacci sequence defined by

$$F_0 = 0,\, F_1 = 1,\, F_{n+2} = F_{n+1} + F_n,\, n \in \mathbb{N}_0.$$

Definition 1.3. Let $\{U_n\}_{n \geq 0} = \{0, 2, 4, 14, 40, ...\}$ be the sequence defined by

$$U_0 = 0,\, U_{n+1} + U_n = 2 \cdot 3^n,\, n \in \mathbb{N}_0.$$

2. The System: $x_{n+1} = \frac{y_{n-3}y_n x_{n-2}}{y_{n-3}x_{n-2} + y_{n-3}y_n + y_n x_{n-2}}$, $y_{n+1} = \frac{y_{n-2}x_{n-1}}{2y_{n-2} - x_{n-1}}$

In this section, we study the solutions of the system of two difference equations

$$x_{n+1} = \frac{y_{n-3}y_n x_{n-2}}{y_{n-3}x_{n-2} + y_{n-3}y_n + y_n x_{n-2}},\, y_{n+1} = \frac{y_{n-2}x_{n-1}}{2y_{n-2} - x_{n-1}},\, n \in \mathbb{N}_0, \quad (2.1)$$

with non-zero real initials conditions such that $y_{-3}x_{-2} + y_{-3}y_0 + y_0 x_{-2} \neq 0$, $\frac{x_{-1}}{y_{-2}}$, $\frac{x_0}{y_{-1}}$ and $\frac{y_{-3}x_{-2}}{y_{-3}x_{-2} + y_{-3}y_0 + y_0 x_{-2}} \notin \left\{(-1)^{n+1}U_n,\, n \geq 1\right\}$.

The following theorem is devoted to the form of the solutions of the system (2.1).

Theorem 2.1. *Let $\{x_n\}_{n \geq -2}$, $\{y_n\}_{n \geq -3}$ be a solution of (2.1). Then, for $n \geq 0$, we have,*

$$\begin{aligned}
x_{3n-1} &= x_{-1}\left(\frac{1}{3}\right)^n,\, x_{3n} = x_0\left(\frac{1}{3}\right)^n,\, x_{3n+1} = \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2} + y_{-3}y_0 + y_0x_{-2}}\left(\frac{1}{3}\right)^n, \\
y_{3n-2} &= \frac{y_{-2}x_{-1}}{y_{-2}U_n + (-1)^n x_{-1}},\, y_{3n-1} = \frac{y_{-1}x_0}{y_{-1}U_n + (-1)^n x_0}, \\
y_{3n} &= \frac{y_{-3}y_0x_{-2}}{(y_{-3}x_{-2} + y_0y_{-3} + y_0x_{-2})\, U_n + (-1)^n y_{-3}x_{-2}}.
\end{aligned}$$

Proof. Form (2.1), we have

$$x_1 = \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2} + y_{-3}y_0 + y_0x_{-2}}.$$

So, the result holds for $n = 0$. Suppose that $n \geq 1$ and that our assumption holds for $n-1$ that is,

$$x_{3n-4} = x_{-1}\left(\frac{1}{3}\right)^{n-1}, \tag{2.2}$$

$$x_{3n-3} = x_{0}\left(\frac{1}{3}\right)^{n-1}, \tag{2.3}$$

$$x_{3n-2} = \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2}+y_{-3}y_0+y_0x_{-2}}\left(\frac{1}{3}\right)^{n}, \tag{2.4}$$

$$y_{3n-5} = \frac{y_{-2}x_{-1}}{y_{-2}U_{n-1}+(-1)^{n-1}x_{-1}}, \tag{2.5}$$

$$y_{3n-4} = \frac{y_{-1}x_{0}}{y_{-1}U_{n-1}+(-1)^{n-1}x_{0}}, \tag{2.6}$$

$$y_{3n-3} = \frac{y_{-3}y_0x_{-2}}{\left(y_{-3}x_{-2}+y_0y_{-3}+y_0x_{-2}\right)U_{n-1}+(-1)^{n-1}y_{-3}x_{-2}}. \tag{2.7}$$

From (2.1) and (2.2), we have

$$\begin{aligned}
x_{3n-1} &= \frac{y_{3n-5}y_{3n-2}x_{3n-4}}{y_{3n-5}x_{3n-4}+y_{3n-5}y_{3n-2}+y_{3n-2}x_{3n-4}} \\
&= \frac{y_{3n-5}\left(\frac{y_{3n-5}x_{3n-4}}{2y_{3n-5}-x_{3n-4}}\right)x_{3n-4}}{y_{3n-5}x_{3n-4}+y_{3n-5}\left(\frac{y_{3n-5}x_{3n-4}}{2y_{3n-5}-x_{3n-4}}\right)+\left(\frac{y_{3n-5}x_{3n-4}}{2y_{3n-5}-x_{3n-4}}\right)x_{3n-4}} \\
&= \frac{1}{3}x_{3n-4} = x_{-1}\left(\frac{1}{3}\right)^{n}.
\end{aligned}$$

From (2.1) and (2.3), we get

$$\begin{aligned}
x_{3n} &= \frac{y_{3n-4}y_{3n-1}x_{3n-3}}{y_{3n-4}x_{3n-3}+y_{3n-4}y_{3n-1}+y_{3n-1}x_{3n-3}} \\
&= \frac{y_{3n-4}\left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}-x_{3n-3}}\right)x_{3n-3}}{y_{3n-4}x_{3n-3}+y_{3n-4}\left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}-x_{3n-3}}\right)+\left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}-x_{3n-3}}\right)x_{3n-3}} \\
&= \frac{1}{3}x_{3n-3} = x_{0}\left(\frac{1}{3}\right)^{n}.
\end{aligned}$$

From (2.1) and (2.4), we get

$$\begin{aligned}
x_{3n+1} &= \frac{y_{3n-3}y_{3n}x_{3n-2}}{y_{3n-3}x_{3n-2}+y_{3n-3}y_{3n}+y_{3n}x_{3n2}} \\
&= \frac{y_{3n-3}\left(\frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}-x_{3n-2}}\right)x_{3n-2}}{y_{3n-3}x_{3n-2}+y_{3n-3}\left(\frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}-x_{3n-2}}\right)+\left(\frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}-x_{3n-2}}\right)x_{3n-2}} \\
&= \frac{1}{3}x_{3n-2} = \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2}+y_{-3}y_0+y_0x_{-2}}\left(\frac{1}{3}\right)^{n}.
\end{aligned}$$

From (2.1), (2.2) and (2.5), we have

$$\begin{aligned} y_{3n-2} &= \frac{y_{3n-5}x_{3n-4}}{2y_{3n-5}-x_{3n-4}} = \frac{\left(\frac{y_{-2}x_{-1}}{y_{-2}U_{n-1}+(-1)^{n-1}x_{-1}}\right)\left(\frac{x_{-1}}{3^{n-1}}\right)}{2\left(\frac{y_{-2}x_{-1}}{y_{-2}U_{n-1}+(-1)^{n-1}x_{-1}}\right)-\left(\frac{x_{-1}}{3^{n-1}}\right)} \\ &= \frac{y_{-2}x_{-1}}{2\cdot 3^{n-1}y_{-2}-y_{-2}U_{n-1}+(-1)^n x_{-1}} = \frac{y_{-2}x_{-1}}{y_{-2}(2\cdot 3^{n-1}-U_{n-1})+(-1)^n x_{-1}} \\ &= \frac{y_{-2}x_{-1}}{y_{-2}U_n+(-1)^n x_{-1}}. \end{aligned}$$

From (2.1), (2.3) and (2.6), we get

$$\begin{aligned} y_{3n-1} &= \frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}-x_{3n-3}} = \frac{\left(\frac{y_{-1}x_{0}}{y_{-1}U_{n-1}+(-1)^{n-1}x_{0}}\right)\left(\frac{x_{0}}{3^{n-1}}\right)}{2\left(\frac{y_{-1}x_{0}}{y_{-1}U_{n-1}+(-1)^{n-1}x_{0}}\right)-\left(\frac{x_{0}}{3^{n-1}}\right)} \\ &= \frac{y_{-1}x_{0}}{2\cdot 3^{n-1}y_{-1}-y_{-1}U_{n-1}+(-1)^n x_{0}} = \frac{y_{-1}x_{0}}{y_{-1}(2\cdot 3^{n-1}-U_{n-1})+(-1)^n x_{0}} \\ &= \frac{y_{-1}x_{0}}{y_{-1}U_n+(-1)^n x_{0}}. \end{aligned}$$

From (2.1), (2.4) and (2.7), we have

$$\begin{aligned} y_{3n} &= \frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}-x_{3n-2}} \\ &= \frac{\left(\frac{y_{-3}y_0x_{-2}}{(y_{-3}x_{-2}+y_0y_{-3}+y_0x_{-2})U_{n-1}+(-1)^{n-1}y_{-3}x_{-2}}\right)\left(\frac{y_{-3}y_0x_{-2}}{3^{n-1}(y_{-3}x_{-2}+y_{-3}y_0+y_0x_{-2})}\right)}{2\left(\frac{y_{-3}y_0x_{-2}}{(y_{-3}x_{-2}+y_0y_{-3}+y_0x_{-2})U_{n-1}+(-1)^{n-1}y_{-3}x_{-2}}\right)-\left(\frac{y_{-3}y_0x_{-2}}{3^{n-1}(y_{-3}x_{-2}+y_{-3}y_0+y_0x_{-2})}\right)} \\ &= \frac{y_{-3}y_0x_{-2}}{(2\cdot 3^{n-1}-U_{n-1})(y_0x_{-2}+y_0y_{-3}+x_{-2}y_{-3})+(-1)^n x_{-2}y_{-3}} \\ &= \frac{y_{-3}y_0x_{-2}}{U_n(y_0x_{-2}+y_0y_{-3}+x_{-2}y_{-3})+(-1)^n x_{-2}y_{-3}}. \end{aligned}$$

The proof is complete. □

Corollary 2.2. *Every solution of system (2.1) is such that*

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} y_n = 0.$$

3. The System: $x_{n+1} = \frac{y_{n-3}y_nx_{n-2}}{y_{n-3}x_{n-2}+y_{n-3}y_n+y_nx_{n-2}}$, $y_{n+1} = \frac{y_{n-2}x_{n-1}}{2y_{n-2}+x_{n-1}}$

In this section, we get the form of the solutions of the system

$$x_{n+1} = \frac{y_{n-3}y_nx_{n-2}}{y_{n-3}x_{n-2}+y_{n-3}y_n+y_nx_{n-2}},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{2y_{n-2}+x_{n-1}},\ n\in\mathbb{N}_0, \quad (3.1)$$

with non-zero real initials conditions such that $\frac{y_{-1}}{x_0}, \frac{y_{-2}}{x_{-1}} \notin \left\{-\frac{F_{3n}}{F_{3n+1}}, n\geq 1\right\}$ and $\frac{y_{-3}y_0+y_0x_{-2}}{y_{-3}x_{-2}} \notin \left\{-\frac{F_{3n+2}}{F_{3n+1}}, n\geq 0\right\}$.

Theorem 3.1. *Let* $\{x_n\}_{n\geq -2}$, $\{y_n\}_{n\geq -3}$ *be a solution of (3.1). Then for* $n = 0, 1, ...,$

$$x_{3n} = \frac{x_0 y_{-1}}{y_{-1}F_{3n+1} + x_0 F_{3n}},\ x_{3n+1} = \frac{y_0 y_{-3} x_{-2}}{y_{-3}x_{-2}F_{3n+2} + y_0(y_{-3} + x_{-2})F_{3n+1}},$$

$$x_{3n+2} = \frac{x_{-1}y_{-2}}{y_{-2}F_{3n+4} + x_{-1}F_{3n+3}},\ y_{3n} = \frac{y_0 y_{-3} x_{-2}}{y_{-3}x_{-2}F_{3n+1} + y_0(y_{-3} + x_{-2})F_{3n}},$$

$$y_{3n+1} = \frac{y_{-2}x_{-1}}{y_{-2}F_{3n+3} + x_{-1}F_{3n+2}},\ y_{3n+2} = \frac{y_{-1}x_0}{y_{-1}F_{3n+3} + x_0 F_{3n+2}}.$$

Proof. From (3.1), we have

$$x_1 = \frac{y_{-3}y_0 x_{-2}}{y_{-3}x_{-2} + y_{-3}y_0 + y_0 x_{-2}} = \frac{y_{-3}y_0 x_{-2}}{y_{-3}x_{-2}F_2 + y_0(y_{-3} + x_{-2})F_1}, \tag{3.2}$$

$$y_1 = \frac{y_{-2}x_{-1}}{2y_{-2} + x_{-1}} = \frac{y_{-2}x_{-1}}{y_{-2}F_3 + x_{-1}F_2}, \tag{3.3}$$

$$y_2 = \frac{y_{-1}x_0}{2y_{-1} + x_0} = \frac{y_{-1}x_0}{y_{-1}F_3 + x_0 F_2}. \tag{3.4}$$

From (3.1)-(3.4), we get

$$\begin{aligned} x_2 &= \frac{y_{-2}y_1 x_{-1}}{y_{-2}x_{-1} + y_{-2}y_1 + y_1 x_{-1}} = \frac{y_{-2}\left(\frac{y_{-2}x_{-1}}{2y_{-2}+x_{-1}}\right)x_{-1}}{y_{-2}x_{-1} + y_{-2}\left(\frac{y_{-2}x_{-1}}{2y_{-2}+x_{-1}}\right) + \left(\frac{y_{-2}x_{-1}}{2y_{-2}+x_{-1}}\right)x_{-1}} \\ &= \frac{x_{-1}y_{-2}}{3y_{-2} + 2x_{-1}} = \frac{x_{-1}y_{-2}}{y_{-2}F_4 + x_{-1}F_3}, \end{aligned}$$

$$\begin{aligned} x_3 &= \frac{y_{-1}y_2 x_0}{y_{-1}x_0 + y_{-1}y_2 + y_2 x_0} = \frac{y_{-1}\left(\frac{y_{-1}x_0}{2y_{-1}+x_0}\right)x_0}{y_{-1}x_0 + y_{-1}\left(\frac{y_{-1}x_0}{2y_{-1}+x_0}\right) + \left(\frac{y_{-1}x_0}{2y_{-1}+x_0}\right)x_0} \\ &= \frac{x_0 y_{-1}}{3y_{-1} + 2x_0} = \frac{x_0 y_{-1}}{y_{-1}F_4 + x_0 F_3}, \end{aligned}$$

$$\begin{aligned} y_3 &= \frac{y_0 x_1}{2y_0 + x_1} = \frac{y_0\left(\frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2}+y_{-3}y_0+y_0x_{-2}}\right)}{2y_0 + \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2}+y_{-3}y_0+y_0x_{-2}}} = \frac{y_{-3}y_0x_{-2}}{3y_{-3}x_{-2} + 2y_0(y_{-3} + x_{-2})} \\ &= \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2}F_4 + y_0(y_{-3} + x_{-2})F_3}. \end{aligned}$$

Similarly we get

$$x_4 = \frac{y_{-3}y_0x_{-2}}{5x_{-2}y_{-3} + 3y_0x_{-2} + 3y_0y_{-3}} = \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2}F_5 + y_0(y_{-3} + x_{-2})F_4},$$

$$y_4 = \frac{x_{-1}y_{-2}}{8y_{-2} + 5x_{-1}} = \frac{y_{-2}x_{-1}}{y_{-2}F_6 + x_{-1}F_5},$$

$$x_5 = \frac{x_{-1}y_{-2}}{13y_{-2} + 8x_{-1}} = \frac{x_{-1}y_{-2}}{y_{-2}F_7 + x_{-1}F_6},$$

$$y_5 = \frac{x_0 y_{-1}}{8y_{-1} + 5x_0} = \frac{y_{-1}x_0}{y_{-1}F_6 + x_0 F_5}.$$

So, the result holds for $n = 0, 1$. Suppose that $n \geq 2$ and that our assumption holds for $n-2$, $n-1$ that is,

$$x_{3n-6} = \frac{x_0 y_{-1}}{y_{-1}F_{3n-5} + x_0 F_{3n-6}}, \tag{3.5}$$

$$x_{3n-5} = \frac{y_0 y_{-3} x_{-2}}{y_{-3}x_{-2}F_{3n-4} + y_0(y_{-3} + x_{-2})F_{3n-5}}, \tag{3.6}$$

$$x_{3n-4} = \frac{x_{-1} y_{-2}}{y_{-2}F_{3n-2} + x_{-1} F_{3n-3}}, \tag{3.7}$$

$$y_{3n-6} = \frac{y_0 y_{-3} x_{-2}}{y_{-3}x_{-2}F_{3n-5} + y_0(y_{-3} + x_{-2})F_{3n-6}}, \tag{3.8}$$

$$y_{3n-5} = \frac{y_{-2} x_{-1}}{y_{-2}F_{3n-3} + x_{-1} F_{3n-4}}, \tag{3.9}$$

$$y_{3n-4} = \frac{y_{-1} x_0}{y_{-1}F_{3n-3} + x_0 F_{3n-4}}, \tag{3.10}$$

$$x_{3n-3} = \frac{x_0 y_{-1}}{y_{-1}F_{3n-2} + x_0 F_{3n-3}}, \tag{3.11}$$

$$x_{3n-2} = \frac{y_0 y_{-3} x_{-2}}{y_{-3}x_{-2}F_{3n-1} + y_0(y_{-3} + x_{-2})F_{3n-2}}, \tag{3.12}$$

$$x_{3n-1} = \frac{x_{-1} y_{-2}}{y_{-2}F_{3n+1} + x_{-1} F_{3n}}, \tag{3.13}$$

$$y_{3n-3} = \frac{y_0 y_{-3} x_{-2}}{y_{-3}x_{-2}F_{3n-2} + y_0(y_{-3} + x_{-2})F_{3n-3}}, \tag{3.14}$$

$$y_{3n-2} = \frac{y_{-2} x_{-1}}{y_{-2}F_{3n} + x_{-1} F_{3n-1}}, \tag{3.15}$$

$$y_{3n-1} = \frac{y_{-1} x_0}{y_{-1}F_{3n} + x_0 F_{3n-1}}. \tag{3.16}$$

From (3.1), (3.10), (3.11) and (3.16), we get

$$\begin{aligned}
x_{3n} &= \frac{y_{3n-4} y_{3n-1} x_{3n-3}}{y_{3n-4}x_{3n-3} + y_{3n-4}y_{3n-1} + y_{3n-1}x_{3n-3}} \\
&= \frac{x_0 y_{-1}}{F_{3n}y_{-1} + x_0 F_{3n-1} + x_0 F_{3n-3} + x_0 F_{3n-4} + F_{3n-2}y_{-1} + F_{3n-3}y_{-1}} \\
&= \frac{x_0 y_{-1}}{y_{-1}\left(F_{3n} + (F_{3n-2} + F_{3n-3})\right) + x_0\left(F_{3n-1} + (F_{3n-3} + F_{3n-4})\right)} \\
&= \frac{x_0 y_{-1}}{y_{-1}\left(F_{3n} + F_{3n-1}\right) + x_0\left(F_{3n-1} + F_{3n-2}\right)} \\
&= \frac{x_0 y_{-1}}{y_{-1}F_{3n+1} + x_0 F_{3n}}.
\end{aligned}$$

From (3.1), (3.12) and (3.14), we have

$$\begin{aligned}
y_{3n} &= \frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}+x_{3n-2}} \\
&= \frac{y_{-3}y_0x_{-2}}{y_0x_{-2}\left(2F_{3n-2}+F_{3n-3}\right)+y_0y_{-3}\left(2F_{3n-2}+F_{3n-3}\right)+x_{-2}y_{-3}\left(2F_{3n-1}+F_{3n-2}\right)} \\
&= \frac{y_{-3}y_0x_{-2}}{y_0x_{-2}\left(F_{3n-2}+F_{3n-1}\right)+y_0y_{-3}\left(F_{3n-2}+F_{3n-1}\right)+x_{-2}y_{-3}\left(F_{3n-1}+F_{3n}\right)} \\
&= \frac{y_{-3}y_0x_{-2}}{y_0x_{-2}F_{3n}+y_0y_{-3}F_{3n}+x_{-2}y_{-3}F_{3n+1}}.
\end{aligned}$$

From (3.1), (3.12) and (3.14), we get

$$\begin{aligned}
x_{3n+1} &= \frac{y_{3n-3}y_{3n}x_{3n-2}}{y_{3n-3}x_{3n-2}+y_{3n-3}y_{3n}+y_{3n}x_{3n-2}} \\
&= \frac{y_{3n-3}\left(\frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}+x_{3n-2}}\right)x_{3n-2}}{y_{3n-3}x_{3n-2}+y_{3n-3}\left(\frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}+x_{3n-2}}\right)+\left(\frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}+x_{3n-2}}\right)x_{3n-2}} \\
&= \frac{y_{3n-3}x_{3n-2}}{3y_{3n-3}+2x_{3n-2}} \\
&= \frac{\left(\frac{y_0y_{-3}x_{-2}}{y_{-3}x_{-2}F_{3n-2}+y_0(y_{-3}+x_{-2})F_{3n-3}}\right)\left(\frac{y_0y_{-3}x_{-2}}{y_{-3}x_{-2}F_{3n-1}+y_0(y_{-3}+x_{-2})F_{3n-2}}\right)}{3\left(\frac{y_0y_{-3}x_{-2}}{y_{-3}x_{-2}F_{3n-2}+y_0(y_{-3}+x_{-2})F_{3n-3}}\right)+2\left(\frac{y_0y_{-3}x_{-2}}{y_{-3}x_{-2}F_{3n-1}+y_0(y_{-3}+x_{-2})F_{3n-2}}\right)} \\
&= \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2}\left(3F_{3n-1}+2F_{3n-2}\right)+y_0\left(y_{-3}+x_{-2}\right)\left(3F_{3n-2}+2F_{3n-3}\right)} \\
&= \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2}\left(F_{3n-1}+2F_{3n}\right)+y_0\left(y_{-3}+x_{-2}\right)\left(F_{3n-2}+2F_{3n-1}\right)} \\
&= \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2}\left(F_{3n+1}+F_{3n}\right)+y_0\left(y_{-3}+x_{-2}\right)\left(F_{3n}+F_{3n-1}\right)} \\
&= \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2}F_{3n+2}+y_0\left(y_{-3}+x_{-2}\right)F_{3n+1}}.
\end{aligned}$$

From (3.1), (3.13) and (3.15), we have

$$\begin{aligned}
y_{3n+1} &= \frac{y_{3n-2}x_{3n-1}}{2y_{3n-2}+x_{3n-1}} \\
&= \frac{\left(\frac{y_{-2}x_{-1}}{y_{-2}F_{3n}+x_{-1}F_{3n-1}}\right)\left(\frac{x_{-1}y_{-2}}{y_{-2}F_{3n+1}+x_{-1}F_{3n}}\right)}{2\left(\frac{y_{-2}x_{-1}}{y_{-2}F_{3n}+x_{-1}F_{3n-1}}\right)+\frac{x_{-1}y_{-2}}{y_{-2}F_{3n+1}+x_{-1}F_{3n}}} \\
&= \frac{y_{-2}x_{-1}}{y_{-2}\left(F_{3n}+2F_{3n+1}\right)+x_{-1}\left(2F_{3n}+F_{3n-1}\right)} \\
&= \frac{y_{-2}x_{-1}}{y_{-2}\left(F_{3n+2}+F_{3n+1}\right)+x_{-1}\left(F_{3n}+F_{3n+1}\right)} \\
&= \frac{y_{-2}x_{-1}}{y_{-2}F_{3n+3}+x_{-1}F_{3n+2}}
\end{aligned}$$

From (3.1), (3.13) and (3.15), we get

$$\begin{aligned}
x_{3n+2} &= \frac{y_{3n-2}y_{3n+1}x_{3n-1}}{y_{3n-2}x_{3n-1}+y_{3n-2}y_{3n+1}+y_{3n+1}x_{3n-1}} \\
&= \frac{y_{3n-2}\left(\frac{y_{3n-2}x_{3n-1}}{2y_{3n-2}+x_{3n-1}}\right)x_{3n-1}}{y_{3n-2}x_{3n-1}+y_{3n-2}\left(\frac{y_{3n-2}x_{3n-1}}{2y_{3n-2}+x_{3n-1}}\right)+\left(\frac{y_{3n-2}x_{3n-1}}{2y_{3n-2}+x_{3n-1}}\right)x_{3n-1}} \\
&= \frac{y_{3n-2}x_{3n-1}}{3y_{3n-2}+2x_{3n-1}} \\
&= \frac{\left(\frac{y_{-2}x_{-1}}{y_{-2}F_{3n}+x_{-1}F_{3n-1}}\right)\left(\frac{x_{-1}y_{-2}}{y_{-2}F_{3n+1}+x_{-1}F_{3n}}\right)}{3\left(\frac{y_{-2}x_{-1}}{y_{-2}F_{3n}+x_{-1}F_{3n-1}}\right)+2\left(\frac{x_{-1}y_{-2}}{y_{-2}F_{3n+1}+x_{-1}F_{3n}}\right)} \\
&= \frac{y_{-2}x_{-1}}{y_{-2}\left(2F_{3n}+3F_{3n+1}\right)+x_{-1}\left(3F_{3n}+2F_{3n-1}\right)} \\
&= \frac{y_{-2}x_{-1}}{y_{-2}\left(2F_{3n+2}+F_{3n+1}\right)+x_{-1}\left(F_{3n}+2F_{3n+1}\right)} \\
&= \frac{y_{-2}x_{-1}}{y_{-2}\left(F_{3n+2}+F_{3n+3}\right)+x_{-1}\left(F_{3n+2}+F_{3n+1}\right)} \\
&= \frac{y_{-2}x_{-1}}{y_{-2}F_{3n+4}+x_{-1}F_{3n+3}}.
\end{aligned}$$

From (3.1), (3.10), (3.11) and (3.16), we have

$$\begin{aligned}
y_{3n+2} &= \frac{y_{3n-1}x_{3n}}{2y_{3n-1}+x_{3n}} = \frac{y_{3n-1}\left(\frac{y_{3n-4}y_{3n-1}x_{3n-3}}{y_{3n-4}x_{3n-3}+y_{3n-4}y_{3n-1}+y_{3n-1}x_{3n-3}}\right)}{2y_{3n-1}+\frac{y_{3n-4}y_{3n-1}x_{3n-3}}{y_{3n-4}x_{3n-3}+y_{3n-4}y_{3n-1}+y_{3n-1}x_{3n-3}}} \\
&= \frac{y_{3n-4}y_{3n-1}x_{3n-3}}{2x_{3n-3}y_{3n-1}+3x_{3n-3}y_{3n-4}+2y_{3n-1}y_{3n-4}} \\
&= \frac{y_{-1}x_0}{y_{-1}\left(F_{3n}+2F_{3n+1}\right)+x_0\left(F_{3n-1}+2F_{3n}\right)} \\
&= \frac{y_{-1}x_0}{y_{-1}\left(F_{3n+2}+F_{3n+1}\right)+x_0\left(F_{3n+1}+F_{3n}\right)} \\
&= \frac{y_{-1}x_0}{y_{-1}F_{3n+3}+x_0F_{3n+2}}.
\end{aligned}$$

The proof is complete. □

Corollary 3.2. *Let $\{x_n\}_{n\geq -2}$, $\{y_n\}_{n\geq -3}$ be a solution of (3.1) with $y_{-2}x_{-1}$, $y_{-1}x_0$ and $y_{-3}x_{-2}(y_{-3}y_0+y_0x_{-2})$ positive. Then*

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} y_n = 0.$$

4. The system $x_{n+1} = \frac{y_{n-3}y_nx_{n-2}}{y_{n-3}x_{n-2}-y_{n-3}y_n-y_nx_{n-2}}$, $y_{n+1} = \frac{y_{n-2}x_{n-1}}{2y_{n-2}+x_{n-1}}$

In this section we study the dynamic of the solutions of the system

$$x_{n+1} = \frac{y_{n-3}y_nx_{n-2}}{y_{n-3}x_{n-2}-y_{n-3}y_n-y_nx_{n-2}},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{2y_{n-2}+x_{n-1}},\ n=0,1,..., \tag{4.1}$$

with non-zero real initials conditions with $y_{-3}x_{-2} - y_0(y_{-3} + x_{-2}) \neq 0$, $\frac{y_{-2}}{x_{-1}}$, $\frac{y_{-1}}{x_0}$ and $\frac{y_{-3}x_{-2} - y_0(y_{-3}+x_{-2})}{y_{-3}x_{-2}} \notin \left\{-\frac{1}{2n}, n \geq 1\right\}$.

Theorem 4.1. *Let $\{x_n\}_{n\geq -2}$, $\{y_n\}_{n\geq -3}$ be a solution of system (4.1), then $\{x_n\}_{n\geq -2}$ is eventually periodic. That is,*

$$x_{n+3} = x_n,\ n \geq -1.$$

Moreover, if

$$x_{-2} = \frac{2y_0y_{-3}}{y_{-3} - y_0},\ y_{-3} - y_0 \neq 0,$$

then the sequence $\{x_n\}_{n\geq -2}$ will be periodic.

Proof. It follows from (4.1) that

$$\begin{aligned} x_{n+3} &= \frac{y_{n-1}y_{n+2}x_n}{y_{n-1}x_n - y_{n-1}y_{n+2} - y_{n+2}x_n} \\ &= \frac{y_{n-1}\left(\frac{y_{n-1}x_n}{2y_{n-1}+x_n}\right)x_n}{y_{n-1}x_n - y_{n-1}\left(\frac{y_{n-1}x_n}{2y_{n-1}+x_n}\right) - \left(\frac{y_{n-1}x_n}{2y_{n-1}+x_n}\right)x_n} = \frac{\frac{y_{n-1}^2x_n^2}{x_n+2y_{n-1}}}{\frac{y_{n-1}^2x_n}{x_n+2y_{n-1}}} \\ &= x_n. \end{aligned}$$

Now using

$$x_{-2} = \frac{2y_0y_{-3}}{y_{-3} - y_0},$$

we get

$$x_1 = x_{-2}.$$

□

Theorem 4.2. *Let $\{x_n\}_{n\geq -2}$, $\{y_n\}_{n\geq -3}$ be a solution of (4.1). Then for $n = 0, 1, 2, ...$, we have*

$$\begin{aligned} x_{3n} &= x_0, \\ x_{3n+1} &= \frac{y_0y_{-3}x_{-2}}{y_{-3}x_{-2} - y_0(y_{-3} + x_{-2})}, \\ x_{3n+2} &= x_{-1}, \\ y_{3n} &= \frac{y_0y_{-3}x_{-2}}{(2n+1)y_{-3}x_{-2} - 2ny_0(y_{-3} + x_{-2})}, \\ y_{3n+1} &= \frac{y_{-2}x_{-1}}{2(n+1)y_{-2} + x_{-1}}, \\ y_{3n+2} &= \frac{y_{-1}x_0}{2(n+1)y_{-1} + x_0}. \end{aligned}$$

Proof. From (4.1), we have

$$x_1 = \frac{y_0y_{-3}x_{-2}}{y_{-3}x_{-2} - y_0(y_{-3} + x_{-2})},\ x_2 = x_{-1},$$

and

$$y_1 = \frac{y_{-2}x_{-1}}{2y_{-2} + x_{-1}},\ y_2 = \frac{y_{-1}x_0}{2y_{-1} + x_0}.$$

So, the result holds for $n = 0$. Suppose $n > 0$ and that our assumption holds for $n-1$ that is,

$$x_{3n-3} = x_0, \tag{4.2}$$
$$x_{3n-2} = \frac{y_0 y_{-3} x_{-2}}{y_{-3}x_{-2} - y_0(y_{-3}+x_{-2})}, \tag{4.3}$$
$$x_{3n-1} = x_{-1}, \tag{4.4}$$
$$y_{3n-3} = \frac{y_0 y_{-3} x_{-2}}{(2n-1)y_{-3}x_{-2} - 2(n-1)y_0(y_{-3}+x_{-2})}, \tag{4.5}$$
$$y_{3n-2} = \frac{y_{-2}x_{-1}}{2ny_{-2}+x_{-1}}, \tag{4.6}$$
$$y_{3n-1} = \frac{y_{-1}x_0}{2ny_{-1}+x_0}. \tag{4.7}$$

From (4.1) and (4.2), we have

$$\begin{aligned}
x_{3n} &= \frac{y_{3n-4}y_{3n-1}x_{3n-3}}{y_{3n-4}x_{3n-3} - y_{3n-4}y_{3n-1} - y_{3n-1}x_{3n-3}} \\
&= \frac{y_{3n-4}\left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}+x_{3n-3}}\right)x_{3n-3}}{y_{3n-4}x_{3n-3} - y_{3n-4}\left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}+x_{3n-3}}\right) - \left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}+x_{3n-3}}\right)x_{3n-3}} \\
&= \frac{\frac{(y_{3n-4}x_{3n-3})^2}{2y_{3n-4}+x_{3n-3}}}{\frac{y_{3n-4}^2x_{3n-3}}{2y_{3n-4}+x_{3n-3}}} \\
&= x_{3n-3} \\
&= x_0.
\end{aligned}$$

From (4.1), (4.3) and (4.5), we get

$$\begin{aligned}
y_{3n} &= \frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}+x_{3n-2}} \\
&= \frac{\left(\frac{y_0y_{-3}x_{-2}}{(2n-1)y_{-3}x_{-2}-2(n-1)y_0(y_{-3}+x_{-2})}\right)\left(\frac{y_0y_{-3}x_{-2}}{y_{-3}x_{-2}-y_0(y_{-3}+x_{-2})}\right)}{2\left(\frac{y_0y_{-3}x_{-2}}{(2n-1)y_{-3}x_{-2}-2(n-1)y_0(y_{-3}+x_{-2})}\right)+\frac{y_0y_{-3}x_{-2}}{y_{-3}x_{-2}-y_0(y_{-3}+x_{-2})}} \\
&= \frac{\frac{(y_{-3}y_0x_{-2})^2}{[(2n-1)y_{-3}x_{-2}-2(n-1)y_0(y_{-3}+x_{-2})][y_{-3}x_{-2}-y_0(y_{-3}+x_{-2})]}}{\frac{(x_{-2}y_{-3}y_0)[(2n+1)x_{-2}y_{-3}-2ny_0(x_{-2}+y_{-3})]}{[y_0x_{-2}+y_0y_{-3}-x_{-2}y_{-3}][(2n-1)y_{-3}x_{-2}-2(n-1)y_0(y_{-3}+x_{-2})]}} \\
&= \frac{y_{-3}y_0x_{-2}}{(2n+1)x_{-2}y_{-3}-2ny_0\left(x_{-2}+y_{-3}\right)}.
\end{aligned}$$

From (4.1) and (4.3), we have

$$\begin{aligned}
x_{3n+1} &= \frac{y_{3n-3}y_{3n}x_{3n-2}}{y_{3n-3}x_{3n-2} - y_{3n-3}y_{3n} - y_{3n}x_{3n-2}} \\
&= \frac{y_{3n-3}\left(\frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}+x_{3n-2}}\right)x_{3n-2}}{y_{3n-3}x_{3n-2} - y_{3n-3}\left(\frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}+x_{3n-2}}\right) - \left(\frac{y_{3n-3}x_{3n-2}}{2y_{3n-3}+x_{3n-2}}\right)x_{3n-2}} \\
&= \frac{\frac{(y_{3n-3}x_{3n-2})^2}{2y_{3n-3}+x_{3n-2}}}{\frac{y_{3n-3}^2x_{3n-2}}{2y_{3n-3}+x_{3n-2}}} \\
&= x_{3n-2} \\
&= \frac{y_0y_{-3}x_{-2}}{y_{-3}x_{-2} - y_0(y_{-3}+x_{-2})}.
\end{aligned}$$

From (4.1), (4.4) and (4.6), we have

$$\begin{aligned}
y_{3n+1} &= \frac{y_{3n-2}x_{3n-1}}{2y_{3n-2}+x_{3n-1}} = \frac{\left(\frac{y_{-2}x_{-1}}{2ny_{-2}+x_{-1}}\right)x_{-1}}{2\left(\frac{y_{-2}x_{-1}}{2ny_{-2}+x_{-1}}\right)+x_{-1}} = \frac{\frac{y_{-2}x_{-1}^2}{2ny_{-2}+x_{-1}}}{\frac{(2(n+1)y_{-2}+x_{-1})x_{-1}}{2ny_{-2}+x_{-1}}} \\
&= \frac{y_{-2}x_{-1}}{2(n+1)y_{-2}+x_{-1}}.
\end{aligned}$$

From (4.1) and (4.4), we get

$$\begin{aligned}
x_{3n+2} &= \frac{y_{3n-2}y_{3n+1}x_{3n-1}}{y_{3n-2}x_{3n-1} - y_{3n-2}y_{3n+1} - y_{3n+1}x_{3n-1}} \\
&= \frac{y_{3n-2}\left(\frac{y_{3n-2}x_{3n-1}}{2y_{3n-2}+x_{3n-1}}\right)x_{3n-1}}{y_{3n-2}x_{3n-1} - y_{3n-2}\left(\frac{y_{3n-2}x_{3n-1}}{2y_{3n-2}+x_{3n-1}}\right) - \left(\frac{y_{3n-2}x_{3n-1}}{2y_{3n-2}+x_{3n-1}}\right)x_{3n-1}} \\
&= \frac{\frac{(y_{3n-2}x_{3n-1})^2}{2y_{3n-2}+x_{3n-1}}}{\frac{y_{3n-2}^2x_{3n-1}}{2y_{3n-2}+x_{3n-1}}} \\
&= x_{3n-1} \\
&= x_{-1}.
\end{aligned}$$

Finally, from (4.1), (4.2) and (4.5), we have

$$
\begin{aligned}
y_{3n+2} &= \frac{y_{3n-1}x_{3n}}{2y_{3n-1}+x_{3n}} \\
&= \frac{y_{3n-1}\left(\frac{y_{3n-4}y_{3n-1}x_{3n-3}}{y_{3n-4}x_{3n-3}-y_{3n-4}y_{3n-1}-y_{3n-1}x_{3n-3}}\right)}{2y_{3n-1}+\left(\frac{y_{3n-4}y_{3n-1}x_{3n-3}}{y_{3n-4}x_{3n-3}-y_{3n-4}y_{3n-1}-y_{3n-1}x_{3n-3}}\right)} \\
&= \frac{y_{3n-1}\left(\frac{y_{3n-4}\left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}+x_{3n-3}}\right)x_{3n-3}}{y_{3n-4}x_{3n-3}-y_{3n-4}\left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}+x_{3n-3}}\right)-\left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}+x_{3n-3}}\right)x_{3n-3}}\right)}{2y_{3n-1}+\left(\frac{y_{3n-4}\left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}+x_{3n-3}}\right)x_{3n-3}}{y_{3n-4}x_{3n-3}-y_{3n-4}\left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}+x_{3n-3}}\right)-\left(\frac{y_{3n-4}x_{3n-3}}{2y_{3n-4}+x_{3n-3}}\right)x_{3n-3}}\right)} \\
&= \frac{y_{3n-1}x_{3n-3}}{2y_{3n-1}+x_{3n-3}} \\
&= \frac{\left(\frac{y_{-1}x_0}{2ny_{-1}+x_0}\right)x_0}{2\left(\frac{y_{-1}x_0}{2ny_{-1}+x_0}\right)+x_0} \\
&= \frac{\frac{y_{-1}x_0^2}{2ny_{-1}+x_0}}{\frac{(2(n+1)y_{-1}+x_0)x_0}{2ny_{-1}+x_0}} \\
&= \frac{y_{-1}x_0}{2(n+1)y_{-1}+x_0}.
\end{aligned}
$$

The proof is complete. □

Corollary 4.3. *Every solution of system (4.1) is such that*

$$\lim_{n\to+\infty} y_n = 0.$$

5. The system $x_{n+1} = \frac{y_{n-3}y_n x_{n-2}}{y_{n-3}x_{n-2}-y_{n-3}y_n-y_n x_{n-2}}$, $y_{n+1} = \frac{y_{n-2}x_{n-1}}{2y_{n-2}-x_{n-1}}$

Consider the system

$$x_{n+1} = \frac{y_{n-3}y_n x_{n-2}}{y_{n-3}x_{n-2}-y_{n-3}y_n-y_n x_{n-2}},\ y_{n+1} = \frac{y_{n-2}x_{n-1}}{2y_{n-2}-x_{n-1}},\ n\in\mathbb{N}_0 \quad (5.1)$$

with non-zero real initial conditions with $y_0\left(y_{-3}+x_{-2}\right) \notin \left\{\pm y_{-3}x_{-2}, \frac{y_{-3}x_{-2}}{2}\right\}$, $y_{-2} \notin \left\{\frac{x_{-1}}{2}, 2x_{-1}\right\}$, $y_{-1} \notin \left\{\frac{x_0}{2}, 2x_0\right\}$.
The following theorem describes the form of the solutions of system (5.1).

Theorem 5.1. *Let* $\{x_n\}_{n\geq -2}$, $\{y_n\}_{n\geq -3}$ *be a solution of (5.1). Then, for* $n = 0, 1, ...,$

$$\begin{aligned}
x_{6n-1} &= x_{-1}\left(\frac{-1}{3}\right)^n, \; x_{6n} = x_0\left(\frac{-1}{3}\right)^n, \\
x_{6n+1} &= \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2} - y_0\left(y_{-3} + x_{-2}\right)}\left(\frac{-1}{3}\right)^n, \; x_{6n+2} = \frac{y_{-2}x_{-1}}{y_{-2} - 2x_{-1}}\left(\frac{-1}{3}\right)^n, \\
x_{6n+3} &= \frac{y_{-1}x_0}{y_{-1} - 2x_0}\left(\frac{-1}{3}\right)^n, \; x_{6n+4} = -\frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2} + y_0\left(y_{-3} + x_{-2}\right)}\left(\frac{-1}{3}\right)^n, \\
y_{6n-2} &= y_{-2}\left(\frac{-1}{3}\right)^n, \; y_{6n-1} = y_{-1}\left(\frac{-1}{3}\right)^n, \\
y_{6n} &= y_0\left(\frac{-1}{3}\right)^n, \; y_{6n+1} = \frac{y_{-2}x_{-1}}{2y_{-2} - x_{-1}}\left(\frac{-1}{3}\right)^n, \\
y_{6n+2} &= \frac{y_{-1}x_0}{2y_{-1} - x_0}\left(\frac{-1}{3}\right)^n, \; y_{6n+3} = \frac{y_{-3}y_0x_{-2}}{y_{-3}x_{-2} - 2y_0\left(y_{-3} + x_{-2}\right)}\left(\frac{-1}{3}\right)^n.
\end{aligned}$$

Corollary 5.2. *Every solution of system (5.1) is such that*

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} y_n = 0.$$

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