Iranian Journal of Mathematical Sciences and Informatics
Vol. 9, No. 2 (2014), pp 87-100

Some Results on Convexity and Concavity of Multivariate Copulas

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Abstract. This paper provides some results on different types of convexity and concavity in the class of multivariate copulas. We also study their properties and provide several examples to illustrate our results.

Keywords: Componentwise concavity, Copula, Quasi-concavity, Schur-concavity.

2000 Mathematics subject classification: Primary 60E05, secondary 62H20.

1. Introduction

Copulas are multivariate distribution functions having univariate marginals uniformly distributed on the interval [0,1]. The concepts of convexity and concavity of distributions have a great importance in the recent applications of statistics; see e.g., [5, 28]. Recently, many investigations have been devoted to search different concepts of convexity and concavity for bivariate copulas; for example we can mention [3, 9, 14]. This paper is devoted to the study of various types of convexity and concavity in the class of multivariate copulas. The paper is organized as follows: Section 2 contains basic definitions and properties of multivariate copulas that we need to present the main result. The notions of componentwise concavity (convexity) of multivariate copulas is considered in Section 3. Section 4 is devoted to the study of Schur-concavity (Schur-convexity) property of multivariate copulas. A method for constructing multivariate Schur-concave copulas is also given in this section. Some results

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Received 11 January 2011; Accepted 26 September 2012

on weakly schur-concavity and Quasi-concavity (quasi-covexity) of multivariate copulas are given in Sections 4 and 5. For each type of convexity (concavity) property, several examples illustrating our results are provided. Finally, Section 6 is devoted to a short discussion about the given results and the related questions.

2. Preliminaries

Let $n \geq 2$ be a natural number. An *n-dimensional copula* (briefly *n-copula*) is the restriction to $[0,1]^n$ $(= \mathbb{I}^n)$ of a continuous n-dimensional distribution function whose univariate margins are uniform on $\mathbb{I}$. Equivalently, an n-copula is a function $C\colon \mathbb{I}^n \longrightarrow \mathbb{I}$ which satisfies the following properties:

(C1) For every $\mathbf{u} = (u_1, \ldots, u_n)$ in $\mathbb{I}^n$, $C(\mathbf{u}) = 0$ if at least one coordinate of $\mathbf{u}$ is 0, and $C(\mathbf{u}) = u_k$ whenever all coordinates of $\mathbf{u}$ are 1 except u_k; and

(C2) for every $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ in $\mathbb{I}^n$ such that $a_k \leq b_k$ for all $1 \leq k \leq n$, $\sum \mathrm{sgn}(\mathbf{c})C(\mathbf{c}) \geq 0$, where the sum is taken over all the *vertices* $\mathbf{c} = (c_1, \ldots, c_n)$ of $[a_1, b_1] \times \cdots \times [a_n, b_n]$ such that each c_k is equal to either a_k or b_k, and $\mathrm{sgn}(\mathbf{c})$ is 1 if $c_k = a_k$ for an even number of k's, and -1 if $c_k = a_k$ for an odd number of k's.

The importance of copulas is described in the following result (Sklar [27]). Let $\mathbf{x} = (x_1, \cdots, x_n)$ be a point in $\mathbb{R}^n$, and let $\mathbf{X} = (X_1, \cdots, X_n)$ be a continuous random vector with joint distribution function H and respective univariate marginals $F_i(x_i) = P[X_i \leq x_i]$, $i = 1, \cdots, n$. Then there exists an n-copula C (which is uniquely determined on $\mathrm{Range}F_1 \times \cdots \times \mathrm{Range}F_n$) such that $H(\mathbf{x}) = C(F_1(x_1), \cdots, F_n(x_n))$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Let Π^n denote the n-copula of independent continuous random variables, i.e., $\Pi^n(\mathbf{u}) = \prod_{i=1}^{n} u_i$. Any n-copula C satisfies that $W^n(\mathbf{u}) \leq C(\mathbf{u}) \leq M^n(\mathbf{u})$ for each $\mathbf{u}$ in $\mathbb{I}^n$, where $W^n(\mathbf{u}) = \max(\sum_{i=1}^{n} u_i - n + 1, 0)$ and $M^n(\mathbf{u}) = \min(u_1, \cdots, u_n)$. For every $n \geq 2$, M^n is an n-copula; however W^n is an n-copula if and only if $n = 2$. For a complete discussion of copulas, see [26]. An n-copula C is Archimedean if it is of the form

$$C(u_1, ..., u_n) = \phi^{-1}\{\phi(u_1) + ... + \phi(u_n)\}$$

where $\phi^{-1}(0) = 1$ and $\phi^{-1}(x) \to 0$, as $x \to \infty$ and ϕ^{-1} is d-monotone, i.e., $(-1)^k \frac{d^k \phi^{-1}(t)}{dt^k} \geq 0$, for all k. Given two n-copulas C_1 and C_2, let $C_1 \leq C_2$ denote the inequality $C_1(\mathbf{u}) \leq C_2(\mathbf{u})$ for all $\mathbf{u}$.

We recall some concepts of positive dependence [19]. The n-copula C is positive lower orthant dependent (PLOD) if $C \geq \Pi^n$. The corresponding negative lower orthant dependence (NLOD) is defined by reversing the sense of the inequality. The vector $\mathbf{U}$ is said to be positive dependent through the stochastic ordering (PDS) if $P(U_j \leq u_j, j = 1, ..., n, j \neq i | U_i = t)$ is decreasing in t. It is known that (see, e.g., Theorem 2.4 in [19]) PDS implies PLOD. The class of copulas will be denoted by $\mathcal{C}$.

We also use the notion of (multivariate) symmetry in the sequel. Let A be an open interval in $\mathbb{R}$. A function $g : \mathbb{A}^n \to \mathbb{R}$ is said to be symmetric if for each point $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{A}^n$, $g(\mathbf{x}) = g(\Pi\mathbf{x})$, for every permutation matrix Π.

3. Componentwise concavity (convexity) of n-copulas

We start with the classical notion of concavity (convexity) of an $n-$copula.

Definition 3.1. An n-copula C is called (globally) concave if for all $\mathbf{u} = (u_1, ..., u_n)$ and $\mathbf{v} = (v_1, ..., v_n) \in \mathbb{I}^n$ and $\lambda \in \mathbb{I}$,

$$C(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) \geq \lambda C(\mathbf{u}) + (1-\lambda)C(\mathbf{v}). \tag{3.1}$$

An n-copula is called (globally) convex if (3.1) holds with a reverse inequality sign.

Note that for the case $n = 2$ the inequality (3.1) means that

$$C(\lambda u_1 + (1-\lambda)v_1, \lambda u_2 + (1-\lambda)v_2) \geq \lambda C(u_1, u_2) + (1-\lambda)C(v_1, v_2),$$

for all u_1, u_2, v_1, v_2 and λ in $\mathbb{I}$. As mentioned in [26], the only convex 2-copula is W^2 and the only concave 2-copula is M^2. Since W^n is not an n-copula for $n > 2$, then the convex n-copula may not exist. The following example shows that the only concave n-copula is M^n.

Example 3.2. If C is a concave n-copula, setting $\mathbf{u} = (1, 1, ..., 1)$ and $\mathbf{v} = (0, 0, ..., 0)$ in (3.1) yields $C(\lambda, ..., \lambda) \geq \lambda$. Since each n-copula C satisfies $C(\mathbf{u}) \leq M^n(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{I}^n$, we have $C(t, ..., t) = t$ on $\mathbb{I}^n$, and hence C must be M^n.

Thus convexity and concavity are conditions too strong to be of much interest for copulas. Weaker versions of these properties are the componentwise concavity and componentwise convexity; see e.g., [10, 11]. We denote by $\mathcal{C}_{CWC}$ the class of componentwise concave copulas.

Definition 3.3. An n-copula C is called componentwise concave (convex) if it is concave (convex) in each coordinate when the other coordinates are held fixed, that is, for every $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{I}$, and all $i \in \{1, 2, ..., n\}$, the function $g_{\mathbf{u},i} : \mathbb{I} \to \mathbb{I}$, given by $g_{\mathbf{u},i}(t) = C(\mathbf{v})$, where $v_i = t$ and $v_j = u_j$ for $j \in \{1, ..., n\} - \{i\}$ is concave (convex).

The case $n = 2$ is already studied in [14]. It is easy to see that the copula M^n is componentwise concave and the n-copula Π^n is both componentwise concave and convex.

Example 3.4. Let C_θ be a member of Farlie-Gumbel-Morgenstren (F-G-M) family of n-copulas [26] defined, for all $(u_1, ..., u_n) \in \mathbb{I}^n$ and $\theta \in [-1,1]$ by

$$C_\theta(u_1, ..., u_n) = \prod_{i=1}^{n} u_i + \theta \prod_{i=1}^{n} u_i(1-u_i). \tag{3.2}$$

Since

$$\frac{\partial^2 C_\theta(u_1, ..., u_n)}{\partial u_i^2} = -2\theta \prod_{k=1, k\neq i}^{n} u_k(1-u_k),$$

then C_θ is componentwise convex for $\theta \in [-1,0]$ and it is componentwise concave for $\theta \in [0,1]$. Note that the F-G-M copula is positive (resp, negative) lower orthant dependent, when $\theta \in [0,1]$ (resp, $\theta \in [-1,0]$).

Every n-copula C is a Lipschitz function (with constant 1) and for $i = 1, ..., n$, admits partial derivatives $\frac{\partial C(u_1,...,u_n)}{\partial u_i}$ $a.e$ on $\mathbb{I}^n$ [26]. If C is the copula of the vector $(V_1, ..., V_n)$ of uniform [0,1] random variables then for $i = 1, ..., n$, it follows (see [26])

$$\frac{\partial C(v_1, ..., v_n)}{\partial v_i} = P(V_1 \leq v_1, ..., V_{i-1} \leq v_{i-1}, V_{i+1} \leq v_{i+1}, ..., V_n \leq v_n | V_i = v_i). \tag{3.3}$$

For a twice differentiable n-copula C, the componentwise concavity (convexity) means that for each $i = 1, ..., n$, the mapping

$$t \to P(V_j \leq v_j, j = 1, ..., n, j \neq i | V_i = t),$$

is decreasing (increasing) in t, that is, for every $t_1, t_2 \in \mathbb{I}$, with $t_1 \leq t_2$,

$$P(V_j \leq v_j, j = 1, ..., n, j \neq i | V_i = t_1) \geq (\leq) P(V_j \leq v_j, j = 1, ..., n, j \neq i | V_i = t_2).$$

The following result characterizes the componentwise concavity and positive dependence property of an n-copula, whose proof is immediate.

Proposition 3.5. *For any n-copula C, the following conditions are equivalent:*
(i) C is componentwise concave.
(ii) C is PDS.

Corollary 3.6. *If C is a componentwise concave n-copula, then C is PLOD.*

The following result provides a condition for componentwise concavity of Archimedean copulas. Similar proof characterizing the positive dependence property of bivariate Archimedean copulas could be found in [6, 21].

Proposition 3.7. *Let C be an Archimedean n-copula with the strict generator ϕ twice differentiable. Then C is componentwise concave if, and only if $\frac{1}{\phi'}$ is concave, where ϕ' is the derivative of ϕ.*

Proof. Since C is symmetric, the proof is done for the first variable. But C is convex in its first component if, and only if the function $g : \mathbb{I} \to \mathbb{I}$ given by

$$g(t) = \phi^{-1}\{\phi(t) + a\},$$

with $a = \sum_{i=2}^{n} \phi(u_i)$, is concave, which is equivalent to

$$g''(t) = \frac{\phi''(t)\left(\phi'(\phi^{-1}(\phi(t)+a))\right)^2 - (\phi'(t))^2\,\phi''\left(\phi^{-1}(\phi(t)+a)\right)}{\left(\phi'(\phi^{-1}(\phi(t)+a))\right)^3} \leq 0,$$

for each $t \in (0, 1)$. Since ϕ' is negative on $(0, 1)$, the above inequality means that

$$\frac{\phi''(t)}{(\phi'(t))^2} \geq \frac{\phi''\left(\phi^{-1}(\phi(t)+a)\right)}{(\phi'(\phi^{-1}(\phi(t)+a)))^2}. \tag{3.4}$$

Put $s = \phi^{-1}(\phi(t) + a) < t$. Then inequality (3.4) amounts to

$$\left(\frac{1}{\phi'(t)}\right)' \leq \left(\frac{1}{\phi'(s)}\right)',$$

for all $t, s \in (0, 1)$ with $t > s$; i.e., the derivative of the function $\frac{1}{\phi'}$ is non-increasing, which implies the concavity of $\frac{1}{\phi'}$. □

Example 3.8. For every $\alpha > 0$, consider the *Clayton* family of copulas [7, 26] given by $C_\alpha(u_1, ..., u_n) = (u_1^{-\alpha} + ... + u_n^{-\alpha} - n + 1)^{-1/\alpha}$, whose strict generator is $\phi(t) = (t^{-\alpha} - 1)/\alpha$ (the case $\alpha = 0$ corresponds to the product copula). Since $\left(\frac{1}{\phi'(t)}\right)'' = -\alpha(\alpha+1)t^{\alpha-1} < 0$ for all $\alpha > 0$, from Proposition 3.7 the n-copula C_α is componentwise concave.

Example 3.9. Consider the *Frank* family of copulas [15, 26] given by

$$C_\alpha(u_1, ..., u_n) = \log_\alpha\left(1 + \frac{(\alpha^{u_1} - 1)...(\alpha^{u_n} - 1)}{\alpha - 1}\right),$$

with $\alpha \in [0, +\infty)\backslash\{1\}$ (the case $\alpha = 1$ corresponds to the product copula). The strict generator is given by $\phi(t) = \ln(\frac{1-\alpha}{1-\alpha^t})$, for $t \in (0, 1)$. Since $\left(\frac{1}{\phi'(t)}\right)'' = \alpha^{-t}\ln\alpha < 0$ for $\alpha \in (0, 1)$, from Proposition 3.7 we have that the n-copula C_α is a componentwise concave for $\alpha \in (0, 1)$.

It is known that the class of copulas $\mathcal{C}$ is a convex and compact (with respect to the L^∞ norm) subset in the class of all continuous functions from $\mathbb{I}^n$ to $\mathbb{I}$. In the following we give a result about the class $\mathcal{C}_{CWC}$ of componentwise concave copulas.

Proposition 3.10. *The class $\mathcal{C}_{CWC}$ is a convex and compact (with respect to L^∞ norm) subset of $\mathcal{C}$.*

4. Schur–concavity of n-copulas

For ease of reference we recall some definitions and concepts of majorization ordering and Schur conditions from Marshall and Olkin [22]. See also [2, 18]. Let $\mathbf{a} = (a_1, ..., a_n)$ and $\mathbf{b} = (b_1, ..., b_n)$ be two points in $\mathbb{R}^n$ and denote by $a_{[1]}, ..., a_{[n]}$ and $b_{[1]}, ..., b_{[n]}$ the components of $\mathbf{a}$ and $\mathbf{b}$ rearranged in decreasing order.

Definition 4.1. The point $\mathbf{a}$ is said to be majorized by the point $\mathbf{b}$ (written $\mathbf{a} \prec_m \mathbf{b}$) if $\sum_{j=1}^n a_{[j]} = \sum_{j=1}^n b_{[j]}$ and $\sum_{j=1}^k a_{[j]} \le \sum_{j=1}^k b_{[j]}$, for $k = 1, ..., n-1$.

Definition 4.2. A real valued function $g : \mathbb{A} \subset \mathbb{R}^n \to \mathbb{R}$, is Schur-concave (Schur-convex) on $\mathbb{A}$ if for all $\mathbf{a}, \mathbf{b} \in A$, $\mathbf{a} \prec_m \mathbf{b}$ implies $g(\mathbf{a}) \ge (\le) g(\mathbf{b})$.

Proposition 4.3. *Let $\mathbb{A} \subset \mathbb{R}^n$ be a symmetric set, i.e., a set with the property that $\mathbf{x} \in \mathbb{A}$ implies $\Pi\mathbf{x} \in \mathbb{A}$ for all permutation matrix Π and let $g : \mathbb{A} \to \mathbb{R}$. The following conditions hold:*
(i) If g is Schur-concave (Schur-convex) on $\mathbb{A}$, then g is symmetric.
(ii) If g is Schur-concave (Schur-convex) on $\mathcal{D} \bigcap \mathbb{A}$, where $\mathcal{D} = \{\mathbf{x} : x_1 \ge x_2 \ge ... \ge x_n\}$, then g is Schur-concave (Schur-convex) on $\mathbb{A}$.

The next result characterizes the continuously differentiable Schur-concave functions.

Proposition 4.4. *Let $\mathbb{J}$ be an open interval in $\mathbb{R}$ and let $g : \mathbb{J}^n \to \mathbb{R}$ be a continuously differentiable function. Then g is Schur-concave on $\mathbb{J}^n$ if, and only if, (i) g is symmetric; (ii) for all $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{J}^n$ and $i \neq j$, $(x_i - x_j)(\frac{\partial g(\mathbf{x})}{\partial x_i} - \frac{\partial g(\mathbf{x})}{\partial x_j}) \le 0$.*

Remark 4.5. Since g is symmetric, the Schur-concavity condition in Proposition 4.4, can be reduced to $(x_1 - x_2)(\frac{\partial g(\mathbf{x})}{\partial x_1} - \frac{\partial g(\mathbf{x})}{\partial x_2}) \le 0$.

The following result characterizes the Schur-concavity (Schur-convexity) of n-copulas.

Proposition 4.6. *An n-copula C is Schur-concave if, and only if, for all $u_1, ..., u_n$ and $\lambda_{ij} \in \mathbb{I}$ with $\sum_{j=1}^n \lambda_{ij} = 1$, for all $i = 1, ..., n$ and $\sum_{i=1}^n \lambda_{ij} = 1$, for all $j = 1, ..., n$,*

$$C(u_1, ..., u_n) \le C\left(\sum_{j=1}^n \lambda_{1j} u_j, ..., \sum_{j=1}^n \lambda_{nj} u_j\right). \tag{4.1}$$

An n-copula C is Schur-convex if (4.1) holds with the reverse inequality sign.

Proof. Let $\mathbf{u} = (u_1, ..., u_n)$ and $\mathbf{v} = (v_1, ..., v_n)$, be two points in $\mathbb{I}^n$. It follows from Theorem B.6. in [22] that $\mathbf{u} \succ_m \mathbf{v}$ if, and only if, there exists a doubly stochastic matrix $P = [\lambda_{ij}]$, i.e., $\lambda_{ij} \ge 0$, with $\sum_{j=1}^n \lambda_{ij} = 1$, $i = 1, ..., n$ and $\sum_{i=1}^n \lambda_{ij} = 1$, $j = 1, ..., n$, such that $\mathbf{v} = P\mathbf{u}$, or equivalently for all $i = 1, ..., n$,

$v_i = \lambda_{i1}u_1 + ... + \lambda_{in}u_n$. Now the result follows from the definition of Schur–concavity (Schur-convexity). □

Note that for the case $n = 2$, the inequality (4.1) reduces to the inequality

$$C(u, v) \leq C(\lambda u + (1 - \lambda)v, (1 - \lambda)u + \lambda v), \tag{4.2}$$

for all $u, v \in \mathbb{I}$ and $\lambda \in [0, 1]$, which is studied in [3, 9, 26].

The following results can be directly derived.

Proposition 4.7. *Let C be an n-copula, then*
(i) If C Schur-concave (Schur-convex), then C is symmetric.
(ii) If C is Schur-concave (Schur-convex) on $\mathcal{D} = \{\mathbf{u} \in \mathbb{I}^n : u_1 \geq u_2 \geq ... \geq u_n\}$, then C is Schur-concave (Schur-convex) on $\mathbb{I}^n$.

The following example shows the Schur-concavity of the copula M^n.

Example 4.8. Consider the copula M^n. For $u_1, ..., u_n \in \mathbb{I}$, suppose that $\min(u_1, ..., u_n) = u_n$. Using the fact that $\sum_{j=1}^{n} \lambda_{ij} = 1$, for $i = 1, ..., n$, it is follows that

$$\sum_{j=1}^{n-1} \lambda_{ij} u_n \leq \sum_{j=1}^{n-1} \lambda_{ij} u_j, \quad i = 1, ..., n,$$

or equivalently,

$$u_n \leq \sum_{j=1}^{n} \lambda_{ij} u_j, \quad i = 1, ..., n,$$

and then

$$u_n \leq \min\{\sum_{j=1}^{n} \lambda_{1j} u_j, ..., \sum_{j=1}^{n} \lambda_{nj} u_j\}.$$

By changing u_n to arbitrary u_i, one get

$$\min(u_1, ..., u_n) \leq \min\{\sum_{j=1}^{n} \lambda_{1j} u_j, ..., \sum_{j=1}^{n} \lambda_{nj} u_j\}.$$

That is, M^n is a Schur-concave n-copula.

Proposition 4.9. *An n-copula C is Schur-concave on $\mathbb{I}^n$ if, and only if,*
(i) C is symmetric;
(ii) for all $\mathbf{x} \in \mathcal{D} = \{\mathbf{x} \in \mathbb{I}^n : x_1 \geq x_2 \geq ... \geq x_n\}$, $\frac{\partial C(\mathbf{x})}{\partial x_1} \leq \frac{\partial C(\mathbf{x})}{\partial x_2}$.

Example 4.10. Consider the F-G-M copula defined by (3.2). For all $u_1, ..., u_n \in [0, 1]$, as a consequence of the inequality $|1 - u_1 - u_2 + 2u_1u_2| \leq 1$, one has $(u_1 - u_2)\left(\frac{\partial C(\mathbf{u})}{\partial u_1} - \frac{\partial C(\mathbf{u})}{\partial u_2}\right) = -(u_1 - u_2)^2 \Pi_{j=3}^{n}\{1 + \theta(1 - u_1 - u_2 + 2u_1u_2) \times \Pi_{j=3}^{n}(1 - u_j)\} \leq 0$, whence C is Schur-concave.

The following result provides the Schur-concavity of the Archimedean n-copulas.

Proposition 4.11. *Every Archimedean n-copula is Schur-concave.*

Proof. Let C be an Archimedean copula with generator ϕ. Let $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{I}^n$ and let $\lambda_{ij}, i, j = 1, ..., n$ be as in Proposition 4.6. Since ϕ is convex and $\sum_{i=1}^{n} \lambda_{ij} = 1$, $j = 1, ..., n$, we have

$$\begin{aligned} \sum_{i=1}^{n} \phi\left(\sum_{j=1}^{n} \lambda_{ij} v_j\right) &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{ij}\phi(v_j) \\ &= \sum_{j=1}^{n} \phi(v_j). \end{aligned}$$

Since ϕ^{-1} is non-increasing we get the required result. □

As a consequence of Proposition 4.11, since the copula Π^n is Archimedean with generator $\phi(t) = -\log(t)$, it is Schur-concave.

Remark 4.12. When $n = 2$, as shown in [9] the copula W^2 is the only Schur-convex copula (and since W^2 is Archimedean, it is also a Schur-concave copula). Since W^n is not an n-copula for $n > 2$, then the Schur-convex n-copula may not exist.

Example 4.13. Consider the n-copula C defined by

$$C(u_1, ..., u_n) = \alpha M^n(u_1, ..., u_n) + (1 - \alpha)\Pi^n(u_1, ..., u_n),$$

for $\alpha \in \mathbb{I}$, which is a member of the Fréchet family of copulas [26]. It is a Schur-concave n-copula, because it is a convex sum of Schur-concave copulas.

For any $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{I}^n$, the k-dimensional marginal C_k, $k = 2, ..., n-1$, of a symmetric n-copula C is defined by

$$C_k(u_1, ..., u_k) = C(u_1, .., u_k, 1, ..., 1).$$

Proposition 4.14. *If C is Schur-concave (Schur-convex), then C_k, $k = 2, ..., n-1$, is Schur-concave (Schur-convex) as well.*

As the following example shows the converse is not true in general.

Example 4.15. Consider the function C defined by

$$C(u_1, u_2, u_3) = (\alpha+\beta-1)u_1u_2u_3 + (1-\alpha)u_2M^2(u_1, u_3) + (1-\beta)u_1W^2(u_2, u_3).$$

As shown in [8], a sufficient condition for C to be a 3-copula is that $\alpha + \beta \geq 1$, where $\alpha, \beta \in \mathbb{I}$. The bivariate margins of C are given by $C_{12}(u_1, u_2) = u_1u_2$, $C_{13}(u_1, u_3) = \alpha u_1u_3 + (1 - \alpha)M^2(u_1, u_3)$ and $C_{23}(u_2, u_3) = \beta u_2u_3 + (1 - \beta)W^2(u_2, u_3)$. All the bivariate margins are Schur-concave but the 3-copula C is not Schur-concave, since it is not symmetric.

Let $h : \mathbb{I} \to \mathbb{I}$ be an increasing, continuous and concave function with $h(0) = 0$ and $h(1) = 1$. For a given n-copula C the h-transform (or distortion) of C, is defined by

$$C_h(u_1, ..., u_n) = h^{-1}\{C(h(u_1), ..., h(u_n))\}. \tag{4.3}$$

These transformations play an important role in statistics; see, e.g., [16, 24]. The following result shows that the Schur-concavity property of a copula is preserved under transformations.

Proposition 4.16. *Let C be a Schur-concave n-copula. Then the h-transform of C defined by (4.3) is Schur-concave as well.*

Proof. Let $u_1, ..., u_n$ and λ_{ij} with $\sum_{j=1}^n \lambda_{ij} = 1$, for all $i = 1, ..., n$ and $\sum_{i=1}^n \lambda_{ij} = 1$, for all $j = 1, ..., n$, be in [0,1]. Since h is concave, we have

$$h(\sum_{j=1}^n \lambda_{ij} u_j) \geq \sum_{j=1}^n \lambda_{ij} h(u_j),$$

for all $i = 1, ..., n$. Moreover, since C is Schur-concave, we have

$$C\left(\sum_{j=1}^n \lambda_{1j} h(u_j), ..., \sum_{j=1}^n \lambda_{nj} h(u_j)\right) \geq C(h(u_1), ..., h(u_n)).$$

But C is increasing in each argument so that

$$C_h\left(\sum_{j=1}^n \lambda_{1j} u_j, ..., \sum_{j=1}^n \lambda_{nj} u_j\right) \geq C_h(u_1, ..., u_j),$$

which completes the proof. ☐

We will denote by $\mathcal{C}_{SC}$ the class of all Schur-concave n-copulas.

Proposition 4.17. *The class $\mathcal{C}_{SC}$ is a convex and compact (with respect to L^∞ norm) subset of $\mathcal{C}$.*

In what follows we provide a method for constructing n-copulas with the Schur-concavity property. Let $U_1, ..., U_n$ and Z be uniform $[0, 1]$ random variables such that the vectors (U_j, Z), $j = 1, ..., n$, are independent and identically distributed with associated 2-copula D and let $C[D]$, denote the distribution (copula) of the vector $(U_1, ..., U_n)$. Then using (3.3) one has

$$\begin{aligned} P(U_1 \leq u_1, ..., U_n \leq u_n | Z = t) &= \prod_{j=1}^n P(U_j \leq u_j | Z = t) \\ &= \prod_{j=1}^n \frac{\partial D(u_j, t)}{\partial t} dt, \end{aligned}$$

and thus

$$\begin{aligned} C[D](u_1,...,u_n) &= P(U_1 \le u_1,...,U_n \le u_n) \\ &= \int_0^1 P(U_1 \le u_1,...,U_n \le u_n | Z = t) dt \\ &= \int_0^1 \prod_{j=1}^n \frac{\partial D(u_j,t)}{\partial t} dt. \end{aligned}$$

Note that the n-copula $C[D]$ is symmetric. The following result provide a class of Schur-concave n-copulas.

Proposition 4.18. *Let D be a 2-copula and consider the function $C[D] : \mathbb{I}^n \to \mathbb{I}$, defined by*

$$C[D](u_1,...,u_n) = \int_0^1 \prod_{j=1}^n \frac{\partial D(u_j,t)}{\partial t} dt.$$

If the mapping $u \to \frac{\partial D(u,t)}{\partial t}$ is log-concave, then the n-copula $C[D]$ is Schur-concave.

Proof. Let $G_t(u) = \frac{\partial D(u,t)}{\partial t}$. From Proposition L.2. in [22], $\prod_{j=1}^n G_t(u_j)$ is Schur-concave if, and only if $G_t(u)$ is log-concave. By using the fact that a mixture of Schur-concave functions is Schur-concave [22], the result follows. □

Example 4.19. Consider the bivariate Marshall-Olkin copula [26] $D(u,t) = \min\{u, u^\alpha t\}$, with $\alpha \in [0,1]$. It is easy to see that $\frac{\partial D(u,t)}{\partial t}$ is logarithmically concave with respect to u. Direct calculations shows that

$$\int_0^1 \prod_{j=1}^n \frac{\partial D(u_j,t)}{\partial t} dt = u_{(1)} \prod_{j=2}^n u_{(j)}^{1-\alpha}, \tag{4.4}$$

where $u_{(1)} \le u_{(2)} \le ... \le u_{(n)}$ denote the components of $(u_1,...,u_n) \in \mathbb{I}^n$ rearranged in increasing order. The copulas of type (4.4) studied in [12].

5. Weakly Schur-concavity

A weakening of condition (4.2), has been considered, mainly in a context different from that of copulas [20],

$$C(u,v) \le C(\frac{u+v}{2}, \frac{u+v}{2}),$$

for all $u, v \in \mathbb{I}$; see also [13].

The concept of weakly Schur-concave bivariate copulas can be generalized to the multivariate setting as follows.

Definition 5.1. An n-copula C is said to be weakly Schur-concave if

$$C(u_1,...,u_n) \le C(\bar{u},...,\bar{u}),$$

for all $u_1,...,u_n \in \mathbb{I}$, where $\bar{u} = \sum_{i=1}^n u_i/n$.

Note that all Schur-concave copulas are weakly Schur-concave. These concepts can be characterized in terms of the concept of mean function [17, 25].

Definition 5.2. Let $y = g(x_1, ..., x_n)$ be a function of n variables $x_1, ..., x_n$. A mean function of $x_1, ..., x_n$ with respect to the function g is a number M_g such that, if each of $x_1, ..., x_n$ is replaced by M_g, the function value is unchanged, i.e.,

$$g(x_1, ..., x_n) = g(M_g, ..., M_g).$$

The weakly Schur-concavity of n-copulas can be characterized as follows.

Proposition 5.3. *Let C be an n-copula with associated mean function M_C. Then C is weakly Schur-concave if, and only if for all $u_1, ..., u_n \in \mathbb{I}$*

$$M_C(u_1, ..., u_n) \leq \bar{u},$$

where $\bar{u} = \frac{1}{n}\sum_{j=1}^{n} u_j$.

The following result whose proof is similar to that in Proposition 4.16 shows that the weakly Schur-concavity property of a copula is preserved under concave transformations.

Proposition 5.4. *Let C be a weakly Schur-concave n-copula. Then the h-transform of C defined by (4.3) is weakly Schur-concave as well.*

We will denote by $\mathcal{C}_{WSC}$ the class of all weakly Schur-concave n-copulas.

Proposition 5.5. *The class $\mathcal{C}_{WSC}$ is a convex and compact (with respect to L^∞ norm) subset of $\mathcal{C}$.*

6. Quasi–concavity of n-copulas

A 2-dimensional copula C is said to be quasi-concave [1, 4, 26] if for all $(u, v), (u', v') \in \mathbb{I}^2$ and all $\lambda \in \mathbb{I}$,

$$C(\lambda u + (1-\lambda)u', \lambda v + (1-\lambda)v') \geq \min\{C(u, v), C(u', v')\}.$$

The n-dimensional ($n \geq 2$) extension of quasi-concavity is as follows [29]:

Definition 6.1. An n-copula C is called quasi-concave if for all $\mathbf{u} = (u_1, ..., u_n)$ and $\mathbf{v} = (v_1, ..., v_n)$ in $\mathbb{I}^n$ and $\lambda \in \mathbb{I}$,

$$C(\lambda \mathbf{u} + (1-\lambda)\mathbf{v}) \geq \min\{C(\mathbf{u}), C(\mathbf{v})\}. \tag{6.1}$$

Condition (6.1) is equivalent to requiring that upper-level sets of C, i.e.,

$$U_q = \{\mathbf{u} \in \mathbb{I}^n : C(\mathbf{u}) \geq q\},$$

are convex for all q.

Example 6.2. As for all q, the set $U_q = \{\mathbf{u} \in \mathbb{I}^n : u_1 \geq q, ..., u_n \geq q\}$ is convex, then the n-copula M^n turns out to be a quasi-concave.

Remark 6.3. Note that the only quasi-convex copula is W^2 (see, [3]). Since W^2 is an Archimedean copula, it is also Quasi-concave. Since W^n is not a copula for $n > 2$, the n-copulas with the quasi-convexity property does not exist.

Proposition 6.4. *Every Archimedean n-copula is quasi-concave.*

Proof. Let C be an Archimedean n-copula with generator ϕ. Since ϕ is convex, then for $\mathbf{u} \in \mathbb{I}^n$, the function $g(\mathbf{u}) = \phi(u_1) + ... + \phi(u_n)$ is convex too, so that

$$\begin{aligned} g(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) &\leq \lambda g(\mathbf{u}) + (1-\lambda)g(\mathbf{v}) \\ &\leq \lambda\max\{g(\mathbf{u}), g(\mathbf{v})\} + (1-\lambda)\max\{g(\mathbf{u}), g(\mathbf{v})\} \\ &= \max\{g(\mathbf{u}), g(\mathbf{v})\}. \end{aligned}$$

Since ϕ^{-1} is non-increasing, we have

$$\begin{aligned} \phi^{-1}\{g(\lambda\mathbf{u} + (1-\lambda)\mathbf{v})\} &\geq \phi^{-1}\{\max(g(\mathbf{u}), g(\mathbf{v}))\} \\ &= \max\{\phi^{-1}(g(\mathbf{u})), \phi^{-1}(g(\mathbf{v}))\} \\ &\geq \min\{\phi^{-1}(g(\mathbf{u})), \phi^{-1}(g(\mathbf{v}))\}, \end{aligned}$$

which gives the required result. □

Proposition 6.5. *Let $h : \mathbb{I} \to \mathbb{I}$ be an increasing, continuous and concave function with $h(0) = 0$ and $h(1) = 1$. If C is a quasi-concave n-copula, then the h-transform of C, given by (4.3), is also quasi-concave.*

Proof. Let $\mathbf{u}, \mathbf{v}$ be in $\mathbb{I}^n$ and $\lambda \in \mathbb{I}$. Because h is concave, we have

$$h(\lambda u_i + (1-\lambda)v_i) \geq \lambda h(u_i) + (1-\lambda)h(v_i),$$

for all $i = 1, ..., n$. Let $e_i = \lambda u_i + (1-\lambda)v_i$ and $f_i = \lambda h(u_i) + (1-\lambda)h(v_i)$ for all $i = 1, ..., n$. Moreover, since C is increasing in each variable and quasi-concave, we have

$$\begin{aligned} C(e_1, ..., e_n) &\geq C(f_1, ..., f_n) \\ &\geq \min\{C(h(u_1), ..., h(u_n)), C(h(v_1), ..., h(v_n))\}. \end{aligned}$$

But h is increasing so that

$$C_h(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) \geq \min\{C_h(\mathbf{u}), C_h(\mathbf{v})\},$$

which completes the proof. □

We will denote by $\mathcal{C}_{QC}$ the class of all quasi-concave n-copulas.

Proposition 6.6. *The class $\mathcal{C}_{QC}$ is a convex and compact (with respect to L^∞ norm) subset of $\mathcal{C}$.*

7. Concluding remarks

In this paper we provided some results on different types of concavity and convexity properties in the class of multivariate copulas. As two of the reviewers mentioned, many questions suggest themselves for further study. We present a few. (i) Geometrical interpretations for different types of convexity/concavity concepts for bivariate copulas can be found in the literature, see, e.g, Section 3.4.3 in [26]. Is it possible to provide geometric interpretations for some of these concepts in multivariate setting? (ii) For the case $n = 2$, the relations among the considered convexity/concavity notions could be found in [3, 4, 14]. For example: Quasi-concavity and symmetry imply Schur-concavity and componentwise concavity implies quasi-concavity. Does it occur in higher dimensions? (iii) In bivariate case, the preservation of componentwise concavity, Schur-concavity and weakly Schur-concavity with respect to the ordinal sum is studied in [9, 13, 14]. Does any of the introduced convexity/concavity notion preserve under multivariate ordinal sum in the sense of [23]?

Acknowledgements

The authors wish to thank Prof. Fabrizio Durante and three anonymous referees for their helpful comments on an earlier version of this paper. The authors also thank the research council of Yazd University for support.

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