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Application of the Norm Estimates for Univalence of Analytic Functions

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Abstract. By using norm estimates of the pre-Schwarzian derivatives for certain family of analytic functions, we shall give simple sufficient conditions for univalence of analytic functions.

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1. Introduction

Let $\mathcal{H}$ Denote the class of all analytic functions in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, let $\mathcal{H}[a, n]$ and $\mathcal{A}_n$ denote the following classes of analytic functions

$$\mathcal{A}_n = \{f \in \mathcal{H} : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ..., z \in \Delta\}$$

and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + ..., z \in \Delta\}.$$

Set $\mathcal{A} := \mathcal{A}_1$.

Also let S denote the class of all univalent functions in $\mathcal{A}$. We denote by S^* the familiar class of functions in $\mathcal{A}$ which are starlike (with respect to origin).

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Suppose f and g belongs to $\mathcal{H}$, we say that $f(z)$ is subordinate to $g(z)$; written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$ on Δ. In particular, if $g(z)$ is univalent in Δ, then it is known that

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad and \quad f(\Delta) \subseteq g(\Delta).$$

For $\alpha \in \mathbb{C}$, let $J_\alpha[f]$ denote the nonlinear integral operator defined by

$$J_\alpha[f](z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt.$$

In [8], Kim and Merkes showed that $J_\alpha(S) = \{J_\alpha[f] : f \in S\} \subseteq S$ when $|\alpha| < 1/4$. Also Aksent'ev and Nezhmetdinov proved in [1] that $J_\alpha[S^*] \subseteq S$ precisely when $|\alpha| < 1/2$ or $\alpha \in [1/2, 1/3]$. Also such results for other spaces were investigated in [4, 10, 11].

For a constant $\beta \in \mathbb{C}, \lambda > 0$ and $\sigma \geq 0$ consider the classes $\mathcal{U}(\beta, \lambda)$ and $\mathcal{U}_\sigma(n, \lambda)$ defined by

$$\mathcal{U}(\beta, \lambda) = \{f \in \mathcal{A}_n : |zf''(z) - \beta(f'(z) - 1)| < \lambda\}$$

and

$$\mathcal{U}_\sigma(n, \lambda) = \{f \in \mathcal{U}(n, \lambda) : |f^{(n+1)}(0)| \leq (n+1)!\sigma\}.$$

Recently, the class $\mathcal{U}(\beta, \lambda)$ and $\mathcal{U}_\sigma(n, \lambda)$ have been studied by Miller and Mocanu [9] and Kuroki and Owa [6]. It is shown that in [9], $\mathcal{U}(\beta, \lambda) \subseteq S^*$ for $0 \leq \beta < n$ and $\lambda = n - \beta$.

Let $f : \Delta \to \mathbb{C}$ be analytic and locally univalent. The pre-Schwarzian derivative $\mathcal{T}_f$ of f is defined by

$$\mathcal{T}_f(z) = \frac{f''(z)}{f'(z)}.$$

Also, the quantity

$$||f|| = \sup_{z \in \Delta} (1 - |z|^2)|\mathcal{T}_f(z)|$$

is called the norm of $\mathcal{T}_f$.

In this paper we find conditions on parameters β, λ, σ and α such that

$$J_\alpha[\mathcal{U}(\beta, \lambda)] \subseteq S \qquad and \qquad J_\alpha[\mathcal{U}_\sigma(n, \lambda)] \subseteq S.$$

For proving our results we need the following two lemmas.

Lemma 1.1. *(see[2,3]) Let f be analytic and locally univalent in Δ. Then if $||f|| < 1$ then f is univalent, and the constant 1 is sharp.*

Lemma 1.2. *(see [5]) Let $h(z)$ be a convex univalent function with $h(0) = a$ and let $Re\gamma > 0$. If $p(z) \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}dt.$$

This result is sharp.

2. Main Results

Theorem 2.1. *Let λ, μ, σ be non-negative numbers with $\mu = \sigma + \frac{\lambda}{n+2} \leq 1$. For a function $f \in \mathcal{U}_\sigma(n, \lambda)$, we have*

$$||J_\alpha[f]|| \leq \frac{2|\alpha|\mu}{1+\sqrt{1-\mu^2}}. \tag{2.1}$$

for every $\alpha \in \mathbb{C}$. In the case $n = 1$ the equality holds above precisely when $f(z) = z + az^2$ with $a = \mu < 1$.

Proof. Taking a logarithmic differentiation, we obtain, $\mathcal{T}_{J_\alpha[f]}(z) = \alpha\mathcal{T}_{J_1[f]}(z)$ and thus

$$||J_\alpha[f]|| = |\alpha|||J_1[f]||.$$

Therefore it suffices to consider $||J_1[f]||$. Let $f(z) = z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+ ...$ be in $\mathcal{U}_\sigma(n, \lambda)$ and $G(z) = J_1[f](z)$. If we set $p(z) = f'(z) - (1+n)\frac{f(z)}{z}$, then we have

$$p(z) = -n + a_{n+2}z^{n+1} + ...$$

and we observe that $p(z) \in \mathcal{H}[-n, n+1]$. Further it is easy to see that, $f \in \mathcal{U}_\sigma(n, \lambda)$ is equivalent to

$$p(z) + zp'(z) \prec -n + \lambda z. \tag{2.2}$$

Applying Lemma 1.2 to the (2.2), we obtain

$$p(z) \prec \frac{1}{(n+1)z^{1/(n+1)}} \int_0^z [-n + \lambda t]t^{1/(n+1)-1}dt = -n + \frac{\lambda}{n+2}z.$$

Hence

$$f'(z) - (n+1)\frac{f(z)}{z} = -n + \frac{\lambda}{n+2}\omega(z), \tag{2.3}$$

where ω is an analytic function in Δ with $|\omega(z)| \leq 1$ and $\omega(0) = \omega'(0) = ... = \omega^{(n)}(0) = 0$. By setting $g(z) = \frac{f(z)}{z} - 1$, we may rewrite the relation (2.3) as

$$zg'(z) - ng(z) = \frac{\lambda}{n+2}\omega(z).$$

Solving this differential equation we have

$$g(z) = a_{n+1}z^n + \frac{\lambda}{n+2}\int_0^1 \frac{\omega(tz)}{t^{n+1}}dt. \tag{2.4}$$

Since $|a_{n+1}|+\frac{\lambda}{n+2}\leq\mu$, and in view of Schwarz's lemma $|\omega(z)|\leq|z|^{n+1}$ for $z\in\Delta$, (2.4) implies that

$$|g(z)|\leq|z|^n\left(|a_{n+1}|+\frac{\lambda}{n+2}|z|\right)<\mu.$$

So $\mathcal{F}(z):=f(z)/z$ is subordinate to the function $q(z):=1+\mu z$ and this means that $\mathcal{F}(z)=q(\omega_1(z))$ where $\omega_1(z)$ is Schwarz function. Hence

$$||G||=\sup_{z\in\Delta}(1-|z|^2)\left|\frac{\mathcal{F}'(z)}{\mathcal{F}(z)}\right|=\sup_{z\in\Delta}(1-|z|^2)\frac{|q'(\omega_1(z))||\omega_1'(z)|}{|q(\omega_1(z))|}$$

But, by the Schwarz-Pick lemma, we know that

$$|\omega_1'(z)|\leq\frac{1-|\omega_1(z)|^2}{1-|z|^2}.$$

Therefore we obtain

$$||G||\leq\sup_{z\in\Delta}(1-|z|^2)\frac{|q'(\omega_1(z))|(1-|\omega_1(z)|^2)}{(1-|z|^2)|q(\omega_1(z))|}\leq\sup_{z\in\Delta}(1-|z|^2)\left|\frac{q'(z)}{q(z)}\right|.$$

Since

$$\frac{q'(z)}{q(z)}=\frac{\mu}{1+\mu z},$$

a computation shows that

$$\sup_{z\in\Delta}(1-|z|^2)\left|\frac{q'(z)}{q(z)}\right|=\mu\sup_{0<t<1}\frac{1-t^2}{1-\mu t}=\frac{2\mu}{1+\sqrt{1-\mu^2}}.$$

Thus inequality (2.1) follows. Now for function $f(z)=z+az^2$ with $0<a<1$ we have

$$||J_\alpha[f]||=\sup_{z\in\Delta}(1-|z|^2)\frac{|\alpha||a|}{|1+az|}.$$

But

$$(1-|z|^2)\frac{|\alpha||a|}{|1+az|}\leq(1-|z|^2)|\alpha|\frac{|a|}{1-|a||z|},$$

and so

$$\sup_{z\in\Delta}(1-|z|^2)\frac{|\alpha||a|}{|1+az|}\leq\sup_{z\in\Delta}(1-|z|^2)\frac{|\alpha||a|}{1-|a||z|}=\frac{2|\alpha|a}{1+\sqrt{1-a^2}}.$$

We note that in the last equality sup has taken on the point $|z|=\frac{1-\sqrt{1-a^2}}{a}$. On the other hand by putting $z=t$ we obtain

$$|\alpha|a\frac{1-t^2}{1+at}\leq\sup_{z\in\Delta}(1-|z|^2)\frac{|\alpha||a|}{|1+az|}.$$

By putting $t=\frac{1-\sqrt{1-a^2}}{-a}$ on the left hand side we have $|\alpha|a\frac{1-t^2}{1+at}=|\alpha|\frac{2a}{1+\sqrt{1-a^2}}$.

Hence equality holds for the function $f(z)=z+az^2$. □

Corollary 2.2. *Let λ, σ be non-negative numbers with $\sigma + \frac{\lambda}{n+2} \leq 1$ and $\alpha \in \mathbb{C}$ satisfy the condition*

$$|\alpha| \leq \frac{1 + \sqrt{1 - (\sigma + \frac{\lambda}{n+2})^2}}{2(\sigma + \frac{\lambda}{n+2})}.$$

Then $J_\alpha(\mathcal{U}_\sigma(n, \lambda)) \subseteq S$.

Letting $a_{n+1} = 0$ in the corollary 2.1, we obtain the following corollary.

Corollary 2.3. *Let $0 < \lambda \leq n + 2$ and α be complex number which satisfy the condition*

$$|\alpha| \leq \frac{n + 2 + \sqrt{(n+2)^2 - \lambda^2}}{2\lambda}.$$

Then $J_\alpha(\mathcal{U}_0(n, \lambda)) \subseteq S$.

By putting $\lambda = n + 2$ in the corollary 2.2 we obtain the following example.

Example 2.4. Let α, c be complex numbers with $|c| < 1$ and $|\alpha| \leq 1/2$. Then the function

$$F(z) = \int_0^z (1 + cu^{n+1})^\alpha du$$

is univalent in Δ.

Example 2.5. Let the function $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2}$ be such that $|a_{n+1}| + |a_{n+2}| < 1$ and α be complex number satisfy $|\alpha| \leq 1/2$. Then function

$$G(z) = \int_0^z (1 + a_{n+1}u^n + a_{n+2}u^{n+1})^\alpha du$$

is univalent in Δ.

Theorem 2.6. *Let β be a complex number with $0 \leq Re\beta < n$ and λ be non-negative number which satisfy the condition $0 < \lambda \leq (n - Re\beta)(n + 1)$. For a function $f \in \mathcal{U}(\beta, \lambda)$ and for any $\alpha \in \mathbb{C}$ we have*

$$||J_\alpha[f]|| \leq \frac{2|\alpha|\mu}{1 + \sqrt{1 - \mu^2}}, \tag{2.5}$$

where $\mu = \lambda/(n + 1)(n - Re\beta)$. If, in addition, $\lambda < 2(1 - Re\beta)$, then equality holds for the case $n = 1$ when $f(z) = z + az^2$ for a constant a with $a = \mu$.

Proof. Suppose that $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ...$ be in $\mathcal{U}(\beta, \lambda)$ and set $\mathcal{F} = J_1[f]$. By letting $p(z) := f'(z) - (1 + \beta)\frac{f(z)}{z} = -\beta + (n - \beta)a_{n+1}z^n + ...$ it is obvious that $p(z) \in \mathcal{H}[-\beta, n]$.

Further, $f \in \mathcal{U}(\beta, \lambda)$ is seen to be equivalent to

$$p(z) + zp'(z) \prec -\beta + \lambda z. \tag{2.6}$$

Applying lemma 1.2, to (2.6) we obtain

$$p(z) \prec \frac{1}{nz^{\frac{1}{n}}} \int_0^z (-\beta + \lambda t)t^{\frac{1}{n}-1}dt,$$

or equivalently

$$f'(z) - (1+\beta)\frac{f(z)}{z} \prec -\beta + \frac{\lambda}{n+1}z. \tag{2.7}$$

We may write the last subordination as

$$f'(z) - (1+\beta)\frac{f(z)}{z} = -\beta + \frac{\lambda}{n+1}\omega(z), \tag{2.8}$$

where ω is an analytic function with $\omega(0) = \omega'(0) = ... = \omega^{n-1}(0) = 0$ and $|\omega(z)| < 1$ for $z \in \Delta$.

If we consider $g(z) = \frac{f(z)}{z} - 1$, then (2.8) becomes

$$zg'(z) - \beta g(z) = \frac{\lambda}{n+1}\omega(z).$$

An algebraic computation yields that

$$g(z) = \frac{\lambda}{n+1}\int_0^1 \frac{\omega(tz)}{t^{\beta+1}}dt. \tag{2.9}$$

Since $\omega(0) = \omega'(0) = ... = \omega^{n-1}(0) = 0$ and $|\omega(z)| < 1$, Schwarz's lemma gives that $|\omega(z)| < |z|^n$ for $z \in \Delta$ and therefore

$$\left|\frac{f(z)}{z} - 1\right| < \frac{\lambda}{(n+1)(n-Re\beta)}|z|^n < \frac{\lambda}{(n+1)(n-Re\beta)} = \mu.$$

Now following the same as proof of Theorem 2.1 we get our result. □

By putting $\beta = 0$ in the Theorem 2.2 we obtain the following corollary.

Corollary 2.7. *Let $0 < \lambda \leq n(n+1)$. If $f(z) \in \mathcal{A}_n$ satisfy the condition*

$$|zf''(z)| < \lambda,$$

then $||J_\alpha[f]|| \leq \frac{2|\alpha|\mu}{1+\sqrt{1-\mu^2}}$, where $\mu = \frac{\lambda}{n(n+1)}$. The result is sharp in the case $n = 1$ for the function $f(z) = z + az^2$ with $|a| = \mu$.

We remark that the special case of corollary 2.3 was obtained in [7]. (see Theorem 2.7)

Corollary 2.8. *Let $0 < \lambda \leq (n-Re\beta)(n+1)$ and α be a complex number with*

$$|\alpha| \leq \frac{(n-Re\beta)(n+1) + \sqrt{(n-Re\beta)^2(n+1)^2 - \lambda^2}}{2\lambda}.$$

Then $J_\alpha(\mathcal{U}(\beta,\lambda)) \subseteq S$.

Let $0 \leq Re\beta < n$ and a function $g(z) \in \mathcal{H}$ satisfy the condition

$$|g(z)| \leq \frac{4(n+1)(n-Re\beta)}{5}.$$

Also let $f(z) \in \mathcal{A}_n$ satisfy the differential equation

$$zf''(z) - \beta(f'(z) - 1) = z^n g(z). \tag{2.10}$$

Then, it is clear that

$$|zf''(z) - \beta(f'(z) - 1)| = |z|^n |g(z)| \leq \frac{4(n+1)(n - Re\beta)}{5}.$$

Hence, from corollary 2.4, we observe that for $|\alpha| \leq 1$, we have $J_\alpha[f] \in S$.

By letting $\alpha = 1$ we have the following example

Example 2.9. Let β be a complex number with $0 \leq Re\beta < n$ and $g(z) \in \mathcal{H}$ satisfy

$$|g(z)| \leq 4(n+1)(n - Re\beta).$$

Then the function $F(z) \in \mathcal{A}_n$ satisfying the differential equation

$$z^2 F'''(z) + (2-\beta) z F''(z) - \beta F'(z) + \beta = z^n g(z) \tag{2.11}$$

is univalent in Δ.

It is easy to see that the solution of (2.11) is

$$F(z) = z + z^{n+1} \int_0^1 \int_0^1 \int_0^1 g(rstz) r^{n-\beta-1} t^n s^n dr ds dt.$$

So we may rewrite example 2.3 in the following equivalent form

Example 2.10. Let β be a complex number with $0 \leq Re\beta < n$ and $g(z) \in \mathcal{H}$ satisfy

$$|g(z)| \leq 4(n+1)(n - Re\beta).$$

Then the function $F(z) \in \mathcal{A}_n$ defined by

$$F(z) = z + z^{n+1} \int_0^1 \int_0^1 \int_0^1 g(rstz) r^{n-\beta-1} t^n s^n dr ds dt.$$

is univalent in Δ.

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References

1. L. A. Aksentev, I. R. Nezhmetdinov, Sufficient conditions for univalence of certain integral representation, *Trudy Sem. Kraev. Zadacham*, **18**, (1982), 3-11.
2. J. Becker, Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen, *J. Reine Angew. Math*, **255**, (1972), 23-43.
3. J. Becker, Ch. Pommerenke, Schlichtheitskriterien und Jordangebiete, *J. Reine Angew. Math*, **354**, (1984), 74-94.
4. M. R. Eslachi, S. Amani, The best uniform polynomial approximation of two classes of rational functions, *Iranian Journal of Mathematical Sciences and Informatics*, **7**(2), (2012), 93-102.
5. D. J. Hallenbeck, St. Ruscheweyh, Subordination by convex functions, *Proc. Amer. Math. Soc*, **52**, (1975), 191-195.

6. K. Kuroki, S. Owa, Double integral operators concerning starlike of order β, *International Journal of Differential Equations*, (2009), 1-13.
7. Y. C. Kim, S. Ponnusamy, T. Sugawa, Geometric properties of nonlinear integral transforms of analytic functions, *Proc. Japan Acad. Ser A*, **80**, (2004), 57-60.
8. Y. J. Kim, E. P. Merkes, On an integral of powers of a spirallike function, *Kyungpook Math. J*, **12**, (1972), 249-253.
9. S. S. Miller, P. T. Mocanu, Double integral starlike operators, *Integral Transforms and Special Functions*, **19** (7-8), (2008), 591-597.
10. M. Obradovic, Simple sufficient conditions for univalence, *Mat. Vesnik*, **49**, (1997), 241-244.
11. A. Taghavi, R. Hosseinzadeh, Uniform boundedness principle for operators on hypervector spaces, *Iranian Journal of Mathematical Sciences and Informatics*, **7** (2), (2012), 9-16.