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Local Cohomology with Respect to a Cohomologically Complete Intersection Pair of Ideals

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Abstract. Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring of dimension n. Let $\mathrm{H}^i_{I,J}(R)$ be the local cohomology with respect to a pair of ideals I, J and c be the $\inf\{i|\mathrm{H}^i_{I,J}(R) \neq 0\}$. A pair of ideals I, J is called cohomologically complete intersection if $\mathrm{H}^i_{I,J}(R) = 0$ for all $i \neq c$. It is shown that, when $\mathrm{H}^i_{I,J}(R) = 0$ for all $i \neq c$, (i) a minimal injective resolution of $\mathrm{H}^c_{I,J}(R)$ presents like that of a Gorenstein ring; (ii) $\mathrm{Hom}_R(\mathrm{H}^c_{I,J}(R), \mathrm{H}^c_{I,J}(R)) \simeq R$, where $(R, \mathfrak{m})$ is a complete ring. Also we get an estimate of the dimension of $\mathrm{H}^i_{I,J}(R)$.

Keywords: Vanishing, Local cohomology, Gorenstein ring.

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1. Introduction

Throughout this paper, R is a commutative Noetherian ring and I, J are ideals of R. The generalized local cohomology module with respect to a pair of ideals I, J of R was introduced by Takahashi–Yoshino–Yoshizawa [6].

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We are concerned with the subsets

$$W(I,J) = \{\mathfrak{p} \in \operatorname{Spec}(R) | I^n \subseteq \mathfrak{p} + J \text{ for an integer n} \gg 1\}$$

of Spec (R) and $\tilde{W}(I,J) = \{\mathfrak{a} \trianglelefteq R | I^n \subseteq \mathfrak{a} + J$ for an integer $n \gg 1\}$. In general, $W(I,J)$ is closed under specialization, but not necessarily a closed subset of Spec(R). For an R–module M, we consider the (I,J)–torsion submodule $\Gamma_{I,J}(M)$ of M which consists of all elements x of M with $\operatorname{Supp}(Rx) \subseteq W(I,J)$. Furthermore, for an integer i, we define the local cohomology functor $\mathrm{H}^i_{I,J}(-)$ with respect to (I,J) to be the i–th right derived functor of $\Gamma_{I,J}(-)$. Note that if $J = 0$ then $\mathrm{H}^i_{I,J}(-)$ coincides with the ordinary local cohomology functor $\mathrm{H}^i_I(-)$ with the support in the closed subset $V(I)$. On the other hand, if J contains I then $\Gamma_{I,J}$ is the identity functor and $\mathrm{H}^i_{I,J}(-) = 0$ for all $i > 0$.
Recently some interesting results for ideals with $c = \operatorname{ht} I = \operatorname{cd} I$, the so called cohomologically complete intersections have been proved. Hellus–Stückrad [3] have shown that if $(R, \mathfrak{m})$ is a complete local ring, then the endomorphism ring $\operatorname{Hom}_R(\mathrm{H}^c_I(R), \mathrm{H}^c_I(R))$ is isomorphic to R. In [2] and [5], Schenzel by a functorial proof and a slight extension, proved that $\operatorname{Hom}_R(\mathrm{H}^c_I(R), \mathrm{H}^c_I(R)) \simeq R$ if and only if $\mathrm{H}^i_I(R) = 0$, $i = n, n-1$.

The endomorphism ring $\operatorname{Hom}_R(\mathrm{H}^c_{I,J}(R), \mathrm{H}^c_{I,J}(R))$, when $c = \inf\{i | \mathrm{H}^i_{I,J}(R) \neq 0\}$ and $(R, \mathfrak{m})$ is a Gorenstein ring , is the main subject of our investigation. First as a generalization of the concept of cohomologically complete intersection, a pair of ideals I, J is called cohomologically complete intersection whenever $c = \inf\{i | \mathrm{H}^i_{I,J}(R) \neq 0\} = \operatorname{cd}(I,J)$, in which $\operatorname{cd}(I,J) = \sup\{i | \mathrm{H}^i_{I,J}(R) \neq 0\}$. Then we show that for this certain class of ideals, $\mathrm{H}^d_{\mathfrak{m}}(\mathrm{H}^c_{I,J}(R)) \cong E$ and $\mathrm{H}^i_{\mathfrak{m}}(\mathrm{H}^c_{I,J}(R)) = 0$ for all $i \neq d$, where E denotes the injective hull of the residue field $R/\mathfrak{m}$. Next by this fact, we prove that $\operatorname{Hom}_R(\mathrm{H}^c_{I,J}(R), \mathrm{H}^c_{I,J}(R))$ is isomorphic to R provided R is a complete local ring. Moreover we show that the natural homomorphism $\operatorname{Hom}_R(\mathrm{H}^c_{I,J}(R), \mathrm{H}^c_{I,J}(R)) \longrightarrow \operatorname{Hom}_R(\mathrm{H}^c_{I',J}(R), \mathrm{H}^c_{I',J}(R))$ is a monomorphism when R is a complete ring and $I \subseteq I'$ with $c = \inf\{i | \mathrm{H}^i_{I,J}(R) \neq 0\} = \inf\{i | \mathrm{H}^i_{I',J}(R) \neq 0\}$. As a consequence, if $\mathrm{H}^i_{I',J}(R) = 0$ for all $i \neq c$, then there exists the natural monomorphism $\operatorname{Hom}_R(\mathrm{H}^c_{I,J}(R), \mathrm{H}^c_{I,J}(R)) \longrightarrow R$.
In this paper we shall use the notion of the dimension $\dim X$ for R–modules X which are not necessarily finitely generated. This is defined by $\dim X = \dim \operatorname{Supp}_R X$, where the dimension of the support is understood in the Zariski topology of $\operatorname{Spec} R$. In particular, $\dim X < 0$ means $X = 0$. We prove that $\dim \mathrm{H}^i_{I,J}(R) \leq n - i$ for all $i \geq c$ and $\dim \mathrm{H}^c_{I,J}(R) = n - c$, when $n = \dim R$ and I, J are proper ideals of R with $c = \inf\{i | \mathrm{H}^i_{I,J}(R) \neq 0\}$.

2. Main Results

Let $(R, \mathfrak{m})$ be a local Gorenstein ring and $n = \dim R$. Let $R \stackrel{\sim}{\longrightarrow} \dot{E}$ denote a minimal injective resolution of R as an R–module. Let $I, J \subset R$ be two ideals and $c = \inf\{i | \mathrm{H}^i_{I,J}(R) \neq 0\}$ and $d = n - c$. The local cohomology modules $\mathrm{H}^i_{I,J}(R)$, $i \in \mathbb{Z}$, are–by definition– the cohomology modules of the complex $\Gamma_{I,J}(\dot{E})$. Because of $\Gamma_{I,J}(E(R/\mathfrak{p})) = 0$ for all $\mathfrak{p} \notin W(I,J)$, it follows that $\Gamma_{I,J}(E^i) = 0$ for all $i < c$. Therefore $\mathrm{H}^c_{I,J}(R) = \mathrm{Ker}\,(\Gamma_{I,J}(\dot{E})^c \longrightarrow \Gamma_{I,J}(\dot{E})^{c+1})$. This observation provides an embedding $\mathrm{H}^c_{I,J}(R)[-c] \longrightarrow \Gamma_{I,J}(\dot{E})$ of complexes of R–modules.

Definition 2.1. The cokernel of the embedding $\mathrm{H}^c_{I,J}(R)[-c] \longrightarrow \Gamma_{I,J}(\dot{E})$ is defined as $\dot{C}(I,J)$, the generalized truncation complex. So there is a short exact sequence of complexes of R–modules

$$(*) \qquad 0 \longrightarrow \mathrm{H}^c_{I,J}(R)[-c] \longrightarrow \Gamma_{I,J}(\dot{E}) \longrightarrow \dot{C}(I,J) \longrightarrow 0.$$

In particular it follows that $\mathrm{H}^i(\dot{C}(I,J)) = 0$ for $i \leq c$ or $i > n$ and $\mathrm{H}^i(\dot{C}(I,J)) \cong \mathrm{H}^i_{I,J}(R)$ for $c < i \leq n$.

Next Lemma is a generalization of [2, Lemma 2.2].

Lemma 2.2. *With the previous notation there are an exact sequence*

$$0 \longrightarrow H^{n-1}_{\mathfrak{m}}(\dot{C}(I,J)) \longrightarrow H^d_{\mathfrak{m}}(H^c_{I,J}(R)) \longrightarrow E \longrightarrow H^n_{\mathfrak{m}}(\dot{C}(I,J)) \longrightarrow 0,$$

isomorphisms $H^{i-c}_{\mathfrak{m}}(H^c_{I,J}(R)) \cong H^{i-1}_{\mathfrak{m}}(\dot{C}(I,J))$ *for* $i < n$ *and the vanishing* $H^{i-c}_{\mathfrak{m}}(H^c_{I,J}(R)) = 0$ *for* $i > n$.

Proof. Take the short exact sequence of the generalized truncation complex and apply the derived functor $R\Gamma_{\mathfrak{m}}(-)$. In the derived category this provides a short exact sequence of complexes

$$0 \longrightarrow R\Gamma_{\mathfrak{m}}(\mathrm{H}^c_{I,J}(R))[-c] \longrightarrow R\Gamma_{\mathfrak{m}}(\Gamma_{I,J}(\dot{E})) \longrightarrow R\Gamma_{\mathfrak{m}}(\dot{C}(I,J)) \longrightarrow 0.$$

We know that $\mathrm{H}^i(\Gamma_{\mathfrak{m}}(\mathrm{H}^c_{I,J}(R)))[-c] = \mathrm{H}^{i-c}(\Gamma_{\mathfrak{m}}(\mathrm{H}^c_{I,J}(R))$. Since $\Gamma_{I,J}(\dot{E})$ is a complex of injective R–modules we might use $\Gamma_{\mathfrak{m}}(\Gamma_{I,J}(\dot{E}))$ as a representative of $R\Gamma_{\mathfrak{m}}(\Gamma_{I,J}(\dot{E}))$. But there is an equality for the composite of section functors $\Gamma_{\mathfrak{m}}(\Gamma_{I,J}(-)) = \Gamma_{\mathfrak{m}}(-)$. Now $\Gamma_{\mathfrak{m}}(E(R/\mathfrak{p})) = 0$ for any prime ideal $\mathfrak{p} \neq \mathfrak{m}$ while $\Gamma_{\mathfrak{m}}(E) = E$. So there is an isomorphism of complexes $\Gamma_{\mathfrak{m}}(\dot{E}) \cong E[-n]$. With these observation, the above short exact sequence induces the exact sequence of the statement and the isomorphisms $\mathrm{H}^{i-c}_{\mathfrak{m}}(\mathrm{H}^c_{I,J}(R)) \cong \mathrm{H}^{i-1}_{\mathfrak{m}}(\dot{C}(I,J))$ for $i < n$ by view of the corresponding long exact cohomology sequence. Moreover by [2, Lemma 1.2] we obtain the vanishing of $\mathrm{H}^i_{\mathfrak{m}}(\mathrm{H}^c_{I,J}(R))$ for all $i > n$.

□

As a consequence there is the following necessary condition for a pair of ideals I, J to be a cohomologically complete intersection.

Corollary 2.3. *Let* $(R, \mathfrak{m})$ *be a local Gorenstein ring and* $n = \dim R$. *Let* $I, J \subset R$ *be two ideals with* $c = \inf\{i | H^i_{I,J}(R) \neq 0\}$ *and* $d = n - c$. *Suppose that* $H^i_{I,J}(R) = 0$ *for all* $i \neq c$. *Then* $H^d_{\mathfrak{m}}(H^c_{I,J}(R)) \cong E$ *and* $H^i_{\mathfrak{m}}(H^c_{I,J}(R)) = 0$ *for all* $i \neq d$, *where* E *denotes the injective hull of the residue field* $R/\mathfrak{m}$.

Proof. By the assumption we have the vanishing of $\mathrm{H}^i_{I,J}(R)$ for all $i \neq c$. Therefore the generalized truncation complex $\dot{C}(I,J)$ is a bounded exact complex. In order to compute the $\mathrm{H}^i_{\mathfrak{m}}(\dot{C}(I,J))$ consider the following spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\mathfrak{m}}(H^q(\dot{C}(I,J)) \Longrightarrow E_\infty^{p+q} = \mathrm{H}^{p+q}_{\mathfrak{m}}(\dot{C}(I,J)).$$

Hence, by the exactness of the truncation complex and because of the vanishing of the initial terms, we have $\mathrm{H}^i_{\mathfrak{m}}(\dot{C}(I,J)) = 0$ for all $i \in \mathbb{Z}$. Hence the claim is true by Lemma 2.2. □

Theorem 2.4. *Let $(R,\mathfrak{m})$ be an n–dimensional local Gorenstein ring. Let I, J be two ideals of R. Let $c = \inf\{i | H^i_{I,J}(R) \neq 0\}$ and $d = n - c$. Then the following hold:*

(a) *There are natural isomorphisms,*

$$\varprojlim_{\mathfrak{a}\in\tilde{W}(I,J)} Ext^c_R(H^c_{\mathfrak{a}}(R), R) \cong Ext^c_R(H^c_{I,J}(R), R) \cong Hom_R(H^c_{I,J}(R), H^c_{I,J}(R)).$$

(b) *If R is in addition complete, then*

$$\varprojlim_{\mathfrak{a}\in\tilde{W}(I,J)} Ext^c_R(H^c_{\mathfrak{a}}(R), R) \cong Hom_R(H^d_{\mathfrak{m}}(H^c_{I,J}(R), E).$$

Moreover if $H^i_{I,J}(R) = 0$ for all $i \neq c$, then the endomorphism ring $Hom_R(H^c_{I,J}(R), H^c_{I,J}(R))$ is isomorphic to R.

Proof. (a) Let $R \xrightarrow{\sim} \dot{E}$ be a minimal injective resolution of R as an R–module. Consider the exact sequence

$$0 \longrightarrow \mathrm{H}^c_{I,J}(R) \longrightarrow \Gamma_{I,J}(\dot{E})^c \longrightarrow \Gamma_{I,J}(\dot{E})^{c+1}.$$

Since $\Gamma_{I,J}(\dot{E})$ is a submodule of $\dot{E}$, it induces a natural commutative diagram with exact rows;

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}_R(\mathrm{H}^c_{I,J}(R), \mathrm{H}^c_{I,J}(R)) & \to & \mathrm{Hom}_R(\mathrm{H}^c_{I,J}(R), \Gamma_{I,J}(\dot{E}))^c & \to & \mathrm{Hom}_R(\mathrm{H}^c_{I,J}(R), \Gamma_{I,J}(\dot{E}))^{c+1} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{Ext}^c_R(\mathrm{H}^c_{I,J}(R), R) & \to & \mathrm{Hom}_R(\mathrm{H}^c_{I,J}(R), \dot{E})^c & \to & \mathrm{Hom}_R(\mathrm{H}^c_{I,J}(R), \dot{E})^{c+1}. \end{array}$$

The last two vertical homomorphisms are isomorphisms, because $\mathrm{Hom}_R(X, E_R(R/\mathfrak{p})) = 0$ for an R–module X with $\mathrm{Supp}_R X \subset W(I,J)$ and $\mathfrak{p} \notin W(I,J)$. Therefore the first vertical map is also an isomorphism. Moreover $\mathrm{Hom}_R(\mathrm{H}^c_{I,J}(R), \mathrm{H}^c_{I,J}(R)) \cong \varprojlim_{\mathfrak{a}\in\tilde{W}(I,J)} \mathrm{Hom}_R(\mathrm{H}^c_{\mathfrak{a}}(R), \mathrm{H}^c_{I,J}(R))$. Therefore we need only show that $\varprojlim_{\mathfrak{a}\in\tilde{W}(I,J)} \mathrm{Ext}^c_R(\mathrm{H}^c_{\mathfrak{a}}(R), R) \cong \varprojlim_{\mathfrak{a}\in\tilde{W}(I,J)} \mathrm{Hom}_R(\mathrm{H}^c_{\mathfrak{a}}(R), \mathrm{H}^c_{I,J}(R))$. Assume that $\mathfrak{a} \in \tilde{W}(I,J)$. Consider the following natural commutative diagram with exact rows;

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}_R(\mathrm{H}^c_{\mathfrak{a}}(R), \mathrm{H}^c_{I,J}(R)) & \to & \mathrm{Hom}_R(\mathrm{H}^c_{\mathfrak{a}}(R), \Gamma_{I,J}(\dot{E}))^c & \to & \mathrm{Hom}_R(\mathrm{H}^c_{\mathfrak{a}}(R), \Gamma_{I,J}(\dot{E}))^{c+1} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{Ext}^c_R(\mathrm{H}^c_{\mathfrak{a}}(R), R) & \to & \mathrm{Hom}_R(\mathrm{H}^c_{\mathfrak{a}}(R), \dot{E})^c & \to & \mathrm{Hom}_R(\mathrm{H}^c_{\mathfrak{a}}(R), \dot{E})^{c+1}. \end{array}$$

Since $V(\mathfrak{a}) \subseteq W(I,J)$, by a similar argument as above, the last two vertical homomorphisms are isomorphisms which implies that the first vertical map is also an isomorphism. Therefore their inverse limits are isomorphic and

$$\varprojlim_{\mathfrak{a}\in \tilde{W}(I,J)} \operatorname{Ext}^c_R(\mathrm{H}^c_{\mathfrak{a}}(R), R) \cong \operatorname{Ext}^c_R(\mathrm{H}^c_{I,J}(R), R).$$

For the proof of (b) recall that the local cohomology commutes with direct limit. So, by the definition of $\mathrm{H}^c_{I,J}(R)$ and the Local Duality Theorem , we have the following isomorphisms;

$$\begin{aligned}
\varprojlim_{\mathfrak{a}\in \tilde{W}(I,J)} \operatorname{Ext}^c_R(\mathrm{H}^c_{\mathfrak{a}}(R), R) &\cong \varprojlim_{\mathfrak{a}\in \tilde{W}(I,J)} \varprojlim \operatorname{Ext}^c_R(\operatorname{Ext}^c_R(R/\mathfrak{a}^\alpha, R), R)\\
&\cong \varprojlim_{\mathfrak{a}\in \tilde{W}(I,J)} \operatorname{Hom}(\mathrm{H}^d_{\mathfrak{m}}(\mathrm{H}^c_{\mathfrak{a}}(R)), E)\\
&\cong \operatorname{Hom}_R(\varinjlim_{\mathfrak{a}\in \tilde{W}(I,J)} \mathrm{H}^d_{\mathfrak{m}}(\mathrm{H}^c_{\mathfrak{a}}(R)), E)\\
&\cong \operatorname{Hom}_R(\mathrm{H}^d_{\mathfrak{m}}(\mathrm{H}^c_{I,J}(R)), E).
\end{aligned}$$

Note that $\operatorname{Hom}_R(E, E) \simeq R$ when $(R, \mathfrak{m})$ is a complete local ring. Now the last statement follows immediately by Corollary 2.3. □

Remark 2.5. Let $c = \inf\{i|\mathrm{H}^i_{I,J}(R) \neq 0\}$. Since $V(\mathfrak{a}) \subseteq W(I, J)$, by [6, Theorem 4.1], $\operatorname{grade}_R\mathfrak{a} = \inf\{\operatorname{depth} R_{\mathfrak{p}}|\mathfrak{p} \in V(\mathfrak{a})\} \geq \inf\{\operatorname{depth} R_{\mathfrak{p}}|\mathfrak{p} \in W(I, J)\} = c$. Now $\mathrm{H}^c_{\mathfrak{a}}(R) \neq 0$ implies that $\operatorname{grade}_R\mathfrak{a} = c$. Therefore $\mathrm{H}^c_{I,J}(R) \cong \varinjlim_{\substack{\mathfrak{a}\in \tilde{W}(I,J)\\ \operatorname{grade}_R\mathfrak{a}=c}} \mathrm{H}^c_{\mathfrak{a}}(R)$.

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a local Gorenstein ring and $\dim R = n$. Let I, I', J be proper ideals of R such that $I \subseteq I'$ and $c = \inf\{i|H^i_{I,J}(R) \neq 0\} = \inf\{i|H^i_{I',J}(R) \neq 0\}$. Then the following hold:*

(a) *There is a natural homomorphism*

$$Hom_R(H^c_{I,J}(R), H^c_{I,J}(R)) \longrightarrow Hom_R(H^c_{I',J}(R), H^c_{I',J}(R)).$$

(b) *Let R be in addition complete. Then the homomorphism in (a) is a monomorphism.*

Proof. (a) Let $R \xrightarrow{\sim} \dot{E}$ be a minimal injective resolution of R as an R–module. Then from the exact sequence $0 \longrightarrow \mathrm{H}^c_{I,J}(R) \longrightarrow \Gamma_{I,J}(\dot{E})^c \longrightarrow \Gamma_{I,J}(\dot{E})^{c+1}$, we get the following natural commutative diagram with exact rows;

$$\begin{array}{ccccccc}
0 & \to & \mathrm{H}^c_{I',J}(R) & \to & \Gamma_{I',J}(\dot{E})^c & \to & \Gamma_{I',J}(\dot{E})^{c+1}\\
 & & \downarrow & & \downarrow & & \downarrow\\
0 & \to & \mathrm{H}^c_{I,J}(R) & \to & \Gamma_{I,J}(\dot{E})^c & \to & \Gamma_{I,J}(\dot{E})^{c+1}.
\end{array}$$

Since the vertical homomorphisms are monomorphism, it follows that the natural homomorphism $\mathrm{H}^c_{I',J}(R) \longrightarrow \mathrm{H}^c_{I,J}(R)$ is a monomorphism. Therefore by applying the $\operatorname{Ext}^{\cdot}_R(-, R)$ to the short exact sequence $0 \longrightarrow \mathrm{H}^c_{I',J}(R) \longrightarrow \mathrm{H}^c_{I,J}(R) \longrightarrow X \longrightarrow 0$ we obtain the natural homomorphism

$$\operatorname{Ext}^c_R(\mathrm{H}^c_{I,J}(R), R) \longrightarrow \operatorname{Ext}^c_R(\mathrm{H}^c_{I',J}(R), R).$$

Now by Theorem 2.4 (a) this proves the statement.
In order to prove (b) we claim that $\dim X \leq d-1$. Consider the short exact sequence (†) $0 \longrightarrow \mathrm{H}^c_{I',J}(R) \longrightarrow \mathrm{H}^c_{I,J}(R) \longrightarrow X \longrightarrow 0$ in which $X \cong \mathrm{H}^c_{I,J}(R)/\mathrm{H}^c_{I',J}(R)$. Let $\mathfrak{p} \in W(I,J)$ be such that $\operatorname{ht}\mathfrak{p} = c$. Then by remark 2.5,

$$(\mathrm{H}^c_{I,J}(R))_{\mathfrak{p}} = (\varinjlim_{\substack{\mathfrak{a}\in\tilde{W}(I,J)\\ \operatorname{ht}\mathfrak{a}=c}} \mathrm{H}^c_{\mathfrak{a}}(R))_{\mathfrak{p}} = \varinjlim_{\substack{\mathfrak{a}\in\tilde{W}(I,J)\\ \operatorname{ht}\mathfrak{a}=c}} \mathrm{H}^c_{\mathfrak{a}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = \varinjlim_{\substack{\mathfrak{a}\subseteq\mathfrak{p}\\ \operatorname{ht}\mathfrak{a}=c}} \mathrm{H}^c_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}).$$

Similarly $(\mathrm{H}^c_{I',J}(R))_{\mathfrak{p}} = \varinjlim_{\substack{\mathfrak{a}\subseteq\mathfrak{p},\,\mathfrak{a}\in\tilde{W}(I',J)\\ \operatorname{ht}\mathfrak{a}=c}} \mathrm{H}^c_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})$. Therefore $X_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in W(I,J)$ with $\operatorname{ht}\mathfrak{p} = c$, which implies that $\dim X \leq d-1$. Now by applying the local cohomology with respect to the maximal ideal to the short exact sequence (†) and the fact that $\dim X \leq d-1$ we obtain that the natural homomorphism $\mathrm{H}^d_{\mathfrak{m}}(\mathrm{H}^c_{I',J}(R)) \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(\mathrm{H}^c_{I,J}(R))$ is an epimorphism. Therefore the natural homomorphism $\operatorname{Hom}_R(\mathrm{H}^d_{\mathfrak{m}}(\mathrm{H}^c_{I,J}(R)),E) \longrightarrow \operatorname{Hom}_R(\mathrm{H}^d_{\mathfrak{m}}(\mathrm{H}^c_{I',J}(R)),E)$ is a monomorphism which by Theorem 2.4 (b), this proves the statement in (b). □

The following result is another necessary condition for a pair of ideals I,J to be a cohomologically complete intersection.

Corollary 2.7. *Let the assumption be as Theorem 2.6. Assume in addition that R is complete and $H^i_{I',J}(R) = 0$ for all $i \neq c$. Then there is a natural monomorphism*

$$Hom_R(H^c_{I,J}(R), H^c_{I,J}(R)) \longrightarrow R.$$

Proof. The result follows by Theorem 2.4 and Theorem 2.6. □

The following result is a generalization of Schenzel [5].

Theorem 2.8. *Let $(R,\mathfrak{m})$ be a local Gorenstein ring with $n = dimR$. Let I,J be proper ideals of R such that $c = \inf\{i | H^i_{I,J}(R) \neq 0\}$. Then the following results hold:*

(i) *$dim H^i_{I,J}(R) \leq n-i$ for all $i \geq c$.*
(ii) *$dim H^c_{I,J}(R) = n-c$.*

Proof. (i) Let $R \overset{\sim}{\longrightarrow} \dot{E}$ be a minimal injective resolution of R as an R–module. Then it is known that $E^i = \bigoplus_{\substack{\operatorname{ht}\mathfrak{p}=i\\ \mathfrak{p}\in\operatorname{Spec}(R)}} E(R/\mathfrak{p})$, hence $\Gamma_{I,J}(E^i) = \bigoplus_{\substack{\operatorname{ht}\mathfrak{p}=i\\ \mathfrak{p}\in W(I,J)}} E(R/\mathfrak{p})$. Therefore $\operatorname{Supp}\mathrm{H}^i_{I,J}(R) \subseteq \operatorname{Supp}(\bigoplus_{\substack{\operatorname{ht}\mathfrak{p}=i\\ \mathfrak{p}\in W(I,J)}} E(R/\mathfrak{p})) = \{\mathfrak{q} \in \operatorname{Spec}(R) | \mathfrak{q} \supseteq \mathfrak{p}\}$ for all $i \geq c$. Let $\mathfrak{p} \in \operatorname{Supp}\mathrm{H}^i_{I,J}(R)$ be a prime ideal, so

$\operatorname{ht}\mathfrak{p} \geq i$ which implies that $\dim R/\mathfrak{p} \leq n-i$. Therefore $\dim \mathrm{H}^i_{I,J}(R) \leq n-i$.
(ii) Consider the exact sequence

$$0 \longrightarrow \mathrm{H}^c_{I,J}(R) \longrightarrow \bigoplus_{\substack{\operatorname{ht}\mathfrak{p}=c \\ \mathfrak{p}\in W(I,J)}} E(R/\mathfrak{p}) \longrightarrow \bigoplus_{\substack{\operatorname{ht}\mathfrak{p}=c+1 \\ \mathfrak{p}\in W(I,J)}} E(R/\mathfrak{p}).$$

This implies that $\operatorname{Ass}(\mathrm{H}^c_{I,J}(R)) \subseteq \{\mathfrak{p} \in W(I,J) | \operatorname{ht}\mathfrak{p} = c\}$. Now let $\mathfrak{p} \in W(I,J)$ be a prime ideal with $\operatorname{ht}\mathfrak{p} = c$. Then by above exact sequence, we have

$$(\mathrm{H}^c_{I,J}(R))_{\mathfrak{p}} = E_{R_{\mathfrak{p}}}(k(\mathfrak{p})) \supseteq k(\mathfrak{p}).$$

Therefore $\mathfrak{p} \in \operatorname{Ass}\mathrm{H}^c_{I,J}(R)$ which implies that $\dim \mathrm{H}^c_{I,J}(R) \geq n-c$. Now by part (i) we can conclude the statement. □

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