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Strongly Almost Ideal Convergent Sequences in a Locally Convex Space Defined by Musielak-Orlicz Function

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Abstract. In this article, we introduce a new class of ideal convergent sequence spaces using an infinite matrix, Musielak-Orlicz function and a new generalized difference matrix in locally convex spaces. We investigate some linear topological structures and algebraic properties of these spaces. We also give some relations related to these sequence spaces.

Keywords: I-convergence, Difference space, Musielak-Orlicz function.

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1. Introduction

Kostyrko et al., [25] introduced the notion of I-convergence (I denotes the ideal of the subsets of the set $\mathbb{N}$ of positive integers), which is a generalization of statistical convergence (see [14, 35]) and further studied by many others (see [6, 19, 20, 38, 39, 40]). Recently, Hazarika [21] introduced the notion generalized difference ideal convergent sequences and studied some interesting results. Quite recently, Esi [11] introduced strongly almost ideal convergent sequence spaces in 2-normed spaces defined by an Orlicz function and prove some results related to this notion.

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Before proceeding let us recall a few concepts, which we shall use throughout this paper.

Let X be a non-empty set, then a family of sets $I \subset 2^X$ (the class of all subsets of X) is called an *ideal* if and only if for each $A, B \in I$ we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$ we have $B \in I$. A non-empty family of sets $F \subset 2^X$ is a *filter* on X if and only if $\phi \notin F$ for each $A, B \in F$ we have $A \cap B \in F$ and each $A \in F$ and each $B \supset A$ we have $B \in F$. An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^X$ is called admissible if and only if $\{\{x\} : x \in X\} \subset I$. A non-trivial ideal I is maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals of 2^X can be found in Kostyrko et al., [25]. Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to a real number ℓ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ ([25]). In this case we write $I - \lim x_k = \ell$.

Throughout the article w, ℓ_∞, c, c_0, denote the classes of *all, bounded, convergent, null* sequences of complex numbers, respectively.

The notion of difference sequence space was introduced by Kizmaz [24], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$. The notion was further generalized by Et and Colak [12] introducing the sequence spaces $\ell_\infty(\Delta^p)$, $c(\Delta^p)$, $c_0(\Delta^p)$. For a non negative integer p, the generalized difference sequence spaces are defined as follows. For a given sequence space Z we have

$$Z(\Delta^p) = \{x = (x_k) \in w : (\Delta^p x_k) \in Z\},$$

where $\Delta^p x_k = \Delta^{p-1} x_k - \Delta^{p-1} x_{k+1}$, $\Delta^0 x_k = x_k$, for all $k \in \mathbb{N}$, the difference operator is equivalent to the following binomial representation:

$$\Delta^p x_k = \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} x_{k+\nu} \text{ for all } k \in \mathbb{N}.$$

Taking $p = 1$, we get the spaces $\ell_\infty(\Delta), c(\Delta), c_0(\Delta)$, introduced and studied by Kizmaz [24].

Tripathy and Esi [36] introduced and studied the new type of generalized difference sequence spaces

$$Z(\Delta_i) = \{(x_k) \in w : \Delta_i x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$ where $\Delta_i x = (\Delta_i x_k) = (x_k - x_{k+i})$ for all $k, i \in \mathbb{N}$.

Tripathy et al., [37] further generalized this notion and introduced the following sequence spaces. For $p \geq 1$ and $i \geq 1$,

$$Z(\Delta_i^p) = \{(x_k) \in w : \Delta_i^p x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$.This generalized difference has the following binomial representation,

$$\Delta_i^p x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{p}{\nu} x_{k+i\nu} \ \text{ for all } k \in \mathbb{N}.$$

Dutta [10] introduced the following difference sequence spaces

$$Z(\Delta_{(i)}^p) = \{(x_k) \in w : \Delta_{(i)}^p x_k \in Z\} \text{ for all } p, i \in \mathbb{N},$$

for $Z = \ell_\infty, \overline{c}, \overline{c}_0$ where $\overline{c}, \overline{c}_0$ are the sets of statistically convergent and statistically null sequences, respectively, and $\Delta_{(i)}^p x = (\Delta_{(i)}^p x_k) = (\Delta_{(i)}^{p-1} x_k - \Delta_{(i)}^{p-1} x_{k-i})$ and $\Delta_{(i)}^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$\Delta_{(i)}^p x_k = \sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} x_{k-i\nu}.$$

Basar and Altay [3] introduced the generalized difference matrix $B(r,s) = (b_{pk}(r,s))$ which is a generalization of $\Delta_{(1)}^1$-difference operator as follows:

$$b_{pk}(r,s) = \begin{cases} r, & \text{if } k = p; \\ s, & \text{if } k = p-1; \\ 0, & \text{if } 0 \leq k < p-1 \text{ or } k > p. \end{cases}$$

for all $k, p \in \mathbb{N}, r, s \in \mathbb{R} - \{0\}$.

Basarir and Kayikci [4] have defined the generalized difference matrix B^p of order p, which reduced the difference operator $\Delta_{(1)}^p$ in case $r = 1, s = -1$ and the binomial representation of this operator is

$$B^p x_k = \sum_{\nu=0}^{p} \binom{p}{\nu} r^{p-\nu} s^\nu x_{k-\nu},$$

where $r, s \in \mathbb{R} - \{0\}$ and $p \in \mathbb{N}$.

Recently Basarir et al., [5] introduced the following generalized difference sequence spaces

$$Z(B_{(i)}^p) = \{(x_k) \in w : B_{(i)}^p x_k \in Z\} \text{ for all } p, i \in \mathbb{N},$$

for $Z = \ell_\infty, \overline{c}, \overline{c}_0$ where $\overline{c}, \overline{c}_0$ are the sets of statistically convergent and statistically null sequences, respectively, and $B_{(i)}^p x = (B_{(i)}^p x_k) = (r B_{(i)}^{p-1} x_k +$

$sB^{p-1}_{(i)}x_{k-i})$ and $B^0_{(i)}x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$B^p_{(i)}x_k = \sum_{\nu=0}^{p} \binom{p}{\nu} r^{p-\nu} s^{\nu} x_{k-i\nu}.$$

Let X and Y be two nonempty subsets of the space w of complex sequences. Let $A = (a_{nk}), (n, k = 1, 2, 3, ...)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$ converges for each n. If $x = (x_k) \in X \Rightarrow Ax = (A_n(x)) \in Y$ we say that A defines a (matrix) transformation from X to Y and we denote it by $A : X \to Y$.

A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent if all of its Banach limits coincide. Let $\widehat{c}$ denotes the space of all almost convergent sequences.

Lorentz [29] introduced the following sequence space.

$$\widehat{c} = \left\{ x \in \ell_\infty : \lim_k t_{m,k}(x) \text{ exists uniformly in } m \right\}$$

where $t_{m,k}(x) = \frac{x_k + x_{k+1} + ... + x_{k+m}}{m+1}$.

The following space of strongly almost convergent sequences was introduced by Maddox [30],

$$[\widehat{c}] = \left\{ x \in \ell_\infty : \lim_k t_{m,k}(|x - Le|) \text{ exists uniformly in } m, \text{ for some } L \right\}$$

where $e = (1, 1, 1, ...)$.

It is clear that

$$t_{m,k}(x) = \begin{cases} \frac{1}{m+1} \sum_{i=1}^{m} x_{k+i} & \text{for } m \geq 1; \\ x_k & \text{for } m = 0 \end{cases}$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ as $x > 0$ and $M(x) \to \infty$ as $x \to \infty$ (see [26]).

An Orlicz function M can always be represented in the following integral form:

$$M(x) = \int_0^x p(t)dt$$

where p is the known kernel of M, right differentiable for $t \geq 0, p(0) = 0, p(t) > 0$ for $t > 0$ and $p(t) \to \infty$ as $t \to \infty$.

If convexity of Orlicz function is replaced by $M(x+y) \leq M(x)+M(y)$ then this function is called the modulus function and characterized by Ruckle [34]. An Orlicz function M is said to satisfy $\Delta_2 - condition$ for all values of u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

Let M be an Orlicz function which satisfies $\Delta_2 - condition$ and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.

Two Orlicz functions M_1 and M_2 are said to be *equivalent* if there exist positive constants α, β and x_0 such that

$$M_1(\alpha) \leq M_2(x) \leq M_1(\beta)$$

for all x with $0 \leq x < x_0$.

Lindenstrauss and Tzafriri [28] studied some Orlicz type sequence spaces defined as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$||x|| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$ for $1 \leq p < \infty$.

A sequence $\mathbf{M} = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function* (for details see [9, 18, 22, 23]). Also a Musielak-Orlicz function $\phi = (\phi_k)$ is called a *complementary function* of a Musielak-Orlicz function $\mathbf{M}$ if

$$\phi_k(t) = \sup\{|\ t|s - M_k(s) : s \geq 0\}, \text{for } k = 1, 2, 3, ...$$

For a given Musielak-Orlicz function $\mathbf{M}$, the Musielak-Orlicz sequence space $l_{\mathbf{M}}$ and its subspace $h_{\mathbf{M}}$ are defined as follows:

$$l_{\mathbf{M}} = \{x = (x_k) \in w : I_{\mathbf{M}}(cx) < \infty, \text{for some } c > 0\};$$

$$h_{\mathbf{M}} = \{x = (x_k) \in w : I_{\mathbf{M}}(cx) < \infty, \text{for all } c > 0\},$$

where $I_{\mathbf{M}}$ is a convex modular defined by

$$I_{\mathbf{M}} = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in l_{\mathbf{M}}.$$

We consider $l_{\mathbf{M}}$ equipped with the Luxemburg norm

$$|| \ x \ || = \inf \left\{ k > 0 : I_{\mathbf{M}} \left(\frac{x}{k} \right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$|| \ x \ ||^0 = \inf \left\{ \frac{1}{k} (1 + I_{\mathbf{M}}(kx)) : k > 0 \right\}.$$

The following well-known inequality will be used throughout the article. Let $p = (p_k)$ be any sequence of positive real numbers with $0 \leq p_k \leq \sup_k p_k = G$, $D = \max\{1, 2^{G-1}\}$ then

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k})$$

for all $k \in \mathbb{N}$ and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max\{1, |a|^G\}$ for all $a \in \mathbb{C}$.

Subsequently Orlicz function was used to define sequence spaces by Parashar and Choudhary [33] and many others (see [2, 27, 31, 41]).

Remark 1.1. It is well known if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$, for all λ with $0 < \lambda < 1$.

Definition 1.2. A sequence space E is said to be *solid (or normal)* if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.

Let $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$ and E be a sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in w : (k_n) \in E\}$.

A canonical preimage of a sequence $\{(x_{k_n})\} \in \lambda_K^E$ is a sequence $\{y_n\} \in w$ defined as

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E, i.e. y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

Definition 1.3. A sequence space E is said to be *monotone* if it contains the canonical preimages of its step spaces.

Lemma 1.1. *Every normal space is monotone.*

Throughout the paper X we denote a locally convex Hausdorff topological linear space whose topology is determined by a set Q of continuous seminorms q. Also we denote I is an non-trivial admissible ideal of $\mathbb{N}$.

2. Ideal convergence in a locally convex space

In this section we define I-convergence and almost I-convergence in a locally convex space X and investigate some basic properties.

Definition 2.1. A sequence $x = (x_k)$ in X is said to be *I-convergent* to $\ell \in X$ if for all $q \in Q$ and all $\varepsilon > 0$,

$$\{k \in \mathbb{N} : q(x_k - \ell) \geq \varepsilon\} \in I.$$

In this case we can write $I_q - \lim x_k = \ell$. We denote $I_q = \{k \in \mathbb{N} : q(x_k - \ell) \geq \varepsilon\}$.

Further, since X is Hausdorff, the limit of ideal convergent sequence is unique.

Remark 2.1. We can introduced this concept in TVS-cone Normed Spaces (for detalis on TVS-cone Normed Spaces see [32]) and in 2-inner Product Spaces (for details on 2-inner Product Spaces see [1]).

Definition 2.2. A sequence $x = (x_k)$ in X is said to be almost *I-convergent* to $\ell \in X$ if for all $q \in Q$ and all $\varepsilon > 0$,

$$\{k \in \mathbb{N} : q(t_{m,k}(x) - \ell) \geq \varepsilon\} \in I \text{ for all } m \in \mathbb{N}.$$

In this case we can write $\widehat{I_q} - \lim t_{m,k}(x) = \ell$. We denote $\widehat{I_q} = \{k \in \mathbb{N} : q(t_{m,k}(x) - \ell) \geq \varepsilon\}$ for all $m \in \mathbb{N}$.

Definition 2.3. Let $\mathbf{M}$ be a Musielak-Orlicz function. We say that a sequence $x = (x_k)$ in $\widehat{w}^I(\mathbf{M})$ if and only if there exists $\ell \in X$ such that for all $q \in Q$ and for every $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left[M_k\left(\frac{q(t_{m,k}(x) - \ell)}{\rho}\right)\right] \geq \varepsilon\right\} \in I \text{ for } \rho > 0, \text{ for all } m \in \mathbb{N}. \tag{2.1}$$

When (2.1) holds we write

$$x_k \to \ell((\widehat{w}^I(\mathbf{M}))).$$

The condition (2.1) provides a definition of strong ideal summability for a sequence in a locally convex space.

Theorem 2.1. *Let $A = (a_{nk})$ be a non-negative reguler matrix and $u = (u_k)$ be a bounded sequence of strictly positive real numbers. Let $\boldsymbol{M}$ be a Musielak-Orlicz function. Then $x_k \to \ell(\widehat{w}(\boldsymbol{M}, A, u))$ implies that $x_k \to \ell(\widehat{I}_q(A))$.*

Proof. Let $q \in Q$. Assume that $x_k \to \ell(\widehat{w}(\mathbf{M}, A, u))$, then for $\rho > 0$ we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(t_{m,k}(x) - \ell)}{\rho} \right) \right]^{u_k} = 0 \text{ for } \ell \in \mathbb{C}, \text{ for all } m \in \mathbb{N}.$$

Let $\varepsilon > 0$ be given. For all $m \in \mathbb{N}$. We define

$$K(\varepsilon) = \{k \in \mathbf{N} : q(t_{m,k}(x) - \ell) \geq \varepsilon\}$$

and we write

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(t_{m,k}(x) - \ell)}{r} \right) \right]^{t_k}$$

$$= \sum_{k\in K(\varepsilon)} a_{nk} \left[M_k \left(\frac{q(t_{m,k}(x) - \ell)}{r} \right) \right]^{u_k} + \sum_{k\notin K(\varepsilon)} a_{nk} \left[M_k \left(\frac{q(t_{m,k}(x) - \ell)}{r} \right) \right]^{u_k}$$

$$\geq \left(\sum_{k\in K(\varepsilon)} a_{nk} \right) \left[M_k \left(\frac{\varepsilon}{r} \right) \right]^{u_k}.$$

Then we have $x_k \to \ell(\widehat{I}_q(A))$. □

Theorem 2.2. *Let $A = (a_{nk})$ be a non-negative reguler matrix and $u = (u_k)$ be a bounded sequence of strictly positive real numbers. Let $\boldsymbol{M}$ be a Musielak-Orlicz function. If $x = (x_k) \in \ell_\infty$ and $x_k \to \ell(\widehat{I}_q(A))$, then $x_k \to \ell(\widehat{w}(\boldsymbol{M}, A, u))$.*

Proof. Suppose that $x = (x_k) \in \ell_\infty$ and $x_k \to \ell(\widehat{I}_q(A))$. Then there is a set $K \in F(\widehat{I}_q)$ such that

$$\lim_{k\in K} q(t_{m,k}(x) - \ell) = 0 \text{ for all } m \in \mathbb{N}.$$

Now

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q(t_{m,k}(x) - \ell)}{r} \right) \right]^{u_k}$$

$$= \sum_{k\in K(\varepsilon)} a_{nk} \left[M_k \left(\frac{q(t_{m,k}(x) - \ell)}{r} \right) \right]^{u_k} + \sum_{k\notin K(\varepsilon)} a_{nk} \left[M_k \left(\frac{q(t_{m,k}(x) - \ell)}{r} \right) \right]^{u_k}$$

$$= \sum_{k=1}^{\infty} a_{nk}\chi_K(k) \left[M_k \left(\frac{q(t_{m,k}(x) - \ell)}{r} \right) \right]^{u_k} + \sum_{k=1}^{\infty} a_{nk}\chi_{K^c}(k) \left[M_k \left(\frac{q(t_{m,k}(x) - \ell)}{r} \right) \right]^{u_k}.$$

If we consider the regularity of A, $K^c \in \widehat{I}_q$ and boundedness of (x_k) right side tends to zero. Hence $x_k \to \ell(\widehat{w}(\mathbf{M}, A, u))$. □

3. Strongly ideal convergent sequences in a locally convex space

In this section we define some new classes of strongly I-convergent sequences by using infinite matrix in a locally convex space X and investigate their linear topological structures. Also we find out some relations related to these spaces.

Recall that a mapping $g : X \to \mathbb{R}$ is called a *paranorm* on X if it satisfies the following conditions:
(i) $g(\theta) = 0$ where θ is the zero element of the space;
(ii) $g(x) = g(-x)$;
(iii) $g(x + y) \leq g(x) + g(y)$;
(iv) $\lambda^n \to \lambda(n \to \infty)$ and $g(x^n - x) \to 0(n \to \infty)$ imply $g(\lambda^n x^n - \lambda x) \to 0(n \to \infty)$ for all $x, y \in X$. The ordered a pair $(X; g)$ is called a paranormed space with respect to the paranorm g.

The main aim of this article is to introduce the following sequence spaces and examine some properties of the resulting sequence spaces.

Let I be an admissible ideal of $\mathbb{N}$, $u = (u_k)$ be a bounded sequence of strictly positive real numbers and $A = (a_{nk})$ be an infinite matrix. Let $\mathbf{M}$ be a Musielak-Orlicz function. Further $w(X)$ denotes the space of all X-valued sequences. For each $\varepsilon > 0$, for all $q \in Q$ and for $\rho > 0$ we define the following sequence spaces.

$\widehat{w}^I(A, B^p_{(i)}, \mathbf{M}, u, q) =$

$$\left\{(x_k) \in w(X) : \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x)) - \ell\right)}{\rho}\right)\right]^{u_k} \geq \varepsilon\right\} \in I \text{ for } \ell \in X, \text{ for all } m \in \mathbb{N}\right\},$$

$\widehat{w}^I_0(A, B^p_{(i)}, \mathbf{M}, u, q) =$

$$\left\{(x_k) \in w(X) : \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho}\right)\right]^{u_k} \geq \varepsilon\right\} \in I \text{ for all } m \in \mathbb{N}\right\},$$

$\widehat{w}^I_\infty(A, B^p_{(i)}, \mathbf{M}, u, q) =$

$$\left\{(x_k) \in w(X) : \exists K > 0 s.t. \left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho}\right)\right]^{u_k} \geq K\right\} \in I \text{ for all } m \in \mathbb{N}\right\},$$

$\widehat{w}_\infty(A, B^p_{(i)}, \mathbf{M}, u, q) =$

$$\left\{(x_k) \in w(X) : \sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho}\right)\right]^{u_k} < \infty \text{ for all } m \in \mathbb{N}\right\}.$$

Some classes are obtained by specializing $p, A, \mathbf{M}$ and $u = (u_k)$ for all $k \in \mathbb{N}$. Here are some examples.

(i) If $p = 1$, then above spaces are denoted by $\widehat{w}^I(A, B_{(i)}, \mathbf{M}, u, q)$, $\widehat{w}_0^I(A, B_{(i)}, \mathbf{M}, u, q)$, $\widehat{w}_\infty^I(A, B_{(i)}, \mathbf{M}, u, q)$ and $\widehat{w}_\infty(A, B_{(i)}, \mathbf{M}, u, q)$.

(ii) If $i = 1$ then above spaces are denoted by $\widehat{w}^I(A, B^p, \mathbf{M}, u, q)$, $\widehat{w}_0^I(A, B^p, \mathbf{M}, u, q)$, $\widehat{w}_\infty^I(A, B^p, \mathbf{M}, u, q)$ and $\widehat{w}_\infty(A, B^p, \mathbf{M}, u, q)$.

(iii) If $M_k(x) = x$ for all $x \in [0, \infty), k \in \mathbf{N}$ then we obtain the above spaces as $\widehat{w}^I(A, B_{(i)}^p, u, q)$, $\widehat{w}_0^I(A, B_{(i)}^p, u, q)$, $\widehat{w}_\infty^I(A, B_{(i)}^p, u, q)$ and $\widehat{w}_\infty(A, B_{(i)}^p, u, q)$.

(iv) If $u = (u_k) = (1, 1, 1...)$, then above spaces are denoted by $\widehat{w}^I(A, B_{(i)}^p, \mathbf{M}, q)$, $\widehat{w}_0^I(A, B_{(i)}^p, \mathbf{M}, q)$, $\widehat{w}_\infty^I(A, B_{(i)}^p, \mathbf{M}, q)$ and $\widehat{w}_\infty(A, B_{(i)}^p, \mathbf{M}, q)$.

(v) If we take $A = (C, 1)$),i.e., the Cesàro matrix, then the above classes of sequences are denoted by $\widehat{w}^I(B_{(i)}^p, \mathbf{M}, u, q)$, $\widehat{w}_0^I(B_{(i)}^p, \mathbf{M}, u, q)$, $\widehat{w}_\infty^I(B_{(i)}^p, \mathbf{M}, u, q)$ and $\widehat{w}_\infty(B_{(i)}^p, \mathbf{M}, u, q)$.

(vi) If we take $A = (a_{nk})$ is a de la Vallée Poussin mean, i.e.,

$$a_{nk} = \begin{cases} \frac{1}{\lambda_n}, & \text{if } k \in I_n = [n - \lambda_n + 1, n]; \\ 0, & \text{otherwise.} \end{cases}$$

where (λ_n) is a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$, then the above classes of sequences are denoted by $\widehat{w}^I(\lambda, B_{(i)}^p, \mathbf{M}, u, q)$, $\widehat{w}_0^I(\lambda, B_{(i)}^p, \mathbf{M}, u, q)$, $\widehat{w}_\infty^I(\lambda, B_{(i)}^p, \mathbf{M}, u, q)$ and $\widehat{w}_\infty(\lambda, B_{(i)}^p, \mathbf{M}, u, q)$.

(vii) By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. As a final illustration let

$$a_{nk} = \begin{cases} \frac{1}{h_r}, & \text{if } k \in I_r = (k_{r-1}, k_r]; \\ 0, & \text{otherwise.} \end{cases}$$

Then the above classes of sequences are denoted by $\widehat{w}^I(\theta, B_{(i)}^p, \mathbf{M}, u, q)$, $\widehat{w}_0^I(\theta, B_{(i)}^p, \mathbf{M}, u, q)$, $\widehat{w}_\infty^I(\theta, B_{(i)}^p, \mathbf{M}, u, q)$ and $\widehat{w}_\infty(\theta, B_{(i)}^p, \mathbf{M}, u, q)$.

Theorem 3.1. $\widehat{w}^I(A, B_{(i)}^p, \mathbf{M}, u, q)$, $\widehat{w}_0^I(A, B_{(i)}^p, \mathbf{M}, u, q)$ *and* $\widehat{w}_\infty^I(A, B_{(i)}^p, \mathbf{M}, u, q)$ *are topological linear spaces.*

Proof. We will proved the result for the space $\widehat{w}_0^I(A, B_{(i)}^p, \mathbf{M}, u, q)$ only and the others can be proved in similar way. Let $x = (x_k)$ and $y = (y_k)$ be two elements in $\widehat{w}_0^I(A, B_{(i)}^p, \mathbf{M}, u, q)$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$A_{\frac{\varepsilon}{2}} = \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B_{(i)}^p(x))\right)}{\rho_1} \right) \right]^{u_k} \geq \frac{\varepsilon}{2} \right\} \in I$$

and

$$B_{\frac{\varepsilon}{2}} = \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(y))\right)}{\rho_2} \right) \right]^{u_k} \geq \frac{\varepsilon}{2} \right\} \in I.$$

Let α, β be two scalars in $\mathbb{R}$. Since $B^p_{(i)}$ is linear and the continuity of the Musielak-Orlicz function $\mathbf{M}$, the following inequality holds:

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(\alpha x + \beta y))\right)}{|\alpha|\rho_1 + |\beta|\rho_2} \right) \right]^{u_k}$$

$$\leq D \sum_{k=1}^{\infty} a_{nk} \left[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho_1} \right) \right]^{u_k}$$

$$+ D \sum_{k=1}^{\infty} a_{nk} \left[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(y))\right)}{\rho_2} \right) \right]^{u_k}$$

$$\leq DK \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho_1} \right) \right]^{p_k}$$

$$+ DK \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(y))\right)}{\rho_2} \right) \right]^{u_k},$$

where $K = \max\{1, \left(\frac{|\alpha|\rho_1}{|\alpha|\rho_1 + |\beta|\rho_2}\right), \left(\frac{|\beta|\rho_2}{|\alpha|\rho_1 + |\beta|\rho_2}\right)\}$.

From the above relation we get

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(\alpha x + \beta y))\right)}{(|\alpha|\rho_1 + |\beta|\rho_2)} \right) \right]^{u_k} \geq \varepsilon \right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : DK \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho_1} \right) \right]^{u_k} \geq \frac{\varepsilon}{2} \right\}$$

$$\cup \left\{ n \in \mathbb{N} : DK \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(y))\right)}{\rho_2} \right) \right]^{u_k} \geq \frac{\varepsilon}{2} \right\}. \quad (3.1)$$

Since both of the sets on the right hand of (3.1) are belong to I, this completes the proof of the theorem. □

Remark 3.2. It is easy to verify that the space $\widehat{w}_\infty(A, B^p_{(i)}, \mathbf{M}, u, q)$ is a linear space.

Theorem 3.3. *Let $\mathbf{S} = (S_k)$ and $\mathbf{T} = (T_k)$ be Musielak-Orlicz functions. Then the following holds:*

$$\widehat{w}^I_0(A, B^p_{(i)}, \mathbf{S}, u, q) \cap \widehat{w}^I_0(A, B^p_{(i)}, \mathbf{T}, u, q) \subseteq \widehat{w}^I_0(A, B^p_{(i)}, \mathbf{S} + \mathbf{T}, u, q).$$

Proof. Let $x = (x_k) \in \widehat{w}^I_0(A, B^p_{(i)}, \mathbf{S}, u, q) \cap \widehat{w}^I_0(A, B^p_{(i)}, \mathbf{T}, u, q)$. Then the result follows from the inequality

$$\sum_{k=1}^{\infty} a_{nk} \left[(S_k + T_k) \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho} \right) \right]^{u_k}$$

$$\leq D \sum_{k=1}^{\infty} a_{nk} \left[S_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho} \right) \right]^{u_k} + D \sum_{k=1}^{\infty} a_{nk} \left[T_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho} \right) \right]^{p_k}.$$

□

Theorem 3.4. *Let $\mathbf{S} = (S_k)$ and $\mathbf{T} = (T_k)$ be Musielak-Orlicz functions. Then the following holds:*

$$\widehat{w}^I_0(A, B^p_{(i)}, \mathbf{T}, u, q) \subseteq \widehat{w}^I_0(A, B^p_{(i)}, \mathbf{ST}, u, q)$$

provided $h = \inf u_k > 0$.

Proof. For a given $\varepsilon > 0$, we first choose $\varepsilon_0 > 0$ such that $\sup_n \left(\sum_{k=1}^{n} a_{nk}\right) \max\{\varepsilon_0^h, \varepsilon_0^H\} < \varepsilon$. Using the continuity of $\mathbf{M}$, choose $0 < \delta < 1$ such that $0 < \delta < t$ implies that $S_k(t) < \varepsilon_0$ for all $k \in \mathbb{N}$. Let $x = (x_k) \in \widehat{w}^I_0(A, B^p_{(i)}, \mathbf{T}, u, q)$. For some $\rho > 0$ we denote

$$A_5 = \left\{ n \in \mathbb{N} : \sum_{k=1}^{n} a_{nk} \left[T_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho} \right) \right]^{u_k} \geq \delta^H \right\} \in I, m \in \mathbb{N}.$$

If $n \notin A_5$, then we have

$$\sum_{k=1}^{n} a_{nk} \left[T_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho} \right) \right]^{u_k} < \delta^H$$

$$i.e. \left[T_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho} \right) \right]^{u_k} < \delta^H \text{ for all } k, m \in \mathbb{N}$$

$$i.e. T_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho} \right) < \delta \text{ for all } k, m \in \mathbb{N}$$

$$i.e. S_k\left(T_k\left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho}\right)\right) < \varepsilon_0 \text{ for all } k, m \in \mathbb{N}.$$

Consequently, we get

$$\sum_{k=1}^{n} a_{nk}\left[S_k\left(T_k\left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho}\right)\right)\right]^{u_k} < \sup_n\left(\sum_{k=1}^{n} a_{nk}\right)\max\{\varepsilon_0^h, \varepsilon_0^H\} < \varepsilon, m \in \mathbb{N}.$$

i.e.

$$\sum_{k=1}^{n} a_{nk}\left[S_k\left(T_k\left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho}\right)\right)\right]^{u_k} < \varepsilon, m \in \mathbb{N}.$$

This shows that

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{n} a_{nk}\left[S_k\left(T_k\left(\frac{q\left(t_{m,k}(B^p_{(i)}(x))\right)}{\rho}\right)\right)\right]^{u_k} \geq \varepsilon\right\} \subset A_5 \in I.$$

This completes the proof. □

Theorem 3.5. *The inclusions* $Z(A, B^{p-1}_{(i)}, \mathbf{M}, u, q) \subset Z(A, B^p_{(i)}, \mathbf{M}, u, q)$, *are strict for* $p \geq 1$. *In general* $Z(A, B^j_{(i)}, \mathbf{M}, u, q) \subset Z(A, B^p_{(i)}, \mathbf{M}, u, q)$, *for* $j = 0, 1, 2, \ldots, p-1$ *and the inclusions are strict, where* $Z = \widehat{w}^I_0, \widehat{w}^I, \widehat{w}^I_\infty$.

Proof. We shall give the proof for $\widehat{w}^I_0(A, B^{p-1}_{(i)}, \mathbf{M}, u, q)$ only. The others can be proved by similar arguments. Let $x = (x_k)$ be any element in the space $\widehat{w}^I_0(A, B^{p-1}_{(i)}, \mathbf{M}, u, q)$. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that the set

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q\left(t_{m,k}(B^{p-1}_{(i)} x_k)\right)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I.$$

Since $\mathbf{M}$ is non-decreasing and convex, it follows that

$$\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q\left(t_{m,k}(B^p_{(i)} x_k)\right)}{2\rho}\right)\right]^{p_k}$$

$$= \sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q\left(t_{m,k}(B^{p-1}_{(i)} x_{k+1} - B^{p-1}_{(i)} x_k)\right)}{2\rho}\right)\right]^{p_k}$$

$$\leq D\sum_{k=1}^{\infty}\left[\frac{1}{2}M_k\left(\frac{q\left(t_{m,k}(B^{p-1}_{(i)} x_{k+1})\right)}{\rho}\right)\right]^{p_k}$$

$$+D\sum_{k=1}^{\infty}a_{nk}\left[\frac{1}{2}M_k\left(\frac{q\left(t_{m,k}(B_{(i)}^{p-1}x_k)\right)}{\rho}\right)\right]^{p_k}$$

$$\leq DH\sum_{k=1}^{\infty}a_{nk}\left[M_k\left(\frac{q\left(t_{m,k}(B_{(i)}^{p-1}x_{k+1})\right)}{\rho}\right)\right]^{p_k}$$

$$+DH\sum_{k=1}^{\infty}a_{nk}\left[M_k\left(\frac{q\left(t_{m,k}(B_{(i)}^{p-1}x_k)\right)}{\rho}\right)\right]^{p_k},$$

where $H=\max\{1,(\frac{1}{2})^G\}$. Thus we have

$$\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}a_{nk}\left[M_k\left(\frac{q\left(t_{m,k}(B_{(i)}^{p}x_k)\right)}{2\rho}\right)\right]^{p_k}\geq\varepsilon\right\}$$

$$\subseteq\left\{n\in\mathbb{N}:DH\sum_{k=1}^{\infty}a_{nk}\left[M_k\left(\frac{q\left(t_{m,k}(B_{(i)}^{p-1}x_{k+1})\right)}{\rho}\right)\right]^{p_k}\geq\frac{\varepsilon}{2}\right\}$$

$$\cup\left\{n\in\mathbb{N}:DH\sum_{k=1}^{\infty}a_{nk}\left[M_k\left(\frac{q\left(t_{m,k}(B_{(i)}^{p-1}x_k)\right)}{\rho}\right)\right]^{p_k}\geq\frac{\varepsilon}{2}\right\} \qquad (3.2)$$

Since both the sets in the right side of (3.2) belongs to I, we get

$$\left\{n\in\mathbb{N}:\sum_{k=1}^{\infty}a_{nk}\left[M_k\left(\frac{q\left(t_{m,k}(B_{(i)}^{p}x_k)\right)}{2\rho}\right)\right]^{p_k}\geq\varepsilon\right\}\in I.$$

□

If follow from the following example that the inclusion is strict.

Example 3.1. Let $A=(C,1)$, $M_k(x)=x$, for all $x\in[0,\infty)$, $u_k=1$ for all $k\in\mathbb{N}$ and $r=1,s=-1$. Consider a sequence $x=(x_k)=(k^p)$. Then $x=(x_k)$ belongs to $w_0^I(A,B_{(i)}^p,\mathbf{M},u,q)$ but does not belong to $w_0^I(A,B_{(i)}^{p-1},M,u,q)$, because $B_{(i)}^p x_k=0$ and $B_{(i)}^{p-1}x_k=(-1)^{p-1}(p-1)!$.

Theorem 3.6. (a) *Let* $0<\inf u_k\leq u_k\leq 1$, *then* $\widehat{w}^I(A,B_{(i)}^p,\mathbf{M},u,q)\subset\widehat{w}^I(A,B_{(i)}^p,\mathbf{M},q)$; $\widehat{w}_0^I(A,B_{(i)}^p,\mathbf{M},u,q)\subset\widehat{w}_0^I(A,B_{(i)}^p,\mathbf{M},q)$.

(b) *If* $1<u_k\leq\sup u_k<\infty$, *then* $\widehat{w}^I(A,B_{(i)}^p,\mathbf{M},q)\subset\widehat{w}^I(A,B_{(i)}^p,\mathbf{M},u,q)$; $\widehat{w}_0^I(A,B_{(i)}^p,\mathbf{M},q)\subset\widehat{w}_0^I(A,B_{(i)}^p,\mathbf{M},u,q)$.

Proof. (a) Let $x = (x_k) \in \widehat{w}^I(A, B^p_{(i)}, \mathbf{M}, u, q)$. Since $0 < \inf u_k \le u_k \le 1$, we have

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}x_k) - \ell\right)}{\rho} \right) \right] \le \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{mk}(B^p_{(i)}x_k) - \ell\right)}{\rho} \right) \right]^{p_k}$$

and therefore

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{mk}(B^p_{(i)}x_k) - \ell\right)}{\rho} \right) \right] \ge \varepsilon \right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M \left(\frac{q\left(t_{mk}(B^p_{(i)}x_k) - \ell\right)}{\rho} \right) \right]^{p_k} \ge \varepsilon \right\} \in I.$$

(b) Let $1 < u_k \le \sup u_k < \infty$ and let $x = (x_k) \in \widehat{w}^I(A, B^p_{(i)}, \mathbf{M}, q)$. Then for each $0 < \varepsilon < 1$ there exists a positive integer N such that

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{mk}(B^p_{(i)}x_k) - \ell\right)}{\rho} \right) \right] \le \varepsilon < 1$$

for all $n \ge N$. This implies that

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{mk}(B^p_{(i)}x_k) - \ell\right)}{\rho} \right) \right]^{p_k} \le \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{mk}(B^p_{(i)}x_k) - \ell\right)}{\rho} \right) \right].$$

Thus we have

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{mk}(B^p_{(i)}x_k) - \ell\right)}{\rho} \right) \right]^{p_k} \ge \varepsilon \right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{mk}(B^p_{(i)}x_k) - \ell\right)}{\rho} \right) \right] \ge \varepsilon \right\} \in I.$$

This completes the proof of the theorem. □

Corollary 3.7. *Let $A = (C, 1)$ Cesáro matrix and let M be an Orlicz function.*

(a) *If* $0 < \inf u_k \le u_k \le 1$, *then*
  (i) $\widehat{w}^I(B^p_{(i)}, \mathbf{M}, u, q) \subset \widehat{w}^I(B^p_{(i)}, \mathbf{M}, q)$;
  (ii) $\widehat{w}^I_0(B^p_{(i)}, \mathbf{M}, u, q) \subset \widehat{w}^I_0(B^p_{(i)}, \mathbf{M}, q)$.

(b) *If* $1 < u_k \le \sup u_k < \infty$, *then*
  (i) $\widehat{w}^I(B^p_{(i)}, \mathbf{M}, q) \subset \widehat{w}^I(B^p_{(i)}, \mathbf{M}, u, q)$;
  (ii) $\widehat{w}^I_0(B^p_{(i)}, \mathbf{M}, q) \subset \widehat{w}^I_0(B^p_{(i)}, \mathbf{M}, u, q)$.

Theorem 3.8. *Let* $0 < u_k \leq v_k$ *for all* $k \in \mathbb{N}$ *and* $\left(\frac{v_k}{u_k}\right)$ *is bounded, then* $\widehat{w}^I(A, B^p_{(i)}, \mathbf{M}, v, q) \subseteq \widehat{w}^I(A, B^p_{(i)}, \mathbf{M}, u, q)$.

Proof. The proof of the theorem is straightforward, so we should omitted here. □

Theorem 3.9. *If* $\lim_k u_k > 0$ *and* $x = (x_k) \to x_0(\widehat{w}^I(A, B^p_{(i)}, \mathbf{M}, u, q))$, *then* x_0 *is unique.*

Proof. Let $\lim_k u_k = u_0 > 0$. Suppose that $(x_k) \to x_0(\widehat{w}^I(A, B^p_{(i)}, \mathbf{M}, u, q))$ and $(x_k) \to y_0(\widehat{w}^I(A, B^p_{(i)}, \mathbf{M}, u, q))$.

Then there exist $\rho_1, \rho_2 > 0$ such that for all $m \in \mathbb{N}$

$$B_1 = \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x)) - x_0\right)}{\rho_1} \right) \right]^{u_k} \geq \frac{\varepsilon}{2} \right\} \in I \quad (3.3)$$

and

$$B_2 = \left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x)) - y_0\right)}{\rho_1} \right) \right]^{u_k} \geq \frac{\varepsilon}{2} \right\} \in I. \quad (3.4)$$

Let $\rho = \max\{2\rho_1, 2\rho_2\}$. Then we have

$$\sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\,(x_0 - y_0)}{\rho} \right) \right]^{u_k} \leq$$

$$D \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x)) - x_0\right)}{\rho_1} \right) \right]^{u_k} + D \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x)) - y_0\right)}{\rho_1} \right) \right]^{u_k}.$$

Thus from (3.3) and (3.4) we have for all $m \in \mathbb{N}$

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\,(x_0 - y_0)}{\rho} \right) \right]^{u_k} \geq \varepsilon \right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : D \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x)) - x_0\right)}{\rho_1} \right) \right]^{u_k} \geq \frac{\varepsilon}{2} \right\}$$

$$\cup \left\{ n \in \mathbb{N} : D \sum_{k=1}^{\infty} a_{nk} \left[M_k \left(\frac{q\left(t_{m,k}(B^p_{(i)}(x)) - y_0\right)}{\rho_1} \right) \right]^{u_k} \geq \frac{\varepsilon}{2} \right\} \subseteq B_1 \cup B_2 \in I.$$

Also we have

$$\left[M_k \left(\frac{q\,(x_0 - y_0)}{\rho} \right) \right]^{u_k} \to \left[M_k \left(\frac{q\,(x_0 - y_0)}{\rho} \right) \right]^{u_0} \text{ as } k \to \infty.$$

Therefore we have

$$\left[M_k\left(\frac{q\left(x_0-y_0\right)}{\rho}\right)\right]^{u_k} \to \left[M_k\left(\frac{q\left(x_0-y_0\right)}{\rho}\right)\right]^{u_0} = 0.$$

Hence $x_0 = y_0$. □

Theorem 3.10. *The sequence spaces* $\widehat{w}_0^I(A, B_{(i)}^p, \mathbf{M}, u, q)$ *and* $\widehat{w}_\infty^I(A, B_{(i)}^p, \mathbf{M}, u, q)$ *are normal as well as monotone.*

Proof. We give the proof for only $\widehat{w}_0^I(A, B_{(i)}^p, \mathbf{M}, u, q)$. Let $x = (x_k) \in \widehat{w}_0^I(A, B_{(i)}^p, \mathbf{M}, u, q)$ and $\alpha = (\alpha_k)$ be be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then for given $\varepsilon > 0$, for all $m \in \mathbb{N}$ we have

$$\left\{n \in \mathbb{N} : \sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q\left(t_{mk}(B_{(i)}^p(\alpha_k x_k))\right)}{\rho}\right)\right]^{u_k} \geq \varepsilon\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : E\sum_{k=1}^{\infty} a_{nk}\left[M_k\left(\frac{q\left(t_{mk}(B_{(i)}^p x_k)\right)}{\rho}\right)\right]^{u_k} \geq \varepsilon\right\} \in I,$$

where $E = \max\{1, |\alpha_k|^G\}$.

Hence $(\alpha_k x_k) \in \widehat{w}_0^I(A, B_{(i)}^p, \mathbf{M}, u, q)$. Thus the space $\widehat{w}_0^I(A, B_{(i)}^p, \mathbf{M}, u, q)$. is normal. Also from Lemma 1.1, it follows that $\widehat{w}_0^I(A, B_{(i)}^p, \mathbf{M}, u, q)$ is monotone. □

Theorem 3.11. *Let* $\mathbf{M} = (M_k)$ *be a Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $\widehat{w}_\infty^I(A, B_{(i)}^p, u, q) \subseteq \widehat{w}_\infty^I(A, B_{(i)}^p, \mathbf{M}, u, q)$

(ii) $\widehat{w}_0^I(A, B_{(i)}^p, u, q) \subseteq \widehat{w}_\infty^I(A, B_{(i)}^p, \mathbf{M}, u, q)$

(iii) $\sup_n \sum_{k=1}^n a_{nk}\left[M_k\left(\frac{t}{\rho}\right)\right]^{u_k} < \infty$ $(t, \rho > 0)$.

Proof. (i)⇒(ii) is obvious, because $\widehat{w}_0^I(A, B_{(i)}^p, u, q) \subseteq \widehat{w}_\infty^I(A, B_{(i)}^p, u, q)$.

(ii)⇒(iii). Suppose $\widehat{w}_0^I(A, B_{(i)}^p, u, q) \subseteq \widehat{w}_\infty^I(A, B_{(i)}^p, \mathbf{M}, u, q)$. We assume that (iii) is not satisfied. Then for some $t, \rho > 0$

$$\sup_n \sum_{k=1}^n a_{nk}\left[M_k\left(\frac{t}{\rho}\right)\right]^{u_k} = \infty,$$

and therefore there exists a sequence (n_j) of positive integers such that

$$\sum_{k=1}^{n_j} a_{n_j k}\left[M_k\left(\frac{j^{-1}}{\rho}\right)\right]^{u_k} > j, j = 1, 2, 3, ... \tag{3.5}$$

Define a sequence $x = (x_k)$ by

$$B^p_{(i)} x_k = \begin{cases} \frac{1}{j}, & \text{if } 1 \le k \le n_j, j = 1, 2, 3, ...; \\ 0, & \text{if } k > n_j \end{cases}$$

Then $x = (x_k) \in \widehat{w}^I_0(A, B^p_{(i)}, u, q)$ but by equation (3.5) we have $x = (x_k) \notin \widehat{w}^I_\infty(A, B^p_{(i)}, \mathbf{M}, u, q)$ which contradicts (ii). Hence (iii) must hold.

(iii)⇒(i) Suppose (iii) is satisfied and $x \in \widehat{w}^I_\infty(A, B^p_{(i)}, u, q)$. Suppose that $x \notin \widehat{w}^I_\infty(A, B^p_{(i)}, \mathbf{M}, u, q)$. Then

$$\sup_n \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{q\left(t_{mk}(B^p_{(i)} x_k)\right)}{\rho} \right) \right]^{u_k} = \infty, \text{ for all } m \in \mathbb{N}. \tag{3.6}$$

Put $t = q\left(t_{mk}(B^p_{(i)} x_k)\right)$ for all $k, m \in \mathbb{N}$. Then by the equation (3.6) we have

$$\sup_n \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{t}{\rho} \right) \right]^{u_k} = \infty$$

which contradicts (iii). Hence (i) must hold. □

Theorem 3.12. *Let $\mathbf{M} = (M_k)$ be a Musielak-Orlicz function. Let $1 \le u_k \le sup_k u_k < \infty$. Then the following statements are equivalent:*

(i) $\widehat{w}^I_0(A, B^p_{(i)}, \mathbf{M}, q) \subseteq \widehat{w}^I_0(A, B^p_{(i)}, u, q)$

(ii) $\widehat{w}^I_0(A, B^p_{(i)}, \mathbf{M}, u, q) \subseteq \widehat{w}^I_\infty(A, B^p_{(i)}, u, q)$

(iii) $\inf_n \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{t}{\rho} \right) \right]^{u_k} > 0 \ (t, \rho > 0)$.

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii). Suppose $\widehat{w}^I_0(A, B^p_{(i)}, \mathbf{M}, u, q) \subseteq \widehat{w}^I_\infty(A, B^p_{(i)}, u, q)$. We assume that (iii) does not hold. Then for some $t, \rho > 0$

$$\inf_n \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{t}{\rho} \right) \right]^{u_k} = 0.$$

We can choose an index sequence (n_j) of positive integers such that

$$\sum_{k=1}^{n_j} a_{n_j k} \left[M_k \left(\frac{i}{\rho} \right) \right]^{u_k} > \frac{1}{j}, j = 1, 2, 3, ... \tag{3.7}$$

Define a sequence $x = (x_k)$ by

$$B^p_{(i)} x_k = \begin{cases} j, & \text{if } 1 \le k \le n_j, j = 1, 2, 3, ...; \\ 0, & \text{if } k > n_j \end{cases}$$

Then by equation (3.7) we have $x = (x_k) \in \widehat{w}_0^I(A, B^p_{(i)}, \mathbf{M}, u, q)$ but $x = (x_k) \notin \widehat{w}_\infty^I(A, B^p_{(i)}, u, q)$ which contradicts (ii). Hence (iii) must hold.

(iii)⇒(i) Let (iii) hold and $x \in \widehat{w}_0^I(A, B^p_{(i)}, \mathbf{M}, u, q)$. Then for every $\varepsilon > 0$, for all $m \in \mathbb{N}$ we have

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{q\left(t_{mk}(B^p_{(i)} x_k)\right)}{\rho} \right) \right]^{u_k} \geq \varepsilon \right\} \in I. \tag{3.8}$$

Suppose that $x \notin \widehat{w}_0^I(A, B^p_{(i)}, u, q)$. Then for some integer $\varepsilon_0 > 0$ for all $m \in \mathbb{N}$ we have

$$\left\{ n \in \mathbb{N} : \sum_{k=1}^{n} a_{nk} \left[q\left(t_{mk}(B^p_{(i)} x_k)\right) \right]^{u_k} \geq \varepsilon_0 \right\} \notin I.$$

Therefore we have

$$\left[M_k \left(\frac{\varepsilon_0}{\rho} \right) \right]^{u_k} \leq \left[M_k \left(\frac{q\left(t_{mk}(B^p_{(i)} x_k)\right)}{\rho} \right) \right]^{u_k}$$

and consequently by the relation (3.8) we have

$$\inf_n \sum_{k=1}^{n} a_{nk} \left[M_k \left(\frac{\varepsilon_0}{\rho} \right) \right]^{u_k} = 0$$

which contradicts (iii). Hence $\widehat{w}_0^I(A, B^p_{(i)}, \mathbf{M}, q) \subseteq \widehat{w}_0^I(A, B^p_{(i)}, u, q)$. □

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