

OD-characterization of Almost Simple Groups Related to $D_4(4)$

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ABSTRACT. Let G be a finite group and $\pi_e(G)$ be the set of orders of all elements in G . The set $\pi_e(G)$ determines the prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ whose vertex set is $\pi(G)$. The set of primes dividing the order of G , and two vertices p and q are adjacent if and only if $pq \in \pi_e(G)$. The degree $\deg(p)$ of a vertex $p \in \pi(G)$, is the number of edges incident on p . Let $\pi(G) = \{p_1, p_2, \dots, p_k\}$ with $p_1 < p_2 < \dots < p_k$. We define $D(G) := (\deg(p_1), \deg(p_2), \dots, \deg(p_k))$, which is called the degree pattern of G . The group G is called k -fold OD-characterizable if there exist exactly k non-isomorphic groups M satisfying conditions $|G| = |M|$ and $D(G) = D(M)$. Usually a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, we classify all finite groups with the same order and degree pattern as an almost simple groups related to $D_4(4)$.

Keywords: Degree pattern, k -fold OD-characterizable, Almost simple group.

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1. INTRODUCTION

Let G be a finite group, $\pi(G)$ the set of all prime divisors of $|G|$ and $\pi_e(G)$ be the set of orders of elements in G . The prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ of G is a simple graph with vertex set $\pi(G)$ in which two vertices p and q are joined by an edge (and we write $p \sim q$) if and only if G contains an element of order pq (i.e. $pq \in \pi_e(G)$).

The degree $\deg(p)$ of a vertex $p \in \pi(G)$ is the number of edges incident on p . If $\pi(G) = \{p_1, p_2, \dots, p_k\}$ with $p_1 < p_2 < \dots < p_k$, then we define $D(G) := (\deg(p_1), \deg(p_2), \dots, \deg(p_k))$, which is called the degree pattern of G , and leads a following definition.

Definition 1.1. The finite group G is called k -fold OD-characterizable if there exist exactly k non-isomorphic groups H satisfying conditions $|G| = |H|$ and $D(G) = D(H)$. In particular, a 1-fold OD-characterizable group is simply called OD-characterizable.

The interest in characterizing finite groups by their degree patterns started in [7] by M. R. Darafsheh and et. all, in which the authors proved that the following simple groups are uniquely determined by their order and degree patterns: All sporadic simple groups, the alternating groups A_p with p and $p - 2$ primes and some simple groups of Lie type. Also in a series of articles (see [4, 6, 8, 9, 14, 17]), it was shown that many finite simple groups are OD-characterizable.

Let A and B be two groups then a split extension is denoted by $A : B$. If L is a finite simple group and $\text{Aut}(L) \cong L : A$, then if B is a cyclic subgroup of A of order n we will write $L : n$ for the split extension $L : B$. Moreover if there are more than one subgroup of orders n in A , then we will denote them by $L : n_1, L : n_2$, etc.

Definition 1.2. A group G is said to be an almost simple group related to S if and only if $S \leq G \lesssim \text{Aut}(S)$, for some non-abelian simple group S .

In many papers (see [2, 3, 10, 13, 15, 16]), it has been proved, up to now, that many finite almost simple groups are OD-characterizable or k -fold OD-characterizable for certain $k \geq 2$.

We denote the socle of G by $\text{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of G . For $p \in \pi(G)$, we denote by G_p and $\text{Syl}_p(G)$ a Sylow p -subgroup of G and the set of all Sylow p -subgroups of G respectively, all further unexplained notation are standard and can be found in [11].

In this article our main aim is to show the recognizability of the almost simple groups related to $L := D_4(4)$ by degree pattern in the prime graph and

order of the group. In fact, we will prove the following Theorem.

Main Theorem Let M be an almost simple group related to $L := D_4(4)$. If G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, then the following assertions hold:

- (a) If $M = L$, then $G \cong L$.
- (b) If $M = L : 2_1$, then $G \cong L : 2_1$ or $L : 2_3$.
- (c) If $M = L : 2_2$, then $G \cong L : 2_2$ or $\mathbb{Z}_2 \times L$.
- (d) If $M = L : 2_3$, then $G \cong L : 2_3$ or $L : 2_1$.
- (e) If $M = L : 3$, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.
- (f) If $M = L : 2^2$, then $G \cong L : 2^2, \mathbb{Z}_2 \times (L : 2_1), \mathbb{Z}_2 \times (L : 2_2), \mathbb{Z}_2 \times (L : 2_3), \mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.
- (g) If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1, L : 6, \mathbb{Z}_3 \times (L : 2_1), \mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.
- (h) If $M = L : (D_6)_2$, then $G \cong L : (D_6)_2, \mathbb{Z}_2 \times (L : 3), \mathbb{Z}_3 \times (L : 2_2), (\mathbb{Z}_3 \times L).\mathbb{Z}_2, \mathbb{Z}_6 \times L$ or $D_6 \times L$.
- (i) If $M = L : 6$, then $G \cong L : 6, L : (D_6)_1, \mathbb{Z}_3 \times (L : 2_1), \mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.
- (j) If $M = L : D_{12}$, then $G \cong L : D_{12}, \mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2), \mathbb{Z}_2 \times (L : 6), \mathbb{Z}_3 \times (L : 2^2), (\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2, \mathbb{Z}_4 \times (L : 3), (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3), (\mathbb{Z}_4 \times L).\mathbb{Z}_3, ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3, \mathbb{Z}_6 \times (L : 2_1), \mathbb{Z}_6 \times (L : 2_2), \mathbb{Z}_6 \times (L : 2_3), (\mathbb{Z}_6 \times L).\mathbb{Z}_2, D_6 \times (L : 2_1), D_6 \times (L : 2_2), D_6 \times (L : 2_3), \mathbb{Z}_{12} \times L, (\mathbb{Z}_2 \times \mathbb{Z}_6) \times L, (\mathbb{Z}_2 \times L).D_6, \mathbb{A}_4 \times L, L.\mathbb{A}_4, D_{12} \times L$ or $T \times L$.

2. PRELIMINARY RESULTS

It is well-known that $\text{Aut}(D_4(4)) \cong D_4(4) : D_{12}$ where D_{12} denotes the dihedral group of order 12. We remark that D_{12} has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to $D_6 \cong \mathbb{S}_3$ and one subgroup of order 4 isomorphic to the Klein's four group denoted by 2^2 . The field and the duality automorphisms of $D_4(4)$ are denoted by 2_1 and 2_2 respectively, and we set $2_3 = 2_1 \cdot 2_2$ (field*duality which is called the diagonal automorphism). Therefore up to conjugacy we have the following almost simple groups related to $D_4(4)$.

Lemma 2.1. *If G is an almost simple group related to $L := D_4(4)$, then G is isomorphic to one of the following groups: $L, L : 2_1, L : 2_2, L : 2_3, L : 3, L : 2^2, L : (D_6)_1, L : (D_6)_2, L : 6, L : D_{12}$.*

Lemma 2.2 ([5]). *Let G be a Frobenius group with kernel K and complement H . Then:*

- (a) K is a nilpotent group.
- (b) $|K| \equiv 1 \pmod{|H|}$.

Let $p \geq 5$ be a prime. We denote by \mathfrak{S}_p the set of all simple groups with prime divisors at most p . Clearly, if $q \leq p$, then $\mathfrak{S}_q \subseteq \mathfrak{S}_p$. We list all the simple groups in class \mathfrak{S}_{17} with their order and the order of their outer automorphisms in TABLE 1, taken from [12].

TABLE 1: Simple groups in \mathfrak{S}_p , $p \leq 17$.

| S | $ S $ | $ \text{Out}(S) $ | S | $ S $ | $ \text{Out}(S) $ |
|---------------|---------------------------------------------|-------------------|---------------|-------------------------------------------------------------------|-------------------|
| A_5 | $2^2 \cdot 3 \cdot 5$ | 2 | $G_2(3)$ | $2^6 \cdot 3^6 \cdot 7 \cdot 13$ | 2 |
| A_6 | $2^3 \cdot 3^2 \cdot 5$ | 4 | ${}^3D_4(2)$ | $2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$ | 3 |
| $S_4(3)$ | $2^6 \cdot 3^4 \cdot 5$ | 2 | $L_2(64)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | 6 |
| $L_2(7)$ | $2^3 \cdot 3 \cdot 7$ | 2 | $U_4(5)$ | $2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$ | 4 |
| $L_2(8)$ | $2^3 \cdot 3^2 \cdot 7$ | 3 | $L_3(9)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$ | 4 |
| $U_3(3)$ | $2^5 \cdot 3^3 \cdot 7$ | 2 | $S_6(3)$ | $2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$ | 2 |
| A_7 | $2^3 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $O_7(3)$ | $2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$ | 2 |
| $L_2(49)$ | $2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ | 4 | $G_2(4)$ | $2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ | 2 |
| $U_3(5)$ | $2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ | 6 | $S_4(8)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$ | 6 |
| $L_3(4)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 12 | $O_8^+(3)$ | $2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$ | 24 |
| A_8 | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 2 | $L_5(3)$ | $2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$ | 2 |
| A_9 | $2^6 \cdot 3^4 \cdot 5 \cdot 7$ | 2 | A_{13} | $2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| J_2 | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | 2 | A_{14} | $2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$ | 2 |
| A_{10} | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ | 2 | A_{15} | $2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ | 2 |
| $U_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7$ | 8 | $L_6(3)$ | $2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$ | 4 |
| $S_4(7)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$ | 2 | Suz | $2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $S_6(2)$ | $2^9 \cdot 3^4 \cdot 5 \cdot 7$ | 1 | A_{16} | $2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ | 2 |
| $O_8^+(2)$ | $2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$ | 6 | Fi_{22} | $2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ | 2 |
| $L_2(11)$ | $2^2 \cdot 3 \cdot 5 \cdot 11$ | 2 | $L_2(17)$ | $2^4 \cdot 3^2 \cdot 17$ | 2 |
| M_{11} | $2^4 \cdot 3^2 \cdot 5 \cdot 11$ | 1 | $L_2(16)$ | $2^4 \cdot 3 \cdot 5 \cdot 17$ | 4 |
| M_{12} | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | 2 | $S_4(4)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 17$ | 4 |
| $U_5(2)$ | $2^{10} \cdot 3^5 \cdot 5 \cdot 11$ | 2 | He | $2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$ | 2 |
| M_{22} | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 2 | $O_8^-(2)$ | $2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$ | 2 |
| A_{11} | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$ | 2 | $L_4(4)$ | $2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$ | 4 |
| M^cL | $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ | 2 | $S_8(2)$ | $2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$ | 1 |
| HS | $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ | 2 | $U_4(4)$ | $2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$ | 4 |
| A_{12} | $2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$ | 2 | $U_3(17)$ | $2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$ | 6 |
| $U_6(2)$ | $2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$ | 6 | $O_{10}^-(2)$ | $2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$ | 2 |
| $L_3(3)$ | $2^4 \cdot 3^3 \cdot 13$ | 2 | $L_2(13^2)$ | $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$ | 4 |
| $L_2(25)$ | $2^3 \cdot 3 \cdot 5^2 \cdot 13$ | 4 | $S_4(13)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$ | 2 |
| $U_3(4)$ | $2^6 \cdot 3 \cdot 5^2 \cdot 13$ | 4 | $L_3(16)$ | $2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$ | 24 |
| $S_4(5)$ | $2^6 \cdot 3^2 \cdot 5^4 \cdot 13$ | 2 | $S_6(4)$ | $2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$ | 2 |
| $L_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 13$ | 4 | $O_8^+(4)$ | $2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ | 12 |
| ${}^2F_4(2)'$ | $2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$ | 2 | $F_4(2)$ | $2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$ | 2 |
| $L_2(13)$ | $2^2 \cdot 3 \cdot 7 \cdot 13$ | 2 | A_{17} | $2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ | 2 |
| $L_2(27)$ | $2^2 \cdot 3^3 \cdot 7 \cdot 13$ | 6 | A_{18} | $2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ | 2 |

Definition 2.3. A completely reducible group will be called a CR -group. The center of a CR -group is a direct product of the abelian factor in the decomposition. Hence, a CR -group is centerless, that is, has trivial center, if and only if it is a direct product of non-abelian simple groups. The following Lemma determines the structure of the automorphism group of a centerless CR -group.

Lemma 2.3 ([11]). *Let R be a finite centerless CR -group and write $R = R_1 \times R_2 \times \dots \times R_k$, where R_i is a direct product of n_i isomorphic copies of a simple group H_i , and H_i and H_j are not isomorphic if $i \neq j$. Then $\text{Aut}(R) = \text{Aut}(R_1) \times \text{Aut}(R_2) \times \dots \times \text{Aut}(R_k)$ and $\text{Aut}(R_i) \cong \text{Aut}(H_i) \wr \mathbb{S}_{n_i}$, where in this wreath product $\text{Aut}(H_i)$ appears in its right regular representation and the symmetric group \mathbb{S}_{n_i} in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms $\text{Out}(R) \cong \text{Out}(R_1) \times \text{Out}(R_2) \times \dots \times \text{Out}(R_k)$ and $\text{Out}(R_i) \cong \text{Out}(H_i) \wr \mathbb{S}_{n_i}$.*

3. OD-CHARACTERIZATION OF ALMOST SIMPLE GROUPS RELATED TO $D_4(4)$

In this section, we study the problem of characterizing almost simple groups by order and degree pattern. Especially we will focus our attention on almost simple groups related to $L = D_4(4)$, namely, we will prove the Main Theorem of Sec. 1. We break the proof into a number of separate propositions.

By assumption, we depict all possibilities for the prime graph associated with G by use of the variables for some vertices in each proposition. Also, we need to know the structure of $\Gamma(M)$ to determine the possibilities for G in some proposition, therefore we depict the prime graph of all extension of L in pages 18 to 20. Note that the set of order elements in each of the following propositions is calculated using Magma.

Proposition 3.1. *If $M = L$, then $G \cong L$.*

Proof. By TABLE 1 $|L| = 2^{24}.3^5.5^4.7.13.17^2$. $\pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 21, 30, 34, 51, 63, 65, 85, 255\}$, so $D(L) = (3, 4, 4, 1, 1, 3)$. Since $|G| = |L|$ and $D(G) = D(L)$, we conclude that the prime graph of G has following form:

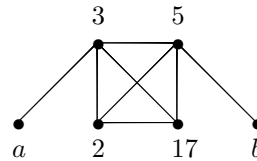


Figure 3.1

where $\{a, b\} = \{7, 13\}$.

We will show that G is isomorphic to $L = D_4(4)$. We break up the proof into a several steps.

Step1. Let K be the maximal normal solvable subgroup of G . Then K is a $\{2, 3, 5\}$ -group. In particular, G is non-solvable.

First we show that K is a $17'$ -group. Assume the contrary and let $17 \in \pi(K)$. Then 13 does not divide the order of K . Otherwise, we may suppose that T is a Hall $\{13, 17\}$ -subgroup of K . It is seen that T is a nilpotent subgroup of order $13 \cdot 17^i$ for $i = 1$ or 2 . Thus, $13 \cdot 17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus $\{17\} \subseteq \pi(K) \subseteq \pi(G) - \{13\}$. Let $K_{17} \in \text{Syl}_{17}(K)$. By Frattini argument, $G = KN_G(K_{17})$. Therefore, $N_G(K_{17})$ contains an element x of order 13. Since G has no element of order $13 \cdot 17$, $\langle x \rangle$ should act fixed point freely on K_{17} , that is implying $\langle x \rangle K_{17}$ is a Frobenius group. By Lemma 2.2(b), $|\langle x \rangle| \mid (|K_{17}| - 1)$. It follows that $13 \mid 17^i - 1$ for $i = 1$ or 2 , which is a contradiction.

Next, we show that K is a p' -group for $p \in \{a, b\}$. Let $p \mid |K|$ and $K_p \in \text{Syl}_p(K)$. Now by Frattini argument, $G = KN_G(K_p)$, so 17 must divide the order of $N_G(K_p)$. Therefore, the normalizer $N_G(K_p)$ contains an element of order 17, say x . So $\langle x \rangle K_p$ is a cyclic subgroup of G of order $17 \cdot p$, and so $p \sim 17$ in $\Gamma(G)$, which is a contradiction. Therefore K is a $\{2, 3, 5\}$ -group. In addition, since K is a proper subgroup of G , it follows that G is non-solvable.

Step 2. The quotient G/K is an almost simple group. In fact, $S \leq G/K \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L := D_4(4)$.

Let $\overline{G} = G/K$. Then $S := \text{Soc}(\overline{G}) = P_1 \times P_2 \times \dots \times P_m$, where P_i 's are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$. If we show that $m = 1$, the proof of Step 2 will be completed.

Suppose that $m \geq 2$. In this case, we claim that 13 does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $\{2, 3\} \subset \pi(P_i)$ for every i (by TABLE 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction. Now, by step 1, we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\text{Aut}(S))$. But $\text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times \dots \times \text{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i 's such that $S = S_1 \times S_2 \times \dots \times S_r$. Therefore, for some j , 13 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i . Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\text{Out}(P_i)|$ is not divisible by 13 (see TABLE 1). Now, by Lemma 2.3, we obtain $|\text{Aut}(S_j)| = |\text{Aut}(P_i)|^t \cdot t!$. Therefore, $t \geq 13$ and so 2^{26} must divide the order of G , which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq \alpha \leq 24$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we deduce that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq G/K \lesssim \text{Aut}(L)$ is completed. As $|G| = |L|$, we deduce $K = 1$, so $G \cong L$ and the proof is completed.

□

Proposition 3.2. *If $M = L : 2_1$, then $G \cong L : 2_1$ or $L : 2_3$.*

Proof. As $|L : 2_1| = 2^{25}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$, then $D(L : 2_1) = (4, 4, 4, 2, 1, 3)$. Since $|G| = |L : 2_1|$ and $D(G) = D(L : 2_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

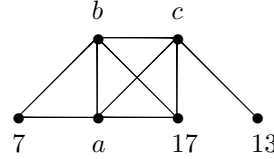


Figure 3.2

where $\{a, b, c\} = \{2, 3, 5\}$.

Step1. Let K be the maximal normal solvable subgroup of G . Then K is a $\{2, 3, 5\}$ -group. In particular, G is non-solvable.

By a similar argument to that in Proposition 3.1, we can obtain this assertion.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group.

The proof is similar to Step 2 of Proposition 3.1.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^\gamma.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \lesssim \text{Aut}(L)$. As $|G| = |L : 2_1| = 2|L|$, we deduce $|K| = 1$ or 2 .

If $|K| = 1$, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$. Obviously, $G \cong L : 2_1$ or $L : 2_3$ because $\deg(2) = 5$ in $\Gamma(L : 2_2)$ (see page 16).

If $|K| = 2$, then $K \leq Z(G)$ and so $\deg(2) = 5$, which is a contradiction. \square

Proposition 3.3. *If $M = L : 2_2$, then $G \cong L : 2_2$ or $\mathbb{Z}_2 \times L$.*

Proof. As $|L : 2_2| = 2^{25}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : 2_2) = (5, 4, 4, 2, 2, 3)$. By assumption $|G| = |L : 2_2|$ and $D(G) = D(L : 2_2)$, so the prime graph of G has following form:

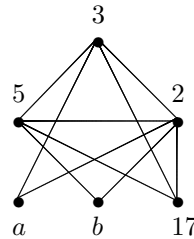


Figure 3.3

where $\{a, b\} = \{7, 13\}$.

Step1. Let K be the maximal normal solvable subgroup of G . Then K is a $\{2, 3, 5\}$ -group. In particular, G is non-solvable.

By similar arguments as in the proof of Step 1 in Proposition 3.1, we conclude that K is a $\{2, 3, 5\}$ -group and G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group.

Let $\bar{G} = \frac{G}{K}$. Then $S := \text{Soc}(\bar{G})$, $S = P_1 \times P_2 \times \dots \times P_m$, where P_i 's are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. We claim a does not divide $|S|$. Assume the contrary and let $a \mid |S|$, we conclude that a just divide the order of one of the simple groups P_i 's. Without loss of generality, we assume that $a \mid |P_1|$. Then the rest of the P_i 's must be $\{2, 3\}$ -group (because only 2 and 3 are adjacent to a in $\Gamma(G)$), this is a contradiction because P_i 's are finite non-abelian simple groups. Now, by Step 1, we observe that $a \in \pi(\bar{G}) \subseteq \pi(\text{Aut}(S))$. But $\text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times \dots \times \text{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i 's such that $S = S_1 \times S_2 \times \dots \times S_r$. Therefore, for some j , a divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i . Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\text{Out}(P_i)|$ is not divisible by a (see TABLE 1), so a does not divide the order of $\text{Aut}(P_i)$. Now, by Lemma 2.3, we obtain $|\text{Aut}(S_j)| = |\text{Aut}(P_i)|^{t!}$. Therefore, $t \geq a$ and so 3^a must divide the order of G , which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$. As $|G| = |L : 2_2| = 2|L|$, we deduce $|K| = 1$ or 2 .

If $|K| = 1$, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$ because $|G| = 2|L|$. It is obvious that $G \cong L : 2_2$, because $\deg(13) = 1$ in $\Gamma(L : 2_1)$ and $\Gamma(L : 2_3)$ (see page 17).

If $|K| = 2$, then $G/K \cong L$ and $K \leq Z(G)$. It follows that G is a central extension of K by L . If G is a non-split extension of K by L , then $|K|$ must divide the Schur multiplier of L , which is 1. But this is a contradiction, so we obtain that G split over $|K|$. Hence $G \cong \mathbb{Z}_2 \times L$. \square

Proposition 3.4. *If $M = L : 2_3$, then $G \cong L : 2_3$ or $L : 2_1$.*

Proof. As $|L : 2_3| = 2^{25} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L : 2_3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$, then $D(L : 2_3) = (4, 4, 4, 2, 1, 3)$. Since $|G| = |L : 2_3|$ and $D(G) = D(L : 2_3)$, we conclude that $\Gamma(G)$ has the following form similarly to Proposition 3.2:

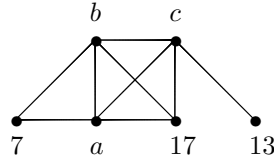


Figure 3.4

where $\{a, b, c\} = \{2, 3, 5\}$.

Step1. Let K be the maximal normal solvable subgroup of G . Then K is a $\{2, 3, 5\}$ -group. In particular, G is non-solvable.

We can prove this by the similar way to that in Proposition 3.2.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group.

By using a similar argument, as in the proof of Proposition 3.2, we can verify that $\frac{G}{K}$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \lesssim \text{Aut}(L)$. As $|G| = |L : 2_3| = 2|L|$, we deduce $|K| = 1$ or 2 .

If $|K| = 1$, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$ because $|G| = 2|L|$. Obviously, $G \cong L : 2_3$ or $L : 2_1$, because $\deg(2) = 5$ in $\Gamma(L : 2_2)$ (see page 16).

If $|K| = 2$, then $K \leq Z(G)$ and so $\deg(2) = 5$, which is a contradiction. \square

Proposition 3.5. *If $M = L : 3$, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.*

Proof. As $|L : 3| = 2^{24} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L : 3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 18, 20, 21, 24, 30, 34, 39, 45, 51, 63, 65, 85, 255\}$, then $D(L : 3) = (3, 5, 4, 1, 2, 3)$. since $|G| = |L : 3|$ and $D(G) = D(L : 3)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : 3)$):

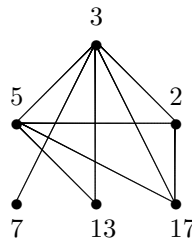


Figure 3.5

Step1. Let K be the maximal normal solvable subgroup of G . Then K is a $\{2, 3\}$ -group. In particular, G is non-solvable.

First, we show that K is a p' -group for $p = 7, 13$ and 17 . Since the proof is quite similar to the proof of Step 1 in Proposition 3.1, so we avoid here full explanation of all details.

Next we consider K is a $5'$ -group. Assume the contrary, $5 \in \pi_e(K)$. Let $K_5 \in \text{Syl}_5(K)$. By Frattini argument, $G = KN_G(K_5)$. Therefore, $N_G(K_5)$ has an element x of order 7. Since G has no element of order 5.7, $\langle x \rangle$ should act fixed point freely on K_5 , implying $\langle x \rangle K_5$ is a Frobenius group. By Lemma 2.2(b), $|\langle x \rangle| \mid (|K_5| - 1)$, which is impossible. Therefore K is a $\{2, 3\}$ -group. In addition since K is a proper subgroup of G , then G is non-solvable and the proof of this step is completed.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group.

In a similar way as in the proof of Step 2 in Proposition 3.1, we conclude that $\frac{G}{K}$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha \cdot 3^\beta \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq \alpha \leq 24$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$. As $|G| = |L : 3| = 3|L|$, we deduce $|K| = 1$ or 3.

If $|K| = 1$, then $G \cong L : 3$.

If $|K| = 3$, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by L . If G is a non-split extension of K by L , then $|K|$ must divide the Schur multiplier of L , which is 1. But this is a contradiction, so we obtain that G split over K . Hence $G \cong \mathbb{Z}_3 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction since L is simple. \square

Proposition 3.6. *If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L : 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.*

Proof. As $|L : 2^2| = 2^{26} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L : 2^2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : 2^2) = (5, 4, 4, 2, 2, 3)$. Since $|G| = |L : 2^2|$ and $D(G) = D(L : 2^2)$, so the prime graph of G has following form similarly to Proposition 3.3:

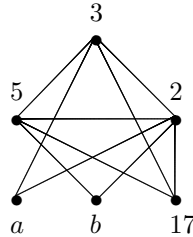


Figure 3.6

where $\{a, b\} = \{7, 13\}$.

Step1. Let K be the maximal normal solvable subgroup of G . Then K is a $\{2, 3, 5\}$ -group. In particular, G is non-solvable.

According to Step 1 in Proposition 3.3, we have K is a $\{2, 3, 5\}$ -group and G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group.

We can prove this by the similar argument in Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq \alpha \leq 26$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \lesssim \text{Aut}(L)$. As $|G| = |L : 2^2| = 4|L|$, we deduce $|K| = 1, 2$ or 4 .

If $|K| = 1$, then $G \cong L : 2^2$.

If $|K| = 2$, then $K \leq Z(G)$. In this case G is a central extension of \mathbb{Z}_2 by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If G splits over K then $G \cong \mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$ or $\mathbb{Z}_2 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1$, $L : 2_2$ and $L : 2_3$, which is impossible.

If $|K| = 4$, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$ or S_3 . Thus $|G/C_G(K)| = 1, 2, 3$ or 6 . If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by L . If G is a non-split extension of K by L , then $|K|$ must divide the Schur multiplier of L , which is 1, but this is a contradiction. Therefore G splits over K . Hence $G \cong K \times L$. So we have $G \cong \mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ because $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $|G/C_G(K)| = 2, 3$ or 6 , then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$. Which is a contradiction, since L is simple. \square

Proposition 3.7. If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1$, $L : 6$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$.

Proof. As $|L : (D_6)_1| = 2^{25} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L : (D_6)_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 39, 42, 45, 51, 60, 63, 65, 85, 255\}$, then $D(L : (D_6)_1) = (4, 5, 4, 2, 2, 3)$. Since $|G| = |L : (D_6)_1|$ and $D(G) = D(L : (D_6)_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

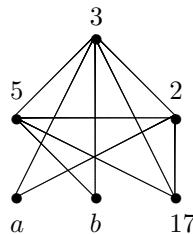


Figure 3.7

where $\{a, b\} = \{7, 13\}$.

Step1. Let K be the maximal normal solvable subgroup of G . Then K is a $\{2, 3, 5\}$ -group. In particular, G is non-solvable.

By the similar argument to that in Step 1 in Proposition 3.1, we can obtain this assertion.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group.

The proof is similar to Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 6$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$. As $|G| = |L : D_6)_1| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6 .

If $|K| = 1$, then $G \cong L : (D_6)_1, L : (D_6)_2$ or $L : 6$ because $|G| = 6|L|$. Obviously, $G \cong L : (D_6)_1$ or $L : 6$ because $\deg(2) = 5$ in $\Gamma(L : (D_6)_2)$.

If $|K| = 2$, then $K \leq Z(G)$ and so $\deg(2) = 5$, which is a contradiction (see page 18).

If $|K| = 3$, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2 . If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. If G splits over K , then $G \cong \mathbb{Z}_3 \times (L : 2_1)$ or $\mathbb{Z}_3 \times (L : 2_3)$ because in $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$ the degree of 2 is 5. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L . If $C_G(K)$ splits over K , then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L , which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6 .

If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2 . If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that $\deg(2) = 5$, a contradiction. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction because L is simple.

If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore, $G \cong D_6 \times L$, which implies that $\deg(2) = 5$, a contradiction. \square

Proposition 3.8. *If $M = L : (D_6)_2$, then $G \cong L : (D_6)_2, \mathbb{Z}_2 \times (L : 3), \mathbb{Z}_3 \times (L : 2_2), (\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2, \mathbb{Z}_6 \times L$ or $S_3 \times L$.*

Proof. As $|L : (D_6)_2| = 2^{25} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$ and $\pi_e(L : (D_6)_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : (D_6)_2) = (5, 5, 4, 2, 3, 3)$. Since $|G| = |L : (D_6)_2|$ and $D(G) = D(L : (D_6)_2)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : (D_6)_2)$):

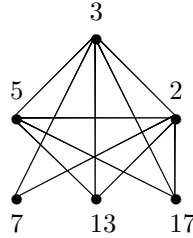


Figure 3.8

Step1. Let K be the maximal normal solvable subgroup of G . Then K is a $\{2, 3\}$ -group. In particular, G is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group.

Let $\bar{G} = \frac{G}{K}$. Then $S := \text{Soc}(\bar{G})$, $S = P_1 \times P_2 \times \dots \times P_m$, where P_i 's are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. By the same argument in Step 2 of Proposition 3.3 and considering 7 instead of a, we get a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha \cdot 3^\beta \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq \alpha \leq 25$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$. As $|G| = |L : (D_6)_2| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6 .

If $|K| = 1$, then $G \cong L : (D_6)_1, L : (D_6)_2$ or $L : 6$ because $|G| = 6|L|$. Obviously $G \cong L : (D_6)_2$ because in $\Gamma(L : (D_6)_1)$ and $\Gamma(L : 6)$, we have $\deg(13) = 2$ (see page 17).

If $|K| = 2$, then $K \leq Z(G)$ and $G/K \cong L : 3$. Hence G is a central extension of K by $L : 3$. If G splits over K , then $G \cong \mathbb{Z}_2 \times (L : 3)$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 3$, which is impossible.

If $|K| = 3$, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2 . If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. If G splits over K , then only $G \cong \mathbb{Z}_3 \times (L : 2_2)$ because $2 \approx 13$ in $\Gamma(\mathbb{Z}_3 \times (L : 2_1))$ and $\Gamma(\mathbb{Z}_3 \times (L : 2_3))$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If

$|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L . If $C_G(K)$ splits over K , then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L , which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6 . If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2 . If $|G/C_G(K)| = 1$, then $K \leq Z(G)$ and $G/K \cong L$. Therefore G is a central extension of K by L . If G is a non-split extension of K by L , then $|K|$ must divide the Schure multiplier of L , which is 1. But this is a contradiction. So we obtain that G splits over K . Hence $G \cong \mathbb{Z}_6 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction because L is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore $G \cong D_6 \times L$. \square

Proposition 3.9. *If $M = L : 6$, then $G \cong L : 6, L : (D_6)_1, \mathbb{Z}_3 \times (L : 2_1), \mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.*

Proof. As $|L : 6| = 2^{25}.3^6.5^4.7.13.17^2$ and $\pi_e(L : 6) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 36, 39, 42, 45, 48, 51, 63, 65, 85, 255\}$, then $D(L : 6) = (4, 5, 4, 2, 2, 3)$. Since $|G| = |L : 6|$ and $D(G) = D(L : 6)$, there exist several possibilities for $\Gamma(G)$ similarly to Proposition 3.7:

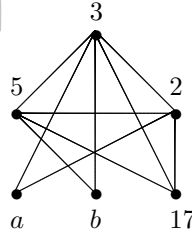


Figure 3.9

where $\{a, b\} = \{7, 13\}$.

Step1. Let K be the maximal normal solvable subgroup of G . Then K is a $\{2, 3, 5\}$ -group. In particular, G is non-solvable.

The proof is similar to that in Proposition 3.3.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group.

Again we refer to Step 2 of proposition 3.3 to get the proof.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^\gamma.7.13.17^2$, where $2 \leq \alpha \leq 25, 1 \leq \beta \leq 6$ and $0 \leq \gamma \leq 4$. Now, using collected results contained

in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$. As $|G| = |L : 6| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6 .

If $|K| = 1$, then $G \cong L : 6$, $L : (D_6)_1$ or $L : (D_6)_2$ because $|G| = 6|L|$. Obviously, $G \cong L : 6$ or $L : (D_6)_1$ because $\deg(2) = 5$ in $\Gamma(L : (D_6)_2)$ (see page 18).

If $|K| = 2$, then $K \leq Z(G)$ and so $\deg(2) = 5$, which is a contradiction.

If $|K| = 3$, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2 . If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. If G splits over K , then $G \cong \mathbb{Z}_3 \times (L : 2_1)$ or $\mathbb{Z}_3 \times (L : 2_3)$ because in $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$ the degree of 2 is 5. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L . If $C_G(K)$ splits over K , then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L , which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6 . If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2 . If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that $\deg(2) = 5$, a contradiction. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction because L is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore, $G \cong D_6 \times L$, which implies that $\deg(2) = 5$, a contradiction. \square

Proposition 3.10. *If $M = L : D_{12}$, then $G \cong L : D_{12}, \mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2), \mathbb{Z}_2 \times (L : 6), \mathbb{Z}_3 \times (L : 2^2), (\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2, \mathbb{Z}_4 \times (L : 3), (\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3), (\mathbb{Z}_4 \times L).\mathbb{Z}_3, ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3, \mathbb{Z}_6 \times (L : 2_1), \mathbb{Z}_6 \times (L : 2_2), \mathbb{Z}_6 \times (L : 2_3), (\mathbb{Z}_6 \times L).\mathbb{Z}_2, S_3 \times (L : 2_1), S_3 \times (L : 2_2), S_3 \times (L : 2_3), \mathbb{Z}_{12} \times L, (\mathbb{Z}_2 \times \mathbb{Z}_6) \times L, D_{12} \times L, (\mathbb{Z}_2 \times L).D_6, A_4 \times L, L.A_4$ or $T \times L$.*

Proof. As $|L : D_{12}| = 2^{26}.3^6.5^4.7.13.17^2$ and $\pi_e(L : (D_{12})) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 48, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : D_{12}) = (5, 5, 4, 2, 3, 3)$. Since $|G| = |L : D_{12}|$ and $D(G) = D(L : D_{12})$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : D_{12})$):

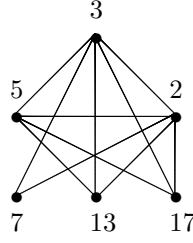


Figure 3.10

Step1. Let K be the maximal normal solvable subgroup of G . Then K is a $\{2, 3\}$ -group. In particular, G is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where S is a finite non-abelian simple group.

To get the proof, follow the way in the proof of Step 2 in proposition 3.5.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha \cdot 3^\beta \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$, where $2 \leq \alpha \leq 26$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \text{Aut}(L)$. As $|G| = |L : D_{12}| = 12|L|$, we deduce $|K| = 1, 2, 3, 4, 6$ or 12 .

If $|K| = 1$, then $G \cong L : D_{12}$.

If $|K| = 2$, then $G/K \cong L : (D_6)_1, L : (D_6)_2$ or $L : 6$ and $K \leq Z(G)$. It follows that G is a central extension of K by $L : (D_6)_1, L : (D_6)_2$ or $L : 6$. If G splits over K , then $G \cong \mathbb{Z}_2 \times (L : (D_6)_1), \mathbb{Z}_2 \times (L : (D_6)_2)$ or $\mathbb{Z}_2 \times (L : 6)$. Otherwise $G \cong \mathbb{Z}_2.(L : (D_6)_1)$ or $\mathbb{Z}_2.(L : (D_6)_2)$.

If $|K| = 3$, then $G/K \cong L : 2^2$. But $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2 . If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2^2$. If G splits over K , then $G \cong \mathbb{Z}_3 \times (L : 2^2)$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2^2$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2^2$, and we obtain $C_G(K)/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. Thus $C_G(K) \cong \mathbb{Z}_3 \times (L : 2_1), \mathbb{Z}_3 \times (L : 2_2)$ or $\mathbb{Z}_3 \times (L : 2_3)$, otherwise we get a contradiction because 3 must divide the Schure multiplier of $L : 2_1, L : 2_2$ or $L : 2_3$, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2$ or $(\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2$.

If $|K| = 4$, then $G/K \cong L : 3$ and $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case we have $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$ or S_3 . Thus $|G/C_G(K)| = 1, 2, 3$ or 6 . If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 3$. If G split over K by $L : 3$, then $G \cong \mathbb{Z}_4 \times (L : 3)$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 3$, which is impossible. If $|G/C_G(K)| \neq 1$, since $|G/C_G(K)| = 2, 3$ or 6 , it follows that $K < C_G(K)$. As L is simple, we conclude that $1 \neq C_G(K)/K$ must

be an extension of L . Hence $|G/C_G(K)| = 3$ and therefore $C_G(K)/K \cong L$. Now, since $K \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of K by L . Thus $C_G(K) \cong \mathbb{Z}_4 \times L$, or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$, otherwise $|K|$ must divide the Schure multiplier of L , which is 1 and it is impossible. Therefore, $G \cong (\mathbb{Z}_4 \times L).\mathbb{Z}_3$ or $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$.

If $|K| = 6$, then $G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ and $K \cong \mathbb{Z}_6$ or D_6 . If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is G is a central extension of \mathbb{Z}_6 by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If G splits over K , we obtain $G \cong \mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$ or $\mathbb{Z}_6 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ or $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, and we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L . Thus $C_G(K) \cong \mathbb{Z}_6 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L . Therefore $G \cong (\mathbb{Z}_6 \times L).\mathbb{Z}_2$. If $K \cong D_6$, then $G/C_G(K) \lesssim D_6$ and so $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is a contradiction. If $|G/C_G(K)| = 2$, then we have $|KC_G(K)| = 6 \cdot |G|/2 = 3|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 3$, then we have $|KC_G(K)| = 6 \cdot |G|/3 = 2|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 6$, then $G/C_G(K) \cong D_6$ and $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. It follows that $C_G(K) \cong L : 2_1, L : 2_2$ or $L : 2_3$ because L is simple. Therefore, $G \cong D_6 \times (L : 2_1)$, $D_6 \times (L : 2_2)$ or $D_6 \times (L : 2_3)$.

Before processing the last case, we recall the following facts.

There exist five non-isomorphic groups of order 12. Two of them are abelian and three are non-abelian. The non-abelian groups are: alternating group A_4 , dihedral group D_{12} and the dicyclic group T with generators a and b , subject to the relations $a^6 = 1$, $a^3 = b^2$ and $b^{-1}ab = a^{-1}$.

If $|K| = 12$, then $G/K \cong L$ and $K \cong \mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, D_{12} , A_4 or T . But $C_G(K)K/K \trianglelefteq G/K \cong L$. If $C_G(K)K/K = 1$, then $C_G(K) \leq K$ and hence $|L| = |G/K| |G/C_G(K)| |Aut(K)|$. Thus $|L| |Aut(K)|$, a contradiction. Therefore, $C_G(K)K/K \neq 1$ and since L is simple group, we conclude that $G = C_G(K)K$ and hence, $G/C_G(K) \cong K/Z(K)$. Now, we should consider the following cases:

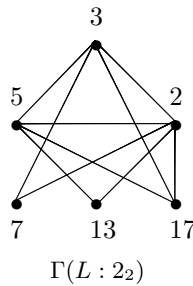
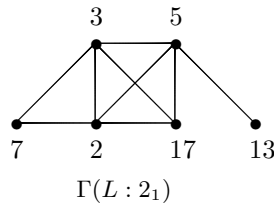
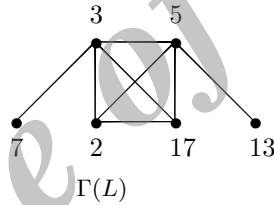
If $K \cong \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$, then $G/C_G(K) = 1$. Therefore $K \leq Z(G)$, that is G is a central extension of \mathbb{Z}_{12} or $\mathbb{Z}_2 \times \mathbb{Z}_6$ by L . If G splits over K , we obtain $G \cong \mathbb{Z}_{12} \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L , which is 1 and it is impossible.

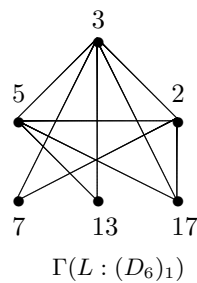
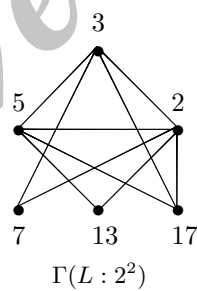
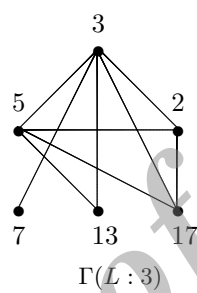
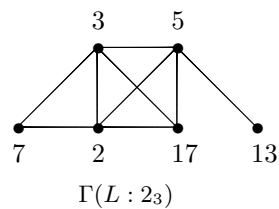
If $K \cong D_{12}$, then $G = K.L$ and $G/C_G(K) \cong D_6$. Since $C_G(K)/Z(K) \cong G/K \cong L$ and $Z(K) \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of $Z(K) \cong \mathbb{Z}_2$ by L . If $C_G(K)$ is a non-split extension, then 2 must divide the Schure multiplier of L , which is 1 and it is impossible. Thus $C_G(K) \cong \mathbb{Z}_2 \times L$ and hence, G is a split extension of K by L . Now, since $\text{Hom}(L, \text{Aut}(D_{12}))$ is trivial, we have $G \cong D_{12} \times L$.

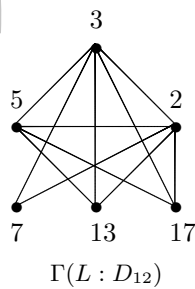
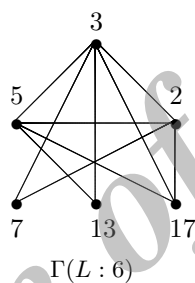
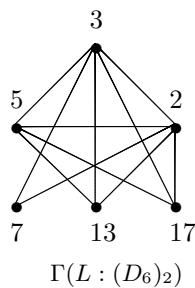
If $K \cong \mathbb{A}_4$, then $G/C_G(K) \cong \mathbb{A}_4$. As $G = C_G(K)K$, It follows that $C_G(K) \cong L$. Therefore $G \cong L \times \mathbb{A}_4$ or $L.\mathbb{A}_4$.

If $K \cong T$, then By the similar way in case $K \cong D_{12}$, we can conclude that G is a split extension of K by L . Also, since $\text{Hom}(L, \text{Aut}(T))$ is trivial, we have $G \cong T \times L$. \square

According to what we said before the proof, here we depict $\Gamma(M)$ by $|M|$ and $\pi_e(M)$, where M is an almost simple group related to $L = D_4(4)$.







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REFERENCES

1. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of finite groups*, Clarendon Press (Oxford), London - New York, 1985.
2. M. R. Darafsheh, G. R. Rezaeezadeh, M. bibak, M. Sajjadi, OD-Characterization of almost simple groups related to ${}^2E_6(2)$, *Advances in Algebra*, **6**, (2013), 45-54.
3. M. R. Darafsheh, G. R. Rezaeezadeh, M. Sajjadi, M. Bibak, OD-Characterization of almost simple groups related to $U_3(17)$, *Quasigroups and related systems*, **21**, (2013), 49-58.
4. M. Foroudi ghasemabadi, N. Ahanjideh, Characterization of the simple groups $D_n(3)$ by prime graph and spectrum, *Iranian Journal of Mathematical Sciences and Informatics*, **7**(1), (2012), 91-106.
5. D. Gorenstein, *Finite groups*, New York: Harper and Row, 1980.
6. A. A. Hoseini, A. R. Moghaddamfar, Recognizing alternating groups A_{p+3} for certain primes p by their orders and degree patterns, *Front. Math. China*, **5**(3), (2010), 541-553.
7. A. R. Moghaddamfar, A. R. Zokayi, M. R. Darafsheh, A characterization of finite simple groups by the degree of vertices of their prime graphs, *Algebra Colloquium*, **12**(3), (2005), 431-442.
8. A. R. Moghaddamfar, A. R. Zokayi, Recognizing finite groups through order and degree patterns, *Algebra Colloquium*, **15**(3), (2008), 449-456.
9. G. R. Rezaeezadeh, M. Bibak, M. Sajjadi, Characterization of Projective Special linear Groups in Dimension three by their orders and degree patterns, *Bulletin of the Iranian Mathematical Society*, in press.
10. G. R. Rezaeezadeh, M. R. Darafsheh, M. Sajjadi, M. Bibak, OD-characterization of almost simple groups related to $L_3(25)$, *Bulletin of the Iranian Mathematical Society*, in press.
11. J. S. Robinson Derek, *A course in the theory of groups*, 2nd ed. New York-Heidelberg-Berlin: Springer-Verlag, 2003.
12. A. Zavarnitsin, Finite simple groups with narrow prime spectrum, *Siberian Electronic Mathematical Reports*, **6**, (2009), 1-12.
13. L. C. Zhang, W. J. Shi, OD-characterization of almost simple groups related to $L_2(49)$, *Archivum Mathematicum (Brno) Tomus*, **44**, (2008), 191-199.
14. L. C. Zhang, W. J. Shi, OD-Characterization of simple K_4 -groups, *Algebra Colloquium*, **16**(2), (2009), 275-282.
15. L. C. Zhang, W. J. Shi, OD-characterization of almost simple groups related to $U_3(5)$, *Acta Mathematica Scientia*, **26**(1), (2010), 161-168.
16. L. C. Zhang, W. J. Shi, OD-characterization of almost simple groups related to $U_6(2)$, *Acta Mathematica Scientia*, **31**(2), (2011), 441-450.
17. L. C. Zhang, W. J. Shi, OD-Characterization of the projective special linear groups $L_2(q)$, *Acta Mathematica Scientia*, **19**, (2012), 509-524.