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OD-characterization of Almost Simple Groups Related to $D_4(4)$

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Abstract. Let G be a finite group and $\pi_e(G)$ be the set of orders of all elements in G. The set $\pi_e(G)$ determines the prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ whose vertex set is $\pi(G)$. The set of primes dividing the order of G, and two vertices p and q are adjacent if and only if $pq \in \pi_e(G)$. The degree $deg(p)$ of a vertex $p \in \pi(G)$, is the number of edges incident on p. Let $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$. We define $D(G) := (deg(p_1), deg(p_2), ..., deg(p_k))$, which is called the degree pattern of G. The group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups M satisfying conditions $|G| = |M|$ and $D(G) = D(M)$. Usually a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, we classify all finite groups with the same order and degree pattern as an almost simple groups related to $D_4(4)$.

Keywords: Degree pattern, k-fold OD-characterizable, Almost simple group.

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1. Introduction

Let G be a finite group, $\pi(G)$ the set of all prime divisors of $|G|$ and $\pi_e(G)$ be the set of orders of elements in G . The prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ of G is a simple graph with vertex set $\pi(G)$ in which two vertices p and q are joined by an edge (and we write $p \sim q$) if and only if G contains an element of order pq (i.e. $pq \in \pi_e(G)$).

The degree $\deg(p)$ of a vertex $p \in \pi(G)$ is the number of edges incident on p. If $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$, then we define $\mathrm{D}(G)$:=($\deg(p_1)$,$\deg(p_2)$,...,$\deg(p_k)$), which is called the degree pattern of G, and leads a following definition.

Definition 1.1. The finite group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups H satisfying conditions $|G| = |H|$ and $D(G) = D(H)$. In particular, a 1-fold OD-characterizable group is simply called OD-characterizable.

The interest in characterizing finite groups by their degree patterns started in [7] by M. R. Darafsheh and et. all, in which the authors proved that the following simple groups are uniquely determined by their order and degree patterns: All sporadic simple groups, the alternating groups A_p with p and $p-2$ primes and some simple groups of Lie type. Also in a series of articles (see [4, 6, 8, 9, 14, 17]), it was shown that many finite simple groups are OD-characterizable.

Let A and B be two groups then a split extension is denoted by $A : B$. If L is a finite simple group and $\mathrm{A}ut(L) \cong L : A$, then if B is a cyclic subgroup of A of order n we will write $L : n$ for the split extension $L : B$. Moreover if there are more than one subgroup of orders n in A, then we will denote them by $L : n_1$, $L : n_2$, etc.

Definition 1.2. A group G is said to be an almost simple group related to S if and only if $S \leq G \lesssim \mathrm{A}ut(S)$, for some non-abelian simple group S.

In many papers (see [2, 3, 10, 13, 15, 16]), it has been proved, up to now, that many finite almost simple groups are OD-characterizable or k-fold OD-characterizable for certain $k \geq 2$.

We denote the socle of G by $\mathrm{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of G. For $p \in \pi(G)$, we denote by G_p and $\mathrm{Syl}_p(G)$ a Sylow p-subgroup of G and the set of all Sylow p-subgroups of G respectively, all further unexplained notation are standard and can be found in [11].

In this article our main aim is to show the recognizability of the almost simple groups related to $L := D_4(4)$ by degree pattern in the prime graph and order of the group. In fact, we will prove the following Theorem.

Main Theorem Let M be an almost simple group related to $L := D_4(4)$. If G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, then the following assertions hold:

(a) If $M = L$, then $G \cong L$.

(b) If $M = L : 2_1$, then $G \cong L : 2_1$ or $L : 2_3$.

(c) If $M = L : 2_2$, then $G \cong L : 2_2$ or $\mathbb{Z}_2 \times L$.

(d) If $M = L : 2_3$, then $G \cong L : 2_3$ or $L : 2_1$.

(e) If $M = L : 3$, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.

(f) If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L : 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

(g) If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1$, $L : 6$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

(h) If $M = L : (D_6)_2$, then $G \cong L : (D_6)_2$, $\mathbb{Z}_2 \times (L : 3)$, $\mathbb{Z}_3 \times (L : 2_2)$, $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$, $\mathbb{Z}_6 \times L$ or $D_6 \times L$.

(i) If $M = L : 6$, then $G \cong L : 6$, $L : (D_6)_1$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

(j) If $M = L : D_{12}$, then $G \cong L : D_{12}$, $\mathbb{Z}_2 \times (L : (D_6)_1)$, $\mathbb{Z}_2 \times (L : (D_6)_2)$, $\mathbb{Z}_2 \times (L : 6)$, $\mathbb{Z}_3 \times (L : 2^2)$, $(\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2$, $(\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2$, $(\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2$, $\mathbb{Z}_4 \times (L : 3)$, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$, $(\mathbb{Z}_4 \times L).\mathbb{Z}_3$, $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$, $\mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$, $\mathbb{Z}_6 \times (L : 2_3)$, $(\mathbb{Z}_6 \times L).\mathbb{Z}_2$, $D_6 \times (L : 2_1)$, $D_6 \times (L : 2_2)$, $D_6 \times (L : 2_3)$, $\mathbb{Z}_{12} \times L$, $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, $(\mathbb{Z}_2 \times L).D_6$, $\mathbb{A}_4 \times L$, $L.\mathbb{A}_4$, $D_{12} \times L$ or $T \times L$.

2. Preliminary Results

It is well-known that $\mathrm{Aut}(D_4(4)) \cong D_4(4) : D_{12}$ where D_{12} denotes the dihedral group of order 12. We remark that D_{12} has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to $D_6 \cong \mathbb{S}_3$ and one subgroup of order 4 isomorphic to the Klein's four group denoted by 2^2. The field and the duality automorphisms of $D_4(4)$ are denoted by 2_1 and 2_2 respectively, and we set $2_3 = 2_1.2_2$ (field$*$duality which is called the diagonal automorphism). Therefore up to conjugacy we have the following almost simple groups related to $D_4(4)$.

Lemma 2.1. *If G is an almost simple group related to L:=$D_4(4)$, then G is isomorphic to one of the following groups: $L, L : 2_1, L : 2_2, L : 2_3, L : 3, L : 2^2, L : (D_6)_1, L : (D_6)_2, L : 6, L : D_{12}$.*

Lemma 2.2 ([5]). *Let G be a Frobenius group with kernel K and complement H. Then:*

(a) K is a nilpotent group.

(b) $|K| \equiv 1(\mathrm{mod}|H|)$.

Let $p \geq 5$ be a prime. We denote by $\mathfrak{S}_p$ the set of all simple groups with prime divisors at most p. Clearly, if $q \leq p$, then $\mathfrak{S}_q \subseteq \mathfrak{S}_p$. We list all the simple groups in class $\mathfrak{S}_{17}$ with their order and the order of their outer automorphisms in TABLE 1, taken from [12].

TABLE 1: Simple groups in $\mathfrak{S}_p$, $p \leq 17$.

S	$\lvert S \rvert$	$\lvert \mathrm{Out}(S) \rvert$	S	$\lvert S \rvert$	$\lvert \mathrm{Out}(S) \rvert$
A_5	$2^2 \cdot 3 \cdot 5$	2	$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4	${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	4
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$L_3(9)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	4
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$S_4(8)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	24
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_5(3)$	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$	2
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	A_{13}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	A_{14}	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	2
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8	$L_6(3)$	$2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	4
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1	A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6	Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2	$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2	He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2	$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2	$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	4
M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2	$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2	$U_4(4)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	4
A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2	$U_3(17)$	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$	6
$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	6	$O_{10}^-(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	2
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2	$L_2(13^2)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$	4
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4	$S_4(13)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$	2
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	$L_3(16)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	24
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2	$S_6(4)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$	2
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4	$O_8^+(4)$	$2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$	12
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2	$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	A_{17}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6	A_{18}	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2

Definition 2.3. A completely reducible group will be called a CR-group. The center of a CR-group is a direct product of the abelian factor in the decomposition. Hence, a CR-group is centerless, that is, has trivial center, if and only if it is a direct product of non-abelian simple groups. The following Lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.3 ([11])**.** *Let R be a finite centerless CR-group and write $R = R_1 \times R_2 \times ... \times R_k$, where R_i is a direct product of n_i isomorphic copies of a simple group H_i, and H_i and H_j are not isomorphic if $i \neq j$. Then $\mathrm{Aut}(R) = \mathrm{Aut}(R_1) \times \mathrm{Aut}(R_2) \times ... \times \mathrm{Aut}(R_k)$ and $\mathrm{Aut}(R_i) \cong \mathrm{Aut}(H_i) \wr \mathbb{S}_{n_i}$, where in this wreath product $\mathrm{Aut}(H_i)$ appears in its right regular representation and the symmetric group $\mathbb{S}_{n_i}$ in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms $\mathrm{Out}(R) \cong \mathrm{Out}(R_1) \times \mathrm{Out}(R_2) \times ... \times \mathrm{Out}(R_k)$ and $\mathrm{Out}(R_i) \cong \mathrm{Out}(H_i) \wr \mathbb{S}_{n_i}$.*

3. OD-Characterization of Almost Simple Groups Related to $D_4(4)$

In this section, we study the problem of characterizing almost simple groups by order and degree pattern. Especially we will focus our attention on almost simple groups related to $L = D_4(4)$, namely, we will prove the Main Theorem of Sec. 1. We break the proof into a number of separate propositions.
By assumption, we depict all possibilities for the prime graph associated with G by use of the variables for some vertices in each proposition. Also, we need to know the structure of $\Gamma(M)$ to determine the possibilities for G in some proposition, therefore we depict the prime graph of all extension of L in pages 18 to 20. Note that the set of order elements in each of the following propositions is calculated using Magma.

Proposition 3.1. *If $M = L$, then $G \cong L$.*

Proof. By TABLE 1 $|L| = 2^{24}.3^5.5^4.7.13.17^2$. $\pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 21, 30, 34, 51, 63, 65, 85, 255\}$, so $D(L) = (3, 4, 4, 1, 1, 3)$. Since $|G| = |L|$ and $D(G) = D(L)$, we conclude that the prime graph of G has following form:

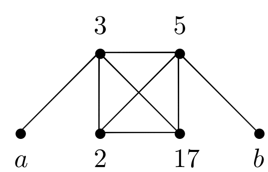


Figure 3.1

where $\{a, b\} = \{7, 13\}$.

We will show that G is isomorphic to $L = D_4(4)$. We break up the proof into a several steps.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$-group. In particular, G is non-solvable.

First we show that K is a $17^{'}$-group. Assume the contrary and let $17 \in \pi(K)$. Then 13 dose not divide the order of K. Otherwise, we may suppose that T is a Hall $\{13,17\}$-subgroup of K. It is seen that T is a nilpotent subgroup of order 13.17^i for $i = 1$ or 2. Thus, $13.17 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Thus $\{17\} \subseteq \pi(K) \subseteq \pi(G) - \{13\}$. Let $K_{17} \in \mathrm{Syl}_{17}(K)$. By Frattini argument, $G = KN_G(K_{17})$. Therefore, $N_G(K_{17})$ contains an element x of order 13. Since G has no element of order 13.17, $\langle\, x\rangle$ should act fixed point freely on K_{17}, that is implying $\langle x\rangle K_{17}$ is a Frobenius group. By Lemma 2.2(b), $|\langle x\rangle|| (|K_{17}| - 1)$. It follows that $13 | 17^i - 1$ for $i = 1$ or 2, which is a contradiction.

Next, we show that K is a $p^{'}$-group for $p \in \{a, b\}$. Let $p||K|$ and $K_p \in \mathrm{Syl}_p(K)$. Now by Frattini argument, $G = KN_G(K_p)$, so 17 must divide the order of $N_G(K_p)$. Therefore, the normalizer $N_G(K_p)$ contains an element of order 17, say x. So $\langle x\rangle K_p$ is a cyclic subgroup of G of order $17.p$, and so $p \sim 17$ in $\Gamma(G)$, which is a contradiction. Therefore K is a $\{2,3,5\}$-group. In addition, since K is a proper subgroup of G, it follows that G is non-solvable.

Step 2. The quotient G/K is an almost simple group. In fact, $S \leq G/K \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group isomorphic to $L := D_4(4)$.

Let $\overline{G} = G/K$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where $P_i^{,}s$ are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. If we show that $m = 1$, the proof of Step 2 will be completed.

Suppose that $m \geq 2$. In this case, we claim that 13 does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $\{2,3\} \subset \pi(P_i)$ for every i (by TABLE 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction. Now, by step 1, we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, 13 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by 13 (see TABLE 1). Now, by Lemma 2.3, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^{t!}.t!$. Therefore, $t \geq 13$ and so 2^{26} must divide the order of G, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 24$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we deduce that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq G/K \lesssim \mathrm{Aut}(L)$ is completed. As $|G| = |L|$, we deduce $K = 1$, so $G \cong L$ and the proof is completed.

□

Proposition 3.2. *If $M = L : 2_1$, then $G \cong L : 2_1$ or $L : 2_3$.*

Proof. As $|L : 2_1| = 2^{25}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$, then $D(L : 2_1) = (4, 4, 4, 2, 1, 3)$. Since $|G| = |L : 2_1|$ and $D(G) = D(L : 2_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

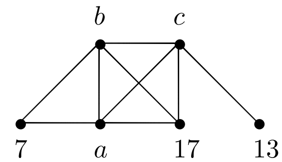


Figure 3.2

where $\{a, b, c\} = \{2, 3, 5\}$.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.
By a similar argument to that in Proposition 3.1, we can obtain this assertion.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.
The proof is similar to Step 2 of Proposition 3.1.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 2_1| = 2|L|$, we deduce $|K| = 1$ or 2.

If $|K| = 1$, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$. Obviously, $G \cong L : 2_1$ or $L : 2_3$ because $deg(2) = 5$ in $\Gamma(L : 2_2)$ (see page 16).

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction. $\square$

Proposition 3.3. *If $M = L : 2_2$, then $G \cong L : 2_2$ or $\mathbb{Z}_2 \times L$.*

Proof. As $|L : 2_2| = 2^{25}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : 2_2) = (5, 4, 4, 2, 2, 3)$. By assumption $|G| = |L : 2_2|$ and $D(G) = D(L : 2_2)$, so the prime graph of G has following form:

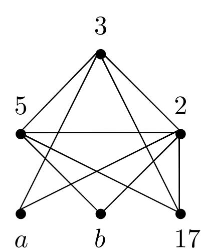


Figure 3.3

where $\{a, b\} = \{7, 13\}$.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$-group. In particular, G is non-solvable.
By similar arguments as in the proof of Step 1 in Proposition 3.1, we conclude that K is a $\{2,3,5\}$-group and G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G})$, $S = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. We claim a does not divide $|S|$. Assume the contrary and let $a \mid |S|$, we conclude that a just divide the order of one of the simple groups P_i's. Without loss of generality, we assume that $a||P_1|$. Then the rest of the P_i's must be $\{2,3\}$-group (because only 2 and 3 are adjacent to a in $\Gamma(G)$), this is a contradiction because P_i's are finite non-abelian simple groups. Now, by Step 1, we observe that $a \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. But $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, a divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_{17}$, it follows that $|\mathrm{Out}(P_i)|$ is not divisible by a (see TABLE 1), so a does not divide the order of $\mathrm{Aut}(P_i)$. Now, by Lemma 2.3, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^{t!}.t!$. Therefore, $t \geq a$ and so 3^a must divide the order of G, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 2_2| = 2|L|$, we deduce $|K| = 1$ or 2.

If $|K| = 1$, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$ because $|G| = 2|L|$. It is obvious that $G \cong L : 2_2$, because $deg(13) = 1$ in $\Gamma(L : 2_1)$ and $\Gamma(L : 2_3)$ (see page 17).

If $|K| = 2$, then $G/K \cong L$ and $K \leq Z(G)$. It follows that G is a central extension of K by L. If G is a non-split extension of K by L, then $|K|$ must divide the Schur multiplier of L, which is 1. But this is a contradiction, so we obtain that G split over $|K|$. Hence $G \cong \mathbb{Z}_2 \times L$. □

Proposition 3.4. *If* $M = L : 2_3$*, then* $G \cong L : 2_3$ *or* $L : 2_1$.

Proof. As $|L : 2_3| = 2^{25}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2_3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$, then $D(L : 2_3) = (4, 4, 4, 2, 1, 3)$. Since $|G| = |L : 2_3|$ and $D(G) = D(L : 2_3)$, we conclude that $\Gamma(G)$ has the following form similarly to Proposition 3.2:

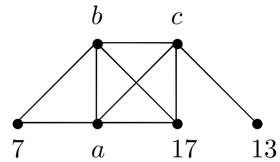


Figure 3.4

where $\{a, b, c\} = \{2, 3, 5\}$.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.
We can prove this by the similar way to that in Proposition 3.2.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.
By using a similar argument, as in the proof of Proposition 3.2, we can verify that $\frac{G}{K}$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 2_3| = 2|L|$, we deduce $|K| = 1$ or 2.

If $|K| = 1$, then $G \cong L : 2_1, L : 2_2$ or $L : 2_3$ because $|G| = 2|L|$. Obviously, $G \cong L : 2_3$ or $L : 2_1$, because $deg(2) = 5$ in $\Gamma(L : 2_2)$ (see page 16).

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction. $\square$

Proposition 3.5. *If $M = L : 3$, then $G \cong L : 3$ or $\mathbb{Z}_3 \times L$.*

Proof. As $|L : 3| = 2^{24}.3^6.5^4.7.13.17^2$ and $\pi_e(L : 3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 18, 20, 21, 24, 30, 34, 39, 45, 51, 63, 65, 85, 255\}$, then $D(L : 3) = (3, 5, 4, 1, 2, 3)$. since $|G| = |L : 3|$ and $D(G) = D(L : 3)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : 3)$):

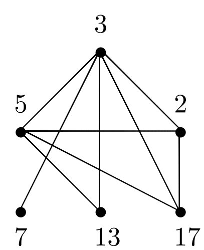


Figure 3.5

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3\}$-group. In particular, G is non-solvable.

First, we show that K is a p'-group for $p = 7, 13$ and 17. Since the proof is quite similar to the proof of Step 1 in Proposition 3.1, so we avoid here full explanation of all details.

Next we consider K is a $5'$-group. Assume the contrary, $5 \in \pi_e(K)$. Let $K_5 \in \mathrm{Syl}_5(K)$. By Frattini argument, $G = KN_G(K_5)$. Therefore, $N_G(K_5)$ has an element x of order 7. Since G has no element of order 5.7, $\langle x \rangle$ should act fixed point freely on K_5, implying $\langle x \rangle K_5$ is a Frobenius group. By Lemma 2.2(b), $|\langle x \rangle| | (|K_5| - 1)$, which is impossible. Therefore K is a $\{2, 3\}$-group.
In addition since K is a proper subgroup of G, then G is non-solvable and the proof of this step is completed.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.
In a similar way as in the proof of Step 2 in Proposition 3.1, we conclude that $\frac{G}{K}$ is an almost simple group.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^4.7.13.17^2$, where $2 \leq \alpha \leq 24$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 3| = 3|L|$, we deduce $|K| = 1$ or 3.

If $|K| = 1$, then $G \cong L : 3$.

If $|K| = 3$, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by L. If G is a non-split extension of K by L, then $|K|$ must divide the Schur multiplier of L, which is 1. But this is a contradiction, so we obtain that G split over K. Hence $G \cong \mathbb{Z}_3 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction since L is simple. $\square$

Proposition 3.6. *If $M = L : 2^2$, then $G \cong L : 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L : 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.*

Proof. As $|L : 2^2| = 2^{26}.3^5.5^4.7.13.17^2$ and $\pi_e(L : 2^2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : 2^2) = (5, 4, 4, 2, 2, 3)$. Since $|G| = |L : 2^2|$ and $D(G) = D(L : 2^2)$, so the prime graph of G has following form similarly to Proposition 3.3:

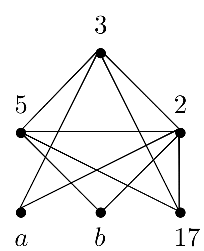


Figure 3.6

where $\{a, b\} = \{7, 13\}$.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3,5\}$-group. In particular, G is non-solvable.
According to Step 1 in Proposition 3.3, we have K is a $\{2,3,5\}$-group and G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.
We can prove this by the similar argument in Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 26$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : 2^2| = 4|L|$, we deduce $|K| = 1, 2$ or 4.

If $|K| = 1$, then $G \cong L : 2^2$.

If $|K| = 2$, then $K \leq Z(G)$. In this case G is a central extension of $\mathbb{Z}_2$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If G splits over K then $G \cong \mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$ or $\mathbb{Z}_2 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible.

If $|K| = 4$, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$ or S_3. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by L. If G is a non-split extension of K by L, then $|K|$ must divide the Schur multiplier of L, which is 1, but this is a contradiction. Therefore G splits over K. Hence $G \cong K \times L$. So we have $G \cong \mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ because $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $|G/C_G(K)| = 2, 3$ or 6, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$. Which is a contradiction, since L is simple. □

Proposition 3.7. *If $M = L : (D_6)_1$, then $G \cong L : (D_6)_1$, $L : 6$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.*

Proof. As $|L : (D_6)_1| = 2^{25}.3^6.5^4.7.13.17^2$ and $\pi_e(L : (D_6)_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 39, 42, 45, 51, 60, 63, 65, 85, 255\}$, then $D(L : (D_6)_1) = (4, 5, 4, 2, 2, 3)$. Since $|G| = |L : (D_6)_1|$ and $D(G) = D(L : (D_6)_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

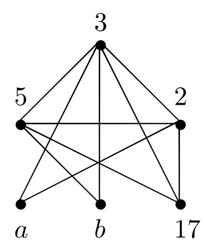


Figure 3.7

where $\{a, b\} = \{7, 13\}$.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.
By the similar argument to that in Step 1 in Proposition 3.1, we can obtain this assertion.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.
The proof is similar to Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 6$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : D_6)_1| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L : (D_6)_1, L : (D_6)_2$ or $L : 6$ because $|G| = 6|L|$. Obviously, $G \cong L : (D_6)_1$ or $L : 6$ because $deg(2) = 5$ in $\Gamma(L : (D_6)_2)$.

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction (see page 18).

If $|K| = 3$, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. If G splits over K, then $G \cong \mathbb{Z}_3 \times (L : 2_1)$ or $\mathbb{Z}_3 \times (L : 2_3)$ because in $\Gamma(\mathbb{Z}_3 \times (L : 2_2))$ the degree of 2 is 5. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. If $C_G(K)$ splits over K, then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6.
If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that $deg(2) = 5$, a contradiction. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction because L is simple.
If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore, $G \cong D_6 \times L$, which implies that $deg(2) = 5$, a contradiction. □

Proposition 3.8. *If $M = L : (D_6)_2$, then $G \cong L : (D_6)_2$, $\mathbb{Z}_2 \times (L : 3)$, $\mathbb{Z}_3 \times (L : 2_2)$, $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$, $\mathbb{Z}_6 \times L$ or $S_3 \times L$.*

Proof. As $|L:(D_6)_2| = 2^{25}.3^6.5^4.7.13.17^2$ and $\pi_e(L:(D_6)_2) = \{1,2,3,4,5,6,7,8,9,10,12,13,14,15,17,18,20,21,24,26,30,34,39,40,42,45,51,60,63,65,68,85,102,126,130,170,255\}$, then $D(L:(D_6)_2) = (5,5,4,2,3,3)$. Since $|G| = |L:(D_6)_2|$ and $D(G) = D(L:(D_6)_2)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L:(D_6)_2)$):

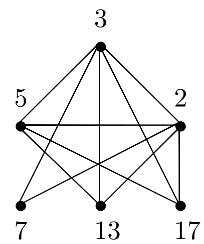


Figure 3.8

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$-group. In particular, G is non-solvable.
The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G})$, $S = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. By the same argument in Step 2 of Proposition 3.3 and considering 7 instead of a, we get a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^4.7.13.17^2$, where $2 \leq \alpha \leq 25$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L: D_6)_2| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L:(D_6)_1, L:(D_6)_2$ or $L:6$ because $|G| = 6|L|$. Obviously $G \cong L:(D_6)_2$ because in $\Gamma(L:(D_6)_1)$ and $\Gamma(L:6)$, we have $deg(13) = 2$(see page 17).

If $|K| = 2$, then $K \leq Z(G)$ and $G/K \cong L:3$. Hence G is a central extension of K by $L:3$. If G splits over K, then $G \cong \mathbb{Z}_2 \times (L:3)$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L:3$, which is impossible.

If $|K| = 3$, then $G/K \cong L:2_1, L:2_2$ or $L:2_3$. But $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L:2_1, L:2_2$ or $L:2_3$. If G splits over K, then only $G \cong \mathbb{Z}_3 \times (L:2_2)$ because $2 \nsim 13$ in $\Gamma(\mathbb{Z}_3 \times (L:2_1))$ and $\Gamma(\mathbb{Z}_3 \times (L:2_3))$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L:2_1, L:2_2$ and $L:2_3$, which is impossible. If

$|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. If $C_G(K)$ splits over K, then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6. If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$ and $G/K \cong L$. Therefore G is a central extension of K by L. If G is a non-split extension of K by L, then $|K|$ must divide the Schure multiplier of L, which is 1. But this is a contradiction. So we obtain that G splits over K. Hence $G \cong \mathbb{Z}_6 \times L$. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction because L is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore $G \cong D_6 \times L$. □

Proposition 3.9. *If $M = L : 6$, then $G \cong L : 6$, $L : (D_6)_1$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_3)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.*

Proof. As $|L : 6| = 2^{25}.3^6.5^4.7.13.17^2$ and $\pi_e(L : 6) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 36, 39, 42, 45, 48, 51, 63, 65, 85, 255\}$, then $D(L : 6) = (4, 5, 4, 2, 2, 3)$. Since $|G| = |L : 6|$ and $D(G) = D(L : 6)$, there exist several possibilities for $\Gamma(G)$ similarly to Proposition 3.7:

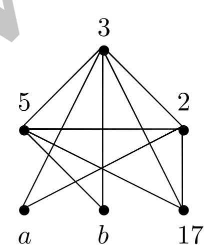


Figure 3.9

where $\{a, b\} = \{7, 13\}$.

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3, 5\}$-group. In particular, G is non-solvable.
The proof is similar to that in Proposition 3.3.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut(S)}$, where S is a finite non-abelian simple group.
Again we refer to Step 2 of proposition 3.3 to get the proof.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^{\gamma}.7.13.17^2$, where $2 \leq \alpha \leq 25$, $1 \leq \beta \leq 6$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L:6| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L:6$, $L:(D_6)_1$ or $L:(D_6)_2$ because $|G| = 6|L|$. Obviously, $G \cong L:6$ or $L:(D_6)_1$ because $deg(2) = 5$ in $\Gamma(L:(D_6)_2)$ (see page 18).

If $|K| = 2$, then $K \leq Z(G)$ and so $deg(2) = 5$, which is a contradiction.

If $|K| = 3$, then $G/K \cong L:2_1, L:2_2$ or $L:2_3$. But $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, G is a central extension of K by $L:2_1, L:2_2$ or $L:2_3$. If G splits over K, then $G \cong \mathbb{Z}_3 \times (L:2_1)$ or $\mathbb{Z}_3 \times (L:2_3)$ because in $\Gamma(\mathbb{Z}_3 \times (L:2_2))$ the degree of 2 is 5. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L:2_1, L:2_2$ and $L:2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L:2_1, L:2_2$ or $L:2_3$, we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. If $C_G(K)$ splits over K, then $C_G(K) \cong \mathbb{Z}_3 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

If $|K| = 6$, then $G/K \cong L$ and $K \cong \mathbb{Z}_6$ or D_6. If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$. It follows that $deg(2) = 5$, a contradiction. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction because L is simple. If $K \cong D_6$, then $K \cap C_G(K) = 1$ and $G/C_G(K) \lesssim D_6$. Thus $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore, $G \cong D_6 \times L$, which implies that $deg(2) = 5$, a contradiction. □

Proposition 3.10. *If $M = L:D_{12}$, then $G \cong L:D_{12}$, $\mathbb{Z}_2 \times (L:(D_6)_1)$, $\mathbb{Z}_2 \times (L:(D_6)_2)$, $\mathbb{Z}_2 \times (L:6)$, $\mathbb{Z}_3 \times (L:2^2)$, $(\mathbb{Z}_3 \times (L:2_1)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L:2_2)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L:2_3)).\mathbb{Z}_2$, $\mathbb{Z}_4 \times (L:3)$, $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L:3)$, $(\mathbb{Z}_4 \times L).\mathbb{Z}_3$, $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$, $\mathbb{Z}_6 \times (L:2_1)$, $\mathbb{Z}_6 \times (L:2_2)$, $\mathbb{Z}_6 \times (L:2_3)$, $(\mathbb{Z}_6 \times L).\mathbb{Z}_2$, $S_3 \times (L:2_1)$, $S_3 \times (L:2_2)$, $S_3 \times (L:2_3)$, $\mathbb{Z}_{12} \times L$, $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, $D_{12} \times L$, $(\mathbb{Z}_2 \times L).D_6$, $\mathbb{A}_4 \times L$, $L.\mathbb{A}_4$ or $T \times L$.*

Proof. As $|L:D_{12}| = 2^{26}.3^6.5^4.7.13.17^2$ and $\pi_e(L:(D_{12})) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 48, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L:D_{12}) = (5, 5, 4, 2, 3, 3)$. Since $|G| = |L:D_{12}|$ and $D(G) = D(L:D_{12})$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L:D_{12})$):

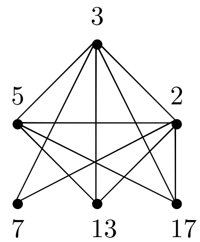


Figure 3.10

Step1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3\}$-group. In particular, G is non-solvable.
The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \le \frac{G}{K} \lesssim$ Aut(S), where S is a finite non-abelian simple group.
To get the proof, follow the way in the proof of Step 2 in proposition 3.5.

By TABLE 1 and Step 1, it is evident that $|S| = 2^{\alpha}.3^{\beta}.5^4.7.13.17^2$, where $2 \le \alpha \le 26$ and $1 \le \beta \le 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L)$. As $|G| = |L : D_{12}| = 12|L|$, we deduce $|K| = 1, 2, 3, 4, 6$ or 12.

If $|K| = 1$, then $G \cong L : D_{12}$.

If $|K| = 2$, then $G/K \cong L : (D_6)_1, L : (D_6)_2$ or $L : 6$ and $K \le Z(G)$. It follows that G is a central extension of K by $L : (D_6)_1, L : (D_6)_2$ or $L : 6$. If G splits over K, then $G \cong \mathbb{Z}_2 \times (L : (D_6)_1)$, $\mathbb{Z}_2 \times (L : (D_6)_2)$ or $\mathbb{Z}_2 \times (L : 6)$. Otherwise $G \cong \mathbb{Z}_2.(L : (D_6)_1)$ or $\mathbb{Z}_2.(L : (D_6)_2)$.

If $|K| = 3$, then $G/K \cong L : 2^2$. But $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \le Z(G)$, that is, G is a central extension of K by $L : 2^2$. If G splits over K, then $G \cong \mathbb{Z}_3 \times (L : 2^2)$, Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2^2$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \ne C_G(K)/K \trianglelefteq G/K \cong L : 2^2$, and we obtain $C_G(K)/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. Since $K \le Z(C_G(K))$, $C_G(K)$ is a central extension of K by $L : 2_1, L : 2_2$ or $L : 2_3$. Thus $C_G(K) \cong \mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_2)$ or $\mathbb{Z}_3 \times (L : 2_3)$, otherwise we get a contradiction because 3 must divide the Schure multiplier of $L : 2_1, L : 2_2$ or $L : 2_3$, which is impossible. Therefore, $G \cong (\mathbb{Z}_3 \times (L : 2_1)).\mathbb{Z}_2, (\mathbb{Z}_3 \times (L : 2_2)).\mathbb{Z}_2$ or $(\mathbb{Z}_3 \times (L : 2_3)).\mathbb{Z}_2$.

If $|K| = 4$, then $G/K \cong L : 3$ and $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this case we have $G/C_G(K) \lesssim \mathrm{Aut}(K) \cong \mathbb{Z}_2$ or S_3. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \le Z(G)$, that is, G is a central extension of K by $L : 3$. If G split over K by $L : 3$, then $G \cong \mathbb{Z}_4 \times (L : 3)$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times (L : 3)$. Otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 3$, which is impossible. If $|G/C_G(K)| \ne 1$, since $|G/C_G(K)| = 2, 3$ or 6, it follows that $K < C_G(K)$. As L is simple, we conclude that $1 \ne C_G(K)/K$ must be an extension of L. Hence $|G/C_G(K)| = 3$ and therefore $C_G(K)/K \cong L$. Now, since $K \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of K by L. Thus $C_G(K) \cong \mathbb{Z}_4 \times L$, or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$, otherwise $|K|$ must divide the Schure multiplier of L, which is 1 and it is impossible. Therefore, $G \cong (\mathbb{Z}_4 \times L).\mathbb{Z}_3$ or $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$.

If $|K| = 6$, then $G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ and $K \cong \mathbb{Z}_6$ or D_6. If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \lesssim \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is G is a central extension of $\mathbb{Z}_6$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If G splits over K, we obtain $G \cong \mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$ or $\mathbb{Z}_6 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of $L : 2_1, L : 2_2$ or $L : 2_3$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$, and we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. Thus $C_G(K) \cong \mathbb{Z}_6 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L. Therefore $G \cong (\mathbb{Z}_6 \times L).\mathbb{Z}_2$. If $K \cong D_6$, then $G/C_G(K) \lesssim D_6$ and so $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is a contradiction. If $|G/C_G(K)| = 2$, then we have $|KC_G(K)| = 6.|G|/2 = 3|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 3$, then we have $|KC_G(K)| = 6.|G|/3 = 2|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 6$, then $G/C_G(K) \cong D_6$ and $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. It follows that $C_G(K) \cong L : 2_1, L : 2_2$ or $L : 2_3$ because L is simple. Therefore, $G \cong D_6 \times (L : 2_1)$, $D_6 \times (L : 2_2)$ or $D_6 \times (L : 2_3)$.

Before processing the last case, we recall the following facts.

There exist five non-isomorphic groups of order 12. Two of them are abelian and three are non-abelian. The non-abelian groups are: alternating group A_4, dihedral group D_{12} and the dicyclic group T with generators a and b, subject to the relations $a^6 = 1$, $a^3 = b^2$ and $b^{-1}ab = a^{-1}$.

If $|K| = 12$, then $G/K \cong L$ and $K \cong \mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, D_{12}, $\mathbb{A}_4$ or T. But $C_G(K)K/K \trianglelefteq G/K \cong L$. If $C_G(K)K/K = 1$, then $C_G(K) \leq K$ and hence $|L| = |G/K| | |G/C_G(K)| | |\mathrm{Aut}(K)|$. Thus $|L| | |\mathrm{Aut}(K)|$, a contradiction. Therefore, $C_G(K)K/K \neq 1$ and since L is simple group, we conclude that $G = C_G(K)K$ and hence, $G/C_G(K) \cong K/Z(K)$. Now, we should consider the following cases:

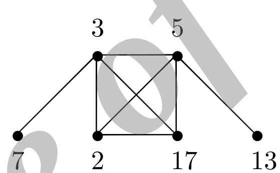
If $K \cong \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$, then $G/C_G(K) = 1$. Therefore $K \leq Z(G)$, that is G is a central extension of $\mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$ by L. If G splits over K, we obtain $G \cong \mathbb{Z}_{12} \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, otherwise we get a contradiction because $|K|$ must divide the Schure multiplier of L, which is 1 and it is impossible.

If $K \cong D_{12}$, then $G = K.L$ and $G/C_G(K) \cong D_6$. Since $C_G(K)/Z(K) \cong G/K \cong L$ and $Z(K) \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of $Z(K) \cong \mathbb{Z}_2$ by L. If $C_G(K)$ is a non-split extension, then 2 must divide the Schure multiplier of L, which is 1 and it is impossible. Thus $C_G(K) \cong \mathbb{Z}_2 \times L$ and hence, G is a split extension of K by L. Now, since $\mathrm{Hom}(L, \mathrm{Aut}(D_{12}))$ is trivial, we have $G \cong D_{12} \times L$.

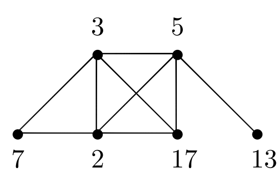
If $K \cong \mathbb{A}_4$, then $G/C_G(K) \cong \mathbb{A}_4$. As $G = C_G(K)K$, It follows that $C_G(K) \cong L$. Therefore $G \cong L \times \mathbb{A}_4$ or $L.\mathbb{A}_4$.

If $K \cong T$, then By the similar way in case $K \cong D_{12}$, we can conclude that G is a split extension of K by L. Also, since $\mathrm{Hom}(L, \mathrm{Aut}(T)$ is trivial, we have $G \cong T \times L$. $\square$

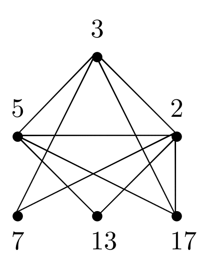
According to what we said before the proof, here we depict $\Gamma(M)$ by $|M|$ and $\pi_e(M)$, where M is an almost simple group related to $L = D_4(4)$.



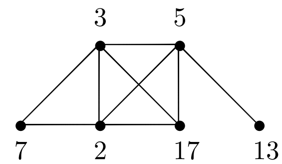
$\Gamma(L)$



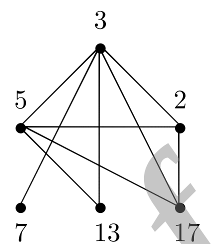
$\Gamma(L : 2_1)$



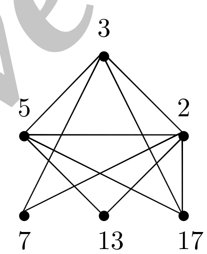
$\Gamma(L : 2_2)$



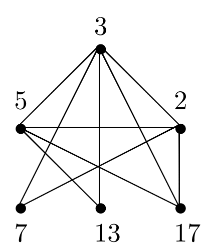
$\Gamma(L:2_3)$



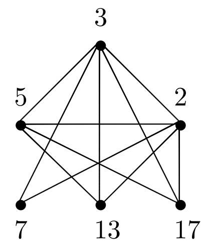
$\Gamma(L:3)$



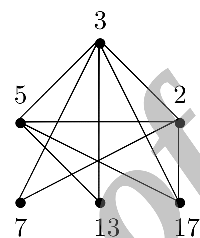
$\Gamma(L:2^2)$



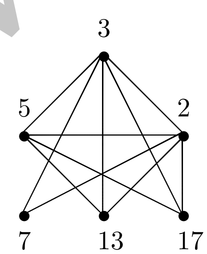
$\Gamma(L:(D_6)_1)$



$\Gamma(L : (D_6)_2)$



$\Gamma(L : 6)$



$\Gamma(L : D_{12})$

4. Acknowledgments

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