Iranian Journal of Mathematical Sciences and Informatics
Vol. 10, No. 1 (2015), pp 95-102
DOI: 10.7508/ijmsi.2015.01.007

Distance-Balanced Closure of Some Graphs

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Abstract. In this paper we prove that any distance-balanced graph G with $\Delta(G) \geq |V(G)| - 3$ is regular. Also we define notion of distance-balanced closure of a graph and we find distance-balanced closures of trees T with $\Delta(T) \geq |V(T)| - 3$.

Keywords: Distances in graphs, Distance-balanced graphs, Distance-balanced closure.

2000 Mathematics subject classification: 05B20, 05E30.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote $|V(G)|$ by n. The set of neighbors of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex v is denoted by $\deg_G(v)$ and minimum degree and maximum degree of G denoted by $\delta(G)$ and $\Delta(G)$,

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Received 26 July 2013; Accepted 11 June 2014

respectively. The distance $d_G(u,v)$ between vertices u and v is the length of a shortest path between u and v in G. The diameter diam(G) of graph G is defined as $\max\{d_G(u,v) : u,v \in V(G)\}$. The notion of distance is studied in several works in graph theory (See [2] and the references therein) and many research works are based on the concepts related to this notion (See for instance [8] and [10]).

For an edge xy of a graph G, W_{xy}^G is the set of vertices which are closer to x than y, more formally

$$W_{xy}^G = \{u \in V(G) | d_G(u,x) < d_G(u,y)\}.$$

Moreover, ${}_xW_y^G$ is the set of vertices of G that have equal distances to x and y, that is

$${}_xW_y^G = \{u \in V(G) | d_G(u,x) = d_G(u,y)\}.$$

These sets play important roles in metric graph theory, see for instance [1, 3, 4, 5]. Since x always belongs to W_{xy}^G, for convenience we let $U_{xy}^G = W_{xy}^G \setminus \{x\}$. *Distance-balanced* graphs are introduced in [9] as graphs for which $|W_{xy}^G| = |W_{yx}^G|$ (or equivalently $|U_{xy}^G| = |U_{yx}^G|$) for every pair of adjacent vertices $x, y \in V(G)$.

In [9], the parameter $b(G)$ of a graph G is introduced as the smallest number of the edges which can be added to G such that the obtained graph is distance-balanced. Since the complete graph is distance-balanced, this parameter is well-defined. We call graph G a *distance-balanced closure* of H if G is distance-balanced and H is a spanning subgraph of G with $|E(G)| = b(H) + |E(H)|$; in other words, a distance-balanced closure of H is a distance-balanced graph G which contains H as a spanning subgraph and has minimum number of edges. As mentioned in [9], the computation of $b(G)$ is quite hard in general but it might be interesting in some special cases. In this paper we compute $b(G)$ for all trees T with $\Delta(T) \geq |V(T)| - 3$. In Section 2, we compute that distance-balanced closure of graphs G with $\Delta(G) = n-1$. In Section 3, and Section 4, we concern graphs G with $\Delta(G) = n-2$ and $\Delta(G) = n-3$, respectively. Then we compute $b(T)$ for all trees T with $\Delta(T) = n-2$ and $\Delta(T) = n-3$.

Here we mention some more definitions and notations about trees. Let P_n denoted the path with n vertices. A tree which has exactly one vertex of degree greater than two is said to be *starlike*. The vertex of maximum degree is called the *central vertex*. We denote by $S(n_1, n_2, \ldots, n_k)$ a starlike tree in which removing the central vertex leaves disjoint paths $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$. We say that $S(n_1, n_2, \ldots, n_k)$ has branches of length $n_1, n_2, \ldots, n_k$. It is obvious that $S(n_1, n_2, \ldots, n_k)$ has $n_1 + n_2 + \ldots + n_k + 1$ vertices. For simplicity a starlike with α_i branches of length n_i $(1 \leq i \leq k)$ is denoted by $S(n_1^{\alpha_1}, n_2^{\alpha_2}, \ldots, n_k^{\alpha_k})$.

2. Distance-Balanced Graphs with Maximum Degree $n-1$

In this section we prove that for any graph G with $\Delta(G) = n-1$, the only distance-balanced closure of G is the complete graph K_n. The following result is very useful in this paper. It is in fact a slight modification of Corollary 2.3 of [9].

Theorem 2.1. *Let G be a graph with diameter at most 2 and H be a distance-balanced graph such that G is a spanning subgraph of H. Then H is a regular graph. Moreover, every regular graph with diameter at most 2 is distance-balanced.*

Corollary 2.2. *For every integer $m \geq 1$, the graph $K_{1,m}$ has a unique distance-balanced closure which is isomorphic to K_{m+1}, hence, $b(K_{1,m}) = \binom{m+1}{2} - m$.*

Proof. Let G be a distance-balanced closure of $K_{1,m}$. By Theorem 2.1, G is a regular graph and since $K_{1,m}$ has a vertex of degree m, G should be m-regular, hence $G \cong K_{m+1}$. □

The following is an immediate conclusion of Theorem 2.1.

Corollary 2.3. *Let G be a graph with n vertices and $\Delta(G) = n-1$. Then the graph G has a unique distance balanced closure. Moreover, this closure is isomorphic to K_n.*

3. Distance-Balanced Graphs with Maximum Degree $n-2$

In this section, we prove that any distance-balanced graph G with $\Delta(G) = n-2$ is a regular graph using this, we construct a distance-balanced closure of T where T is a tree with this property (that is $\Delta(T) = n-2$) and then compute $b(T)$.

The following Lemma will be used occasionally in this paper and the proof is easily deduced from the definition of U^G_{xy}.

Lemma 3.1. *Let x and y be two adjacent vertices of a graph G, then $U^G_{xy} \cap N_G(y) = \emptyset$ (or $U^G_{xy} \subseteq V(G) \setminus N_G[y]$). Furthermore, $N_G[y] \setminus U^G_{yx} \subseteq N_G[x]$.*

Theorem 3.2. *Let $T = S(2, 1^{m-1})$ be a starlike tree and H be a distance-balanced graph containing T as a spanning subgraph. Then* $\mathrm{diam}(H) \leq 2$, *hence, H is an r-regular graph for some $m \leq r \leq m+1$.*

Proof. Suppose that the vertices of T are labeled as shown in Figure 1. If $oy \in E(H)$, then H contains $K_{1,m+1}$ as a spanning subgraph, so by Corollary 2.2, $H \cong K_{m+2}$.

So, we may assume that $oy \notin E(H)$ (and consequently $\mathrm{diam}(H) \neq 1$). We prove $\mathrm{diam}(H) = 2$. For this, it is enough to show that $d(y, x_i) \leq 2$ for $i = 2, \cdots, m$. Let i be an integer with $2 \leq i \leq m$; using the fact that y is the only vertex which is not adjacent to o in H and using Lemma 3.1, we conclude that $U^H_{x_io} \subseteq \{y\}$; If $U^H_{x_io} = \{y\}$, then $x_iy \in E(H)$ and $d_H(x_i, y) = 1$, hence in this case $d_H(x_i, y) \leq 2$. Otherwise, $U^H_{x_io} = \emptyset$, in this case, for every $z \in V(H) \setminus \{o, x_i\}$ we have $d_H(z, x_i) = d_H(z, o)$, particularly, $d_H(z, x_i) = d_H(y, o) = 2$. Hence, $\mathrm{diam}(H) = 2$, as required. The result is now concluded using Theorem 2.1.

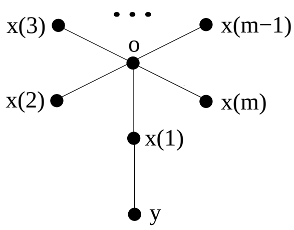

Figure 1

□

Theorem 3.3. *Let $T = S(2, 1^{m-1})$ be a starlike of order $m + 2$. Then*

$$b(T) = \begin{cases} \frac{m^2}{2} - 1 & \text{if } m \text{ is even;} \\ \binom{m+1}{2} & \text{otherwise.} \end{cases}$$

Proof. Suppose that the vertices of T be labeled as in Figure 1, and let $\overline{T}$ be a distance-balanced closure of T. First, suppose that m is an odd integer; Since there is no m-regular graph of order $m + 2$, by Theorem 3.2, $\overline{T} \cong K_{m+2}$.

Now, suppose that m is an even integer. Let $H = K_{m+2}$ be a complete graph with vertex set $V(T)$ and M be a complete matching of H which contains the edge oy. Then $H \setminus M$, is an m-regular graph with diameter 2 and contains T as a spanning subgraph. Hence, by Theorem 2.1 and Theorem 3.2, $\overline{T} = H \setminus M$, is a distance-balanced closure of T and $b(T) = \frac{m^2}{2} - 1$. □

Corollary 3.4. *Let G be a connected graph of order n, with $\Delta(G) = n - 2$ and H be a distance-balanced graph which contains G as a spanning subgraph. Then H is either an $(n - 2)$-regular graph or the complete graph K_n.*

Proof. In this case $S(2, 1^{n-3})$ is an spanning subgraph of G. So, by Theorem 3.2, H is either an $(n - 2)$-regular graph or the complete graph K_n. □

4. Distance-Balanced Graphs with Maximum Degree $n - 3$

In this section we will prove that every distance-balanced graph with $\Delta(G) = n - 3$ is regular. Moreover, by constructing distance-balanced closure of trees with $\Delta(T) = n - 3$ we compute $b(T)$ for these trees.

Theorem 4.1. *Let $T = S(2^2, 1^{m-2})$ be a starlike of order $m+3$ and H be a distance-balanced graph which contains T as a spanning subgraph. Then* $\mathrm{diam}(H) \leq 2$. *Moreover, H is an r-regular graph with $r \geq m$.*

Proof. Suppose the vertices of T are labeled as in Figure 2. If either oy or oz be an edge of H, then by Theorem 3.2, H is an r-regular graph with $r \geq m+1$, which proves this theorem. So, suppose that $oy, oz \notin E(H)$.

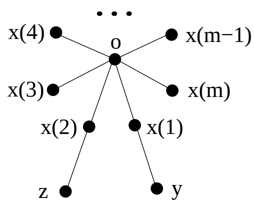

Figure 2

By Lemma 3.1, for each $1 \leq i \leq m$, $U^H_{x_io} \subseteq \{y, z\}$. Now, we prove that $\deg_H(x_i) = m$, for each $i = 1, \cdots, m$. For this, we consider three possible cases:

Case 1. $|U^H_{x_io}| = 0$**:** Then $U^H_{x_io} = \emptyset$ and $U^H_{ox_i} = \emptyset$. Hence, by Lemma 3.1, $N_H(x_i) = \{o, x_1, x_2, \ldots, x_m\} \setminus \{x_i\}$ and $\deg_H(x_i) = m$.

Case 2. $|U^H_{x_io}| = 1$**:** Without loss of generality we can assume that $U^H_{x_io} = \{y\}$. Then there is an integer $1 \leq j \leq m$ such that $U^H_{ox_i} = \{x_j\}$. Since $d_H(o, y) = 2$, $yx_i \in E(H)$. Hence, using Lemma 3.1, $N_H(x_i) = \{o, y, x_1, x_2, \ldots, x_m\} \setminus \{x_j\}$ and $\deg_H(x_i) = m$.

Case 3. $|U^H_{x_io}| = 2$**:** We have $U^H_{x_io} = \{y, z\}$. Since $d_H(o, y) = d_H(o, z) = 2$, we conclude that $yx_i, zx_i \in E(H)$. Since $|U^H_{ox_i}| = 2$, there are integers j and k such that $U^H_{ox_i} = \{x_j, x_k\}$. Hence, by Lemma 3.1, we have $N_H(x_i) = \{o, y, z, x_1, x_2, x_3, \ldots, x_m\} \setminus \{x_i, x_j, x_k\}$ and $\deg_H(x_i) = m$.

Next, we prove that $\deg_H(y) \geq m-3$ and $\deg_H(z) \geq m-3$. From $\deg_H(x_1) = m$ and Lemma 3.1, it concludes that $|U^H_{yx_1}| \leq 2$, hence, $|U^H_{x_1y}| \leq 2$, which means that there are at most two elements in $N_H(x_1) \setminus N_H[y]$. Using this and Lemma 3.1, we provide $\deg_H(y) \geq m-3$. With a similar argument, the inequality $\deg_H(z) \geq m-3$ is concluded.

Now, by using $\deg_H(y), \deg_H(z) \geq m-3$, $\deg_H(x_i) \geq m$, $(i = 1, \cdots, m)$, and $oy, oz \notin E(H)$, hence every two nonadjacent vertices have a common neighbor, provided that $m \geq 7$. This means that $\mathrm{diam}(H) = 2$, which proves the result in case $m \geq 7$, using Theorem 2.1. For the cases, $3 \leq m \leq 6$, through a case by case inspection (by using $\deg_H(x_i) \geq m$, $i = 1, \cdots, m$,) the same result is obtained. ☐

Theorem 4.2. *For the starlike tree $T = S(2^2, 1^{m-2})$ of order $m+3$, $b(G) = \frac{m^2+m-4}{2}$.*

Proof. Let the vertices of T be labeled as in Figure 2 and $\overline{T}$ be a distance-balanced closure of T. Now, we are going to construct $\overline{T}$. Let $H = K_{m+3}$ be a complete graph with the same vertex set as H. Omit the edges of cycles $C_1 = x_1x_2x_3\ldots x_mx_1$ and $C_2 = oyzo$ from H to obtain $\overline{T} = H \setminus (C_1 \cup C_2)$. Now, $\overline{G}$ is an m-regular graph with diameter 2, which contains T as a spanning subgraph, so by Theorem 4.1 and Theorem 2.1, $\overline{T}$ is a distance-balanced closure of T and $b(T) = \frac{m^2+m-4}{2}$. □

Theorem 4.3. *Let T be the tree of Figure 3 and H be a distance-balanced graph which contains T as a spanning subgraph. Then* $\mathrm{diam}(H) \leq 2$*, hence, H is a regular graph. Moreover, $b(T) = \frac{m^2+m-4}{2}$.*

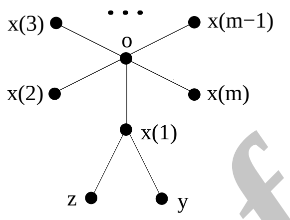

Figure 3

Proof. If either $oy \in E(H)$ or $oz \in E(H)$, then by Theorem 3.2, $\mathrm{diam}(H) \leq 2$ and H is a regular graph. So, suppose that neither oy nor oz is in $E(H)$. Since $|W^H_{x_1y}| = |W^H_{yx_1}|$ and $o \in W^H_{x_1y}$, there exists a vertex x_i, $i \neq 1$, such that $yx_i \in E(H)$. Therefore, graph H contains graph $S(2^2, 1^{m-2})$ as a spanning subgraph and using Theorem 4.1, $\mathrm{diam}(H) \leq 2$ and H is a regular graph. Furthermore, the graph introduced in the proof of Theorem 4.2, is also distance-balanced closure of T. Hence $b(T) = \frac{m^2+m-4}{2}$. □

Theorem 4.4. *Consider the starlike tree $T = S(3, 1^{m-1})$ of order $m+3$ and let H be a distance-balanced graph which contains T as a spanning subgraph. Then H is an r-regular graph for some $m \leq r \leq m+2$.*

Proof. Let the vertices of T be labeled as in Figure 4.

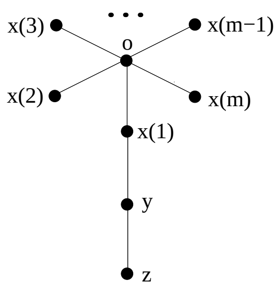

Figure 4

If either $oy \in E(H)$ or $oz \in E(H)$, then by Theorem 3.2, $\mathrm{diam}(H) \leq 2$ and H is a regular graph. If $zx_1 \in E(H)$, then H contains the graph shown in Figure 3 as a spanning subgraph, so by Theorem 4.3, H is a regular graph. So we may assume that $\{oy, oz, x_1z\} \cap E(H) = \emptyset$. Since $|W^H_{yz}| = |W^H_{zy}|$ and $x_1 \in W^H_{yz}$, the vertex z is adjacent to at least one vertex in $\{x_2, x_3, \dots, x_m\}$ (because otherwise according to the structure of T we have $V \setminus \{y, z\} \subseteq U_{yz}$). Hence, H contains the graph $S(2^2, 1^{m-2})$, as a spanning subgraph. So, by Theorem 4.1, $\mathrm{diam} H \leq 2$ and H is a regular graph, as desired. □

Corollary 4.5. *Let G be a connected graph of order n with $\Delta(G) = n-3$. Then every distance-balanced graph H which contains G as a spanning subgraph, is regular.*

Proof. Since $\Delta(G) = n-3$, G contains at least one of the graphs $S(2^2, 1^{n-2})$, $S(3, 1^{n-1})$ or the graph shown in Figure 3, as a spanning subgraph. Hence, the result follows from Theorem 4.2, Theorem 4.6 and Theorem 4.3. □

Theorem 4.6. *For the starlike tree $G = S(3, 1^{m-1})$ of order $m+3$, $b(G) = \frac{m^2+m-4}{2}$.*

Proof. Let the vertices of G be labeled as in Figure 4 and let $\overline{G}$ be a distance-balanced closure of G. Now, we are going to construct $\overline{G}$. Let $H = K_{m+3}$ be a complete graph with the same vertex set as G. Omit the edges of cycles $C_1 = x_3x_4 \dots x_mx_3$ and $C_2 = oyx_2x_1zo$ from H to obtain $\overline{G} = H \setminus (C_1 \cup C_2)$. Then the graph $\overline{G}$ is an n-regular graph with diameter 2, which contains G as a spanning subgraph. So by Theorem 4.4, $\overline{G}$ is a distance-balanced closure of G and $b(G) = \frac{m^2+m-4}{2}$. □

Conclusion. In previous sections, we have proved that any connected distance-balanced graph G with $\Delta(G) \geq |V(G)| - 3$, is a regular graph, moreover, distanced-closure of such a graph G is a smallest regular graph which contains G. This helped us to find a distance-balanced closure of trees T with $\Delta(T) \geq |V(T)| - 3$ and to compute $b(T)$ for such trees.

Acknowledgments

The authors would like to thank the anonymous referees for their useful comments and suggestions.

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