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On Tensor Product of Graphs, Girth and Triangles

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Abstract. The purpose of this paper is to obtain a necessary and sufficient condition for the tensor product of two or more graphs to be connected, bipartite or eulerian. Also, we present a characterization of the duplicate graph $G \oplus K_2$ to be unicyclic. Finally, the girth and the formula for computing the number of triangles in the tensor product of graphs are worked out.

Keywords: Tensor product, Bipartite graph, Connected graph, Eulerian graph, Girth, Cycle, Path.

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1. Introduction

We shall consider only finite, undirected graphs without loops or multiple edges. We follow the terminology of [1]. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. For a connected graph G, nG is the graph with n components, each being isomorphic to G. It is well-known that a graph is *bipartite* if and only if it contains no odd cycle. We now define the tensor product of two graphs [8] as follows: The *tensor product* of two graphs G_1 and G_2 is the graph, denoted by $G_1 \oplus G_2$, with vertex set $V(G_1 \oplus G_2) = V(G_1) \times V(G_2)$, and any two of its vertices (u_1, v_1) and (u_2, v_2) are adjacent, whenever u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2.

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The graphs G_1 and G_2 are called *factors* of the product $G_1 \oplus G_2$. Other popular names for tensor product that have appeared in the literature are *Kronecker product, cross product, direct product, conjenction product.* Sampathkumar [6] defines the tensor product of a graph G by K_2 as the *duplicate graph* of G, and studied its properties and a characterization in great detail. This product is also studied in [5]. Now, we define the two special type of tensor products: $G \oplus nK_2$ and $G \oplus [\oplus_{i=1}^{n} K_2]$ as the *generalized duplicate graphs of a graph* G, for any integer $n \geq 2$, and study their structural properties for our later use.

2. Structural Properties of the Generalized Duplicate Graphs

The following theorem of Weichsel [8] will be useful in the proof of our results.

Theorem 2.1. *If the connected graphs G and H are bipartite, then $G \oplus H$ has exactly two components.*

Next, we present some elementary results of the generalized duplicate graphs for our immediate use.

Theorem 2.2. *For any connected, bipartite graph G, $G \oplus nK_2 = 2nG$ for $n \geq 1$.*

Proof. For $n = 1$, Theorem 2.1 implies that $G \oplus K_2$ has exactly two components. Furthermore, by using the definition of tensor product, we see that each component of $G \oplus K_2$ is isomorphic to G. Therefore, $G \oplus K_2 = 2G$. Moreover, corresponding to $n \geq 1$ copies of K_2, $G \oplus nK_2$ certainly contains exactly $2n$ copies of G. Thus, $G \oplus nK_2 = 2nG$. □

Theorem 2.3. *For any connected graph G, $G \oplus nK_2$ for $n \geq 1$, is bipartite.*

Proof. We discuss two cases depending on G.
Case 1. Suppose G is bipartite. By Theorem 2.2, $G \oplus nK_2 = 2nG$. Since G is bipartite, it follows immediately that $G \oplus nK_2$ is bipartite.
Case 2. Suppose G is non-bipartite. Certainly, G contains a cycle C_m for odd $m \geq 3$. Corresponding to each copy of K_2 in $G \oplus nK_2$, there are exactly n distinct subgraphs in $G \oplus nK_2$, each is isomorphic to $C_m \oplus K_2$. It is shown in [2] that $C_m \oplus K_2$ is isomorphic to C_{2m}. For even $m \geq 4$, it is also shown in [2] that $C_m \oplus K_2 = C_m \cup C_m$. This proves that $G \oplus nK_2$ has no odd cycles. Hence, $G \oplus nK_2$ is bipartite. □

Theorem 2.4. *Let G be a connected, bipartite graph and let $H = \oplus_{i=1}^{n} K_2$. Then $G \oplus H = 2^n G$ for $n \geq 1$.*

Proof. We proceed by induction on n. If $n = 1$, then by Theorem 2.2, $G \oplus H = 2G$. Assume the result holds with at most $n-1$. Consider $G \oplus H = G \oplus [\oplus_{i=1}^{n} K_2]$

$= G \oplus [\oplus_{i=1}^{n-1} K_2 \oplus K_2] = [G \oplus (\oplus_{i=1}^{n-1} K_2)] \oplus K_2$. By induction hypothesis, we have $G \oplus [\oplus_{i=1}^{n-1} K_2] = 2^{n-1}G$. Hence,

$$G \oplus [\oplus_{i=1}^{n} K_2] = 2^{n-1}G \oplus K_2. \cdots\cdots (2.1)$$

In view of Theorem 2.2 (with $n = 1$), $G \oplus K_2 = 2G$. Using this in (2.1), we get $G \oplus H = 2^n G$. □

3. Characterization of Connected Tensor Product of Graphs

Now, we obtain a characterization of connected tensor product of arbitrarily many graphs. We see that Weichsel [6] studied the connectedness of the tensor product of two graphs as follows:

Theorem 3.1. *Let G and H be connected graphs. Then $G \oplus H$ is connected if and only if either G or H contains an odd cycle.*

Now, we present the natural finite extension of Weichsel's Theorem as follows:

Theorem 3.2. *Let G_k $(1 \leq k \leq n\ ;\ n \geq 2)$ be connected graph, and let $G = \oplus_{k=1}^{n} G_k$. Then G is connected if and only if at most one of G_k's is bipartite.*

Proof. Assume that G is connected. We prove by contradiction. If possible, assume that there are at least two distinct graphs G_i and G_j $(1 \leq i, j \leq n)$, which are bipartite. By Theorem 2.1, $G_i \oplus G_j$ contains exactly two components say, F and H. Now, we have $G = \oplus_{k=1}^{n} G_k = (F \oplus M) \cup (H \oplus M)$, where $M = \oplus_{k=1}^{n} G_k$ $(k \neq i, j)$. This shows that G is certainly disconnected, and hence we immediately arrive at a contradiction. Thus, it proves that at most one of G_k's is bipartite.

Conversely, assume that at most one of G_k's is bipartite.

We discuss two cases.

Case 1. None of G_k's is bipartite. Immediately, it follows that each G_k contains an odd cycle.

Case 2. Exactly one of G_k's is bipartite. Without loss of generality, we assume that G_1 is bipartite. The remaining G_i $(2 \leq i \leq n)$ is non-bipartite, and hence each such G_i contains an odd cycle.

In either case, by applying Theorem 3.1 and the mathematical induction on the number of factors, the result follows. □

4. Characterization of Bipartite Tensor Product of Graphs

Now, we shall obtain a necessary and sufficient condition for the tensor product of two or more graphs to be bipartite, (which is proposed in [3]).

Theorem 4.1. *Let G_1 and G_2 be two connected graphs. Then $G_1 \oplus G_2$ is bipartite if and only if at least one of G_1 and G_2 is bipartite.*

Proof. Suppose $G_1 \oplus G_2$ is bipartite. We claim that at least one of G_1 and G_2 is bipartite. If this is not so, then both G_1 and G_2 are non-bipartite. Consequently, there exist two odd cycles C_m (for $m \geq 3$) and C_n (for $n \geq 3$) in G_1 and G_2, respectively. Without loss of generality, we consider $m \leq n$. Let $C_m : u_1, u_2, \ldots, u_m, u_1$ and let $C_n : v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n, v_1$. Then $C_m \oplus C_n$ contains the cycle Z of length n as follows:
$Z : (u_1, v_1), (u_2, v_2), \ldots, (u_m, v_m), (u_{m-1}, v_{m+1}), (u_m, v_{m+2}), (u_{m-1}, v_{m+3}),$
$(u_m, v_{m+4}), \ldots, (u_{m-1}, v_{n-1}), (u_m, v_n), (u_1, v_1)$.
So, $G_1 \oplus G_2$ contains the odd cycle Z. Hence, $G_1 \oplus G_2$ is non-bipartite. This is a contradiction.
Conversely, assume that at least one of G_1 and G_2 is bipartite.
We discuss two cases.
Case 1. Suppose both G_1 and G_2 are bipartite. Then by Theorem 2.1, $G_1 \oplus G_2$ contains exactly two components, say H_1 and H_2. Now, we claim that both H_1 and H_2 are bipartite. On contrary, if one of $H_i's$ is non-bipartite. Without loss of generality, we assume that H_1 is non-bipartite. Then H_1 contains an odd cycle, say
$C : (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n), (u_1, v_1)$ in $G_1 \oplus G_2$, where $u_i \in V(G_1)$ $(1 \leq i \leq n)$, $v_j \in V(G_2)$ $(1 \leq j \leq n)$. Certainly, the first co-ordinate vertices $u_1, u_2, \ldots, u_n, u_1$ of the cycle C forms a closed odd walk W in G_1. Since every closed odd walk in a graph contains an odd cycle, it follows that the walk W contains an odd cycle in G_1. This shows that G_1 is not bipartite. But this contradicts the hypothesis that G_1 is bipartite. Since each H_i is bipartite, it follows that $G_1 \oplus G_2 = H_1 \cup H_2$ is also bipartite.
Case 2. Suppose one of G_1 and G_2 is bipartite, and the other is non-bipartite. Assume that G_1 is bipartite. Since G_2 is non-bipartite, G_2 contains an odd cycle. From Theorem 3.1, $G_1 \oplus G_2$ is connected. Next, we claim that $G_1 \oplus G_2$ is bipartite. If this is not so, then $G_1 \oplus G_2$ is non-bipartite, and hence it contains an odd cycle Z. By repeating the same argument as in Case 1, we obtain $G_1 \oplus G_2$ is bipartite.
In either case, we see that $G_1 \oplus G_2$ is bipartite. □

Next, we obtain the finite extension of the above theorem, and its proof follows by the mathematical induction on the number of factors.

Corollary 4.2. *Let G_k $(1 \leq k \leq n\ ;\ n \geq 2)$ be a connected graph, and let $G = \oplus_{k=1}^{n} G_k$. Then G is bipartite if and only if at least one of G_k's is bipartite.*

5. Characterization of Eulerian Tensor Product of Graphs

An *Euler tour* of a graph G is a closed walk in G that traverses each edge of G exactly once. A graph is eulerian if it contains an Euler tour. It is well-known that a connected graph G is eulerian if and only if every vertex in G has an even degree. For any vertex (u, v) in a tensor product $(G \oplus H)$ of two graphs

G and H, $deg(u,v) = deg(u) \bullet deg(v)$. Now, we present a characterization of eulerian tensor product of two graphs.

Theorem 5.1. *Let G and H be connected graphs such that at most one of them is bipartite. Then $G \oplus H$ is eulerian if and only if at least one of G and H is eulerian.*

Proof. Suppose G is eulerian, and it contains an odd cycle. Then by Theorem 4.1, $G \oplus H$ is connected. Since G is eulerian, $deg(u)$ is even for all vertices u in G. Consequently, for any vertex v of H, the pair (u,v) is a vertex in $G \oplus H$, and $deg(u,v) = deg(u) \bullet deg(v)$, which is an even degree, because $deg(u)$ is even, and $deg(v) \geq 1$. This implies that $G \oplus H$ is eulerian.
Conversely, assume that $G \oplus H$ is eulerian. By definition, $G \oplus H$ is certainly connected. Again by Theorem 4.1, one of G and H contains an odd cycle. To complete the proof, we claim that at least one of G and H is eulerian. On contrary, if possible assume that both G and H are not eulerian graphs. Immediately, there exist at least two odd degree vertices x and y in G and H, respectively. Thus, (x,y) is a vertex in $G \oplus H$, and also $deg(x,y) = deg(x) \bullet deg(y)$, which is odd, because both $deg(x)$ and $deg(y)$ are odd. This shows that $G \oplus H$ is not eulerian, and it contradicts the hypothesis that $G \oplus H$ is eulerian. □

The finite extension of Theorem 5.1 is the following result, and its proof directly follows by the induction on the number of factors.

Corollary 5.2. *Let G_k $(1 \leq k \leq n\ ;\ n \geq 2)$ be a connected graph such that at most one of G_k's is bipartite, and let $G = \oplus_{k=1}^{n} G_k$. Then G is eulerian if and only if at least one of G_k's is eulerian.*

6. Characterization of Unicyclic Duplicate Graph

A *unicyclic graph* is a connected graph which contains exactly one cycle. Next, we obtain a characterization of unicyclic duplicate graph $G \oplus K_2$.

Theorem 6.1. *A non-bipartite graph G is unicyclic if and only if the duplicate graph $G \oplus K_2$ is unicyclic.*

Proof. Suppose a non-bipartite graph G is unicyclic. Then G contains exactly one odd cycle C. Hence by Theorem 4.1, $G \oplus K_2$ is connected. Let C : $u_1, u_2, \ldots, u_{2k+1}, u_1$ for $k \geq 1$. Next, we show that $G \oplus K_2$ is unicyclic. For this, let us consider $V(K_2) = \{v_1, v_2\}$. It is easy to see that the subgraph induced by $C \oplus K_2$ in $G \oplus K_2$ is certainly isomorphic to an even cycle $C_{2(2k+1)}$, where $C_{2(2k+1)} : (u_1,v_1), (u_2,v_2), (u_3,v_1), \ldots, (u_{2k-1},v_1), (u_{2k},v_2), (u_{2k+1},v_1),$
$(u_{2k},v_2), (u_{2k+1},v_1), (u_1,v_2), (u_2,v_1), (u_3,v_2), \ldots, (u_{2k-1},v_2), (u_{2k},v_1),$
$(u_{2k+1},v_2), (u_1,v_1)$. Since G is unicyclic, it follows that $G \oplus K_2$ has no cycles other than $C_{2(2k+1)}$. If this is not so, then there exists another cycle J in $G \oplus K_2$, which is different from $C_{2(2k+1)}$. Consequently, the first co-ordinates

of the vertices of the cycle J, which are in pairs, will form another cycle C' in G. Since $J \neq C_{2(2k+1)}$ in $G \oplus K_2$, it follows $C \neq C'$ in G. This is a contradiction to the fact that G is unicyclic. Therefore, $G \oplus K_2$ is unicyclic.

Conversely, suppose that $G \oplus K_2$ is unicyclic. Let Z be the only one cycle in $G \oplus K_2$. By Theorem 2.3 (with $n = 1$), $G \oplus K_2$ is bipartite. Hence, Z is a unique even cycle. Clearly, we notice that the first co-ordinate vertices of G in Z forms an odd cycle C in G. Since $G \oplus K_2$ is unicyclic, it follows that C is the unique cycle in G. Moreover, since $G \oplus K_2$ is connected, it implies that G is connected. Therefore, G is unicyclic. □

7. The Girth and Triangles in Tensor Product Graphs

The *girth* of a graph G, denoted by $g(G)$, is the length of a shortest cycle in G, if any. Otherwise, it is undefined if G is a forest. It is clear that the girth of a graph G is the minimum of the girths of its components. Firstly, we determine the girth of the generalized duplicate graphs.

Theorem 7.1. *Let G be a connected graph with $g(G) = k$. For any positive integer $n \geq 1$, we have*

$$g(G \oplus n\ K_2) = g(G \oplus [\ \oplus_{i=1}^{n}\ K_2]) = \begin{cases} k & if\ G\ is\ bipartite, \\ min\{2p, q\} & otherwise, \end{cases}$$

where C_p and C_q are the minimal odd and even cycles in a non-bipartite graph G, respectively.

Proof. First, we discuss the result when $n = 1$.
Case 1. Assume G is bipartite. From Theorem 2.2 (with $n = 1$), we have for the duplicate graph $G \oplus K_2 = 2G$. Consequently, $g(G \oplus K_2) = k$.
Case 2. Suppose G is not bipartite. Then G contains an odd cycle. Let C_p for $p \geq 3$, be a minimal odd cycle in G.

Now, there are two possibilities to discuss:
2.1. If G is free-from even cycles, then $C_p \oplus K_2$ contains an even cycle C_{2p} in $G \oplus K_2$.
2.2. If G contains a minimal even cycle C_q, $q \geq 4$, then $C_q \oplus K_2 = 2C_q$ appears in $G \oplus K_2$.
From the above possibilities, it follows that $g(G \oplus K_2)$ is the minimum of $2p$ and q. Thus, $g(G \oplus K_2) = min\{2p, q\}$.
Finally, consider the result when $n \geq 2$. The result follows immediately if we proceed as above by applying Theorem 2.2 or 2.4 repeatedly.

□

Next, we derive a formula (which is proposed in [4]) for computing the number of triangles in the tensor product of two graphs. For this, firstly we establish the following lemma.

Lemma 7.2. *Let G_k $(1 \leq k \leq n\ ;\ n \geq 2)$ be a connected graph. Then the product $\oplus_{k=1}^{n} G_k$ contains a triangle if and only if each G_k contains a triangle.*

Proof. Now, we discuss the case when $n = 2$. Suppose $G_1 \oplus G_2$ contains a triangle T, and let $(a_1, b_1), (a_2, b_2)$ and (a_3, b_3) be any three vertices of T. By definition, $(a_1, b_1)(a_2, b_2), (a_2, b_2)(a_3, b_3)$ and $(a_3, b_3)(a_1, b_1)$ are the edges of T in $G_1 \oplus G_2$ if and only if the edges: $a_1a_2,\ a_2a_3$ and a_3a_1 constitute a triangle T_1 in G_1 and also the edges : $b_1b_2,\ b_2b_3$ and b_3b_1 constitute a triangle T_2 in G_2. But this is so if and only if both G_1 and G_2 have triangles T_1 and T_2, respectively. Finally, we discuss the case when $n \geq 3$. The result follows immediately if we proceed by applying induction on the number of factors. □

Theorem 7.3. *Let G_i $(1 \leq i \leq 2)$ be a connected graph having the number of triangles n_i. Then the product $G_1 \oplus G_2$ contains $6n_1n_2$ triangles.*

Proof. First, let us compute the actual number of triangles in the product $T_1 \oplus T_2$, when T_i is any triangle in G_i (for $i = 1, 2$). It is easy to see that there are exactly 6 distinct triangles in $T_1 \oplus T_2$. But each G_i contains n_i triangles. Consequently, the product $G_1 \oplus G_2$ contains $6n_1n_2$ triangles, and there are no more other triangles because of Lemma 7.2. □

The immediate consequence of the above theorem is the following corollary.

Corollary 7.4. *Let G_k $(1 \leq k \leq n\ ;\ n \geq 2)$ be a connected graph having the number of triangles n_k. Then the product $\oplus_{k=1}^{n} G_k$ contains $6^{n-1}(\Pi_{k=1}^{n} n_k)$ triangles.*

Corollary 7.5. *The number of triangles in $K_m \oplus K_n$ is $\frac{1}{6}[\ mn(m-1)(n-1)(m-2)(n-2)]$.*

Proof. We know that the number of triangles in $K_p = pC_3$. Therefore from Theorem 7.3, the number of triangles in $K_m \oplus K_n$ is $6(mC_3)(nC_3) = \frac{1}{6}[\ mn(m-1)(n-1)(m-2)(n-2)]$. □

Finally to determine the girth of the tensor product of graphs, we need to establish the following lemma.

Lemma 7.6. *Let G_k $(1 \leq k \leq n\ ;\ n \geq 2)$ be a connected, triangle-free graph such that each G_k contains an induced subgraph isomorphic to P_3. Then $g(\oplus_{k=1}^{n}\ G_k) = 4$.*

Proof. We discuss the case when $n = 2$. Let a_i $(1 \leq i \leq 3)$ and b_i $(1 \leq i \leq 3)$ be the vertices of a subgraph isomorphic to P_3 in G_1 and G_2, respectively. Then the subgraph $< \{a_1, a_2, a_3\} > \oplus < \{b_1, b_2, b_3\} >$ is isomorphic to $P_3 \oplus P_3$ in $G_1 \oplus G_2$. It is easy to see that $P_3 \oplus P_3 = K_{1,4} \cup C_4$. Immediately, a 4-cycle C_4 appears as a subgraph in $G_1 \oplus G_2$. However from Lemma 7.2, there is no triangle in $G_1 \oplus G_2$. Consequently, C_4 is the smallest cycle in $G_1 \oplus G_2$.

Therefore, $g(G_1 \oplus G_2) = 4$.
When $n \geq 3$, the result follows easily if we proceed by induction on the number of factors. □

The following result gives the girth of tensor product of arbitrarily many graphs.

Theorem 7.7. *Let G_k $(1 \leq k \leq n\ ;\ n \geq 2)$ be a connected graph of order ≥ 3. Then $g(\oplus_{k=1}^{n} G_k)$ is either 3 or 4.*

Proof. We discuss three cases when $n = 2$.
Case 1. Suppose both G_1 and G_2 have triangles. By Lemma 7.2, $G_1 \oplus G_2$ contains a triangle. Hence, $g(G_1 \oplus G_2) = 3$.
Case 2. Suppose one of G_1 and G_2 is triangle-free. Without loss of generality, we assume that G_1 contains a triangle, and G_2 has an induced subgraph isomorphic to P_3. It is easy to check that $K_3 \oplus P_3$ contains a 4-cycle C_4. Consequently, this C_4 appears in $(G_1 \oplus G_2)$. However again by Lemma 7.2, $G_1 \oplus G_2$ is triangle-free. This implies that C_4 is the smallest cycle in $G_1 \oplus G_2$. Therefore, $g(G_1 \oplus G_2) = 4$.
Case 3. Suppose G_1 and G_2 are triangle-free. Then each G_1 and G_2 contains an induced subgraph isomorphic to P_3. From Lemma 7.6, $g(G_1 \oplus G_2) = 4$.
From the above cases, it follows that $g(G_1 \oplus G_2) = 3$ or 4.
When $n \geq 3$, It is not difficult to prove the result if we proceed by induction on the number of factors. □

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