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A Successive Numerical Scheme for Some Classes of Volterra-Fredholm Integral Equations

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Abstract. In this paper, a reliable iterative approach, for solving a wide range of linear and nonlinear Volterra-Fredholm integral equations is established. First the approach considers a discretized form of the integral terms where considering some conditions on the kernel of the integral equation it is proved that solution of the discretized form converges to the exact solution of the problem. Then the solution of the discretized form is approximated by an iterative scheme. Comparison of the approximate solution with exact solution shows that the used approach is easy and practical for some classes of linear and nonlinear Volterra-Fredholm integral equations.

Keywords: Volterra-Fredholm integral equation, Discretization, Approximation.

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1. Introduction

Integral equations are an important branch of modern mathematics and arise frequently in many applied areas including engineering, mechanics, physics,

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chemistry, astronomy, biology, economics, potential theory and electrostatics [7, 14].

Volterra-Fredholm integral equations are usually difficult to solve analytically and so the numerical approaches are created to overcome the complexities of analytical methods.. Extracting the numerical solutions of Volterra-Fredholm integral equation is a well-studied problem and a large variety of numerical methods have been developed to obtain rapidly and accurately approximate solutions. Collocation methods [5, 11], Taylor series [15], toeplitz matrix method [4], orthogonal polynomial method of type Legendre polynomials [1, 2], particular trapezoidal Nystrom [9, 8] and Adomian decomposition method [6] are several of many approaches that have previously been considered.

In this study, we tend to present a numerical scheme for extracting approximate solutions for the Volterra-Fredholm integral equations as

$$x(s) = f(s) + \int_a^s g(s,t,x(t))dt + \int_a^b h(s,t,x(t))dt, \text{ a.e. on } [a,b], \tag{1.1}$$

by an iterative method and it is supposed that the discussed integral equations have at least one solution. At the beginning, we transform the equation into a discretized form.

2. Integral Equation Transformation

Let $\triangle = \{a = s_0, s_1, \cdots, s_{n-1}, s_n = b\}$ be an equidistant partition of $[a,b]$ where $h = s_{i+1} - s_i$, $i = 0,1,\cdots,n-1$ is the discretization parameter of the partition. Now, if $x^*(t)$ be an analytical solution of (1.1), then for the partition $\triangle$ on $[a,b]$, we have

$$x^*(s_i) = f(s_i) + \int_a^{s_i} g(s_i,t,x^*(t))dt + \int_a^b h(s_i,t,x^*(t))dt, \; i = 0,1,\cdots,n. \tag{2.1}$$

In (2.1), the integral term can be estimated by a numerical method of integration, e.g. Newton-Cotes methods. Therefore, by taking equidistant partition $\triangle$, as above with $h = t_{i+1} - t_i$, $i = 0,1,\cdots,n-1$ and also the known weights w_{i_j}, $j = 0,1,\cdots,i$, for interval $[a,s_i]$ and w_l, $l = 0,1,\cdots,n$, for interval $[a,b]$, equality (2.1) can be written as,

$$x_i^* = f_i + \sum_{j=0}^{i} w_{i_j} g(s_i,t_j,x_j^*) + O(h^{\nu_1}) + \sum_{l=0}^{n} w_l h(s_i,t_l,x_l^*) + O(h^{\nu_2}), \tag{2.2}$$

where $i = 0,1,\cdots,n, x_i^* = x^*(s_i)$, $f_i = f(s_i)$, $i = 0,1,\cdots,n$, and ν_1,ν_2 depend upon the used method of Newton-Cotes for estimating of the integrals in (2.1). From (2.2) we have

$$x_i^* = f_i + \sum_{j=0}^{i} w_{i_j} g(s_i,t_j,x_j^*) + \sum_{l=0}^{n} w_l h(s_i,t_l,x_l^*) + O(h^{\nu}), \; i = 0,1,\cdots,n, \tag{2.3}$$

where $\nu = min(\nu_1, \nu_2)$.
For partition $\triangle$, we consider a nonlinear equations system obtained by neglecting the truncation error of (2.1), as follows,

$$\xi_i = f_i + \sum_{j=0}^{i} w_{i_j} g(s_i, t_j, \xi_j) + \sum_{l=0}^{n} w_l h(s_i, t_l, \xi_l),\ i = 0, 1, \cdots, n, \tag{2.4}$$

and suppose that the exact solution of nonlinear system (2.4) is n-tuple $(\xi_0^*, \xi_1^*, \cdots, \xi_n^*)$. In the following proposition, we seek for the conditions of vanishing $\|x^* - \xi^*\|_\infty$ where x^* and ξ^* are the following vectors:

$$x^* = (x_0^*, x_1^*, \cdots, x_n^*)^T,\ \xi^* = (\xi_0^*, \xi_1^*, \cdots, \xi_n^*)^T.$$

Proposition 2.1. *Suppose,*
(i) $g(s,t,x(s)), h(s,t,x(s)) \in C([a,b] \times [a,b] \times I\!R)$,
(ii) $g_x(s,t,x(s)), h_x(s,t,x(s))$ *exist on* $[a,b] \times [a,b] \times I\!R$ *and* $\gamma_1 < \frac{1}{b-a}$, $\gamma_2 < \frac{1}{b-a}$, *where*

$$\gamma_1 = \sup_{s,t \in [a,b]} |g_x(s,t,x(s))|, \gamma_2 = \sup_{s,t \in [a,b]} |h_x(s,t,x(s))|.$$

Then

$$\|x^* - \xi^*\|_\infty \leq \frac{|O(h^\nu)|}{1 - (\gamma_1 + \gamma_2)(b-a)}. \tag{2.5}$$

Proof. Let

$$|x_p^* - \xi_p^*| = \|x^* - \xi^*\|_\infty,$$

in which $0 \leq p \leq n$. By (2.3) and (2.4), we have

$$x_p^* - \xi_p^* = \sum_{j=0}^{p} w_{p_j} (g(s_p, t_j, x_j^*) - g(s_p, t_j, \xi_j^*)) + \sum_{l=0}^{n} w_l (h(s_p, t_l, x_l^*) - g(s_p, t_l, \xi_l^*)) + O(h^\nu).$$

According to *(ii)*

$$g(s_p, t_j, x_j^*) - g(s_p, t_j, \xi_j^*) = \frac{\partial g}{\partial x}(s_p, t_j, \eta_j)(x_j^* - \xi_j^*),\ j = 0, 1, \cdots, n,$$

$$h(s_p, t_l, x_l^*) - h(s_p, t_l, \xi_l^*) = \frac{\partial h}{\partial x}(s_p, t_l, \zeta_l)(x_l^* - \xi_l^*),\ l = 0, 1, \cdots, n,$$

where for each $j = 0, 1, \cdots, n$, η_j and ζ_j are real numbers between x_j^* and ξ_j^*. Again by *(ii)* and the above equalities, we conclude that

$$|x_p^* - \xi_p^*| \leq \gamma_1 \sum_{j=0}^{p} w_{p_j} |x_j^* - \xi_j^*| + \gamma_2 \sum_{l=0}^{n} w_l |x_l^* - \xi_l^*| + |O(h^\nu)|$$

$$\leq \gamma_1 |x_p^* - \xi_p^*| \sum_{j=0}^{p} w_{p_j} + \gamma_2 |x_p^* - \xi_p^*| \sum_{l=0}^{n} w_l + |O(h^\nu)|.$$

Since $\sum_{j=0}^{p} w_{p_j} \leq b-a$ and $\sum_{l=0}^{n} w_l = b-a$, thus

$$|x_p^* - \xi_p^*| \leq \frac{|O(h^\nu)|}{1-(\gamma_1+\gamma_2)(b-a)}.$$

□

Equation (2.5) leads to the following corollary.

Corollary 2.2. $\|x^* - \xi^*\|_\infty$ *vanishes when* $h \to 0$.

So far, we came to the nonlinear equations system (2.4) with a special form that let us offer a numerical approach for detecting the approximate solution.

3. The Numerical Approach

Iterative methods are widely used for finding approximate solution of nonlinear equations systems [13]; The nonlinear equations system (2.4) also has a structure that permits to approximate its solution by an iterative method. For this purpose, we apply a successive substitution, similar to Gauss-Seidel method of solving linear equations systems, and thereby define an iterative process leading to the sequence of vectors $\{\xi^{(k)}\}$, where the components of the vectors satisfy the iteration formula,

$$\xi_i^{(k+1)} = f_i + \sum_{j=0}^{i} w_{i_j} g(s_i, t_j, \xi_j^{(k)}) + \sum_{l=0}^{n} w_l h(s_i, t_l, \xi_l^{(k)}), \tag{3.1}$$

where$i = 0, 1, \cdots, n,\ \ k = 0, 1, \cdots$. However, we should first study the conditions that guarantee the convergence of the sequence $\{\xi^{(k)}\}$.

Theorem 3.1. *Considering assumptions of Proposition 2.1, the produced sequence* $\{\xi^{(k)}\}$ *from the iteration process (3.1) tends to the exact solution of (2.4), say* ξ^**, for any arbitrary initial vector* $\xi^{(0)}$ *.*

Proof. By (2.4) and (3.1) we have,

$$\begin{aligned}\xi_i^{(k+1)} - \xi_i^* = &\sum_{j=0}^{i} w_{i_j}(g(s_i, t_j, \xi_j^{(k)}) - g(s_i, t_j, \xi_j^*)) \\ &+ \sum_{l=0}^{n} w_l(h(s_i, t_l, \xi_l^{(k)}) - h(s_i, t_l, \xi_l^*)),\ i = 0, 1, \cdots, n,\end{aligned}$$

and according to the condition (ii) of Proposition 2.1,

$$\begin{aligned}\xi_i^{(k+1)} - \xi_i^* = &\sum_{j=0}^{i} w_{i_j} \frac{\partial g}{\partial x}(s_i, t_j, \eta_j^{(k)})(\xi_j^{(k)} - \xi_j^*) \\ &+ \sum_{l=0}^{n} w_l \frac{\partial h}{\partial x}(s_i, t_l, \zeta_l^{(k)})(\xi_l^{(k)} - \xi_l^*),\ i = 0, 1, \cdots, n,\end{aligned}$$

where $\eta_j^{(k)}$ and $\zeta_j^{(k)}$ are real numbers between $\xi_j^{(k)}$ and ξ_j^* for $j = 0, 1, \cdots, n$. Thus, for each $i = 0, 1, \cdots, n$, one may obtain the following inequalities

$$\begin{aligned} |\xi_i^{(k+1)} - \xi_i^*| &\leq \|\xi^{(k)} - \xi^*\|_\infty \sum_{j=0}^{i} w_{i_j} |\frac{\partial g}{\partial x}(s_i, t_j, \eta_j^{(k)})| \\ &\quad + \|\xi^{(k)} - \xi^*\|_\infty \sum_{l=0}^{n} w_l |\frac{\partial h}{\partial x}(s_i, t_l, \zeta_l^{(k)})| \\ &\leq \gamma_1 \|\xi^{(k)} - \xi^*\|_\infty \sum_{j=0}^{i} w_{i_j} + \gamma_2 \|\xi^{(k)} - \xi^*\|_\infty \sum_{l=0}^{n} w_l, \end{aligned}$$

where $i = 0, 1, \cdots, n$. By setting $\lambda_1 = \gamma_1(b-a)$ and $\lambda_2 = \gamma_2(b-a)$ we conclude that

$$\|\xi^{(k+1)} - \xi^*\|_\infty \leq \lambda_1 \|\xi^{(k)} - \xi^*\|_\infty + \lambda_2 \|\xi^{(k)} - \xi^*\|_\infty \leq \lambda \|\xi^{(k)} - \xi^*\|_\infty,$$

where $\lambda = \max\{\lambda_1, \lambda_2\}$. By induction on k, we get

$$\|\xi^{(k+1)} - \xi^*\|_\infty \leq \lambda^k \|\xi^{(0)} - \xi^*\|_\infty,$$

for each $k = 0, 1, \cdots$. Since $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, thus $0 < \lambda < 1$ and $k \to +\infty$ implies that $\|\xi^{(k+1)} - \xi^*\|_\infty$ vanishes. □

4. Algorithm of the Approach

In this section, we try to propose an algorithm on the basis of the above discussions and suppose that we face with the Volterran-Fredholm integral equation (1.1), where its kernels satisfy the conditions of Proposition 2.1. This algorithm is presented in two stages, initialization step and main steps.

Initialization step:

Choose $\epsilon > 0$, an equidistant partition $\triangle = \{a = s_0 = t_0, s_1 = t_1, \cdots, s_{n-1} = t_{n-1}, s_n = t_n = b\}$ on $[a, b]$ with the step size $h = s_{i+1} - s_i,\ i = 0, 1, \cdots, n-1$ and an initial vector $\xi^{(0)} = (\xi_0^{(0)}, \xi_1^{(0)}, \cdots, \xi_n^{(0)})^T$. Set $k = 0$ and go to the main steps.

Main steps:

Step 1. Compute $\xi^{(k+1)}$ by (3.1), and go to Step 2.

Step 2. Compute $\|\xi^{(k+1)} - \xi^{(k)}\|_\infty$ and go to Step 3.

Step 3. If $\|\xi^{(k+1)} - \xi^{(k)}\|_\infty < \epsilon$, stop; Otherwise, set $k = k + 1$ and go to step 1.

In the next section we present some numerical examples to show the efficiency of this approach.

5. Numerical Examples

We suppose $x^*(s)$ be exact solution of Volterran-Fredholm integral equation (1.1) and $\hat{\xi}_i,\ i = 0, 1, \cdots, n$ be a solution obtained by applying the given

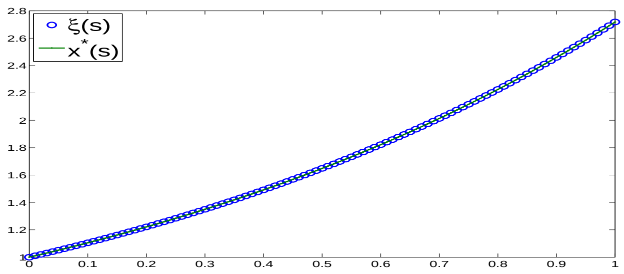

FIGURE 1. circle-wise curve shows approximate solution and dash-dotted curve shows exact solution.

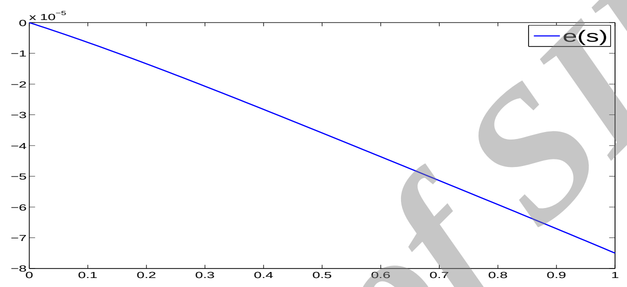

FIGURE 2. The error function of Example 5.1.

algorithm with a known $\epsilon > 0$ and partition $\triangle$. To compare the solutions we define a discrete error function

$$e_{\triangle}(s_i) = x^*(s_i) - \hat{\xi}(s_i), \ i = 0, 1, \cdots, n. \tag{5.1}$$

Example 5.1. In this example, we apply the given scheme to a Volterran-Fredholm integral equation as follows:

$$x(s) = e^s - \frac{s}{2}(e^{2s} + 1) + \int_0^s \frac{se^{2s}}{x^2(t)}dt + \int_0^1 stx(t)dt, \ s \in [0, 1].$$

This integral equation has analytical solution $x(s) = e^s$ on $[0, 1]$. We take $\epsilon = 10^{-6}$ and a partition with the discretization parameter $h = \frac{1}{100}$. The initial vector $\xi^{(0)} = \mathbf{1}$ is considered for starting algorithm. One can compare the exact and approximate solutions of the integral equation in Fig. 1. The error function (5.1) also can be seen in Fig. 2.

Example 5.2. In this example, a Volterran-Fredholm integral equation as follows:

$$x(s) = \frac{s^2}{12}(11 - 4s^2) + \int_0^s 4(s-t)x(t)dt + \int_0^{0.5} 2s^2 x(t)dt, \ s \in [0, 0.5],$$

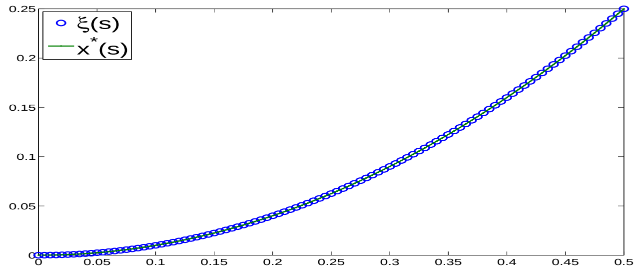

FIGURE 3. circle-wise curve shows approximate solution and dash-dotted curve shows exact solution.

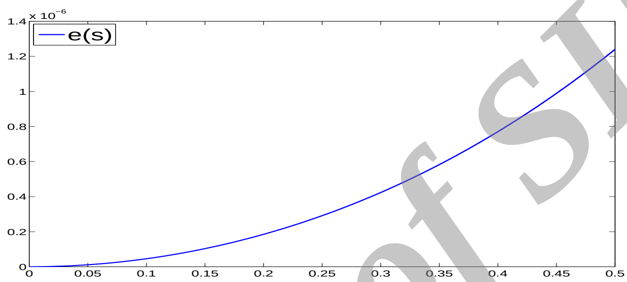

FIGURE 4. The error function of Example 5.2.

has been considered. This integral equation has analytical solution $x(s) = s^2$ on $[0, 0.5]$. Taking $\epsilon = 10^{-6}$, $h = \frac{1}{100}$ and $\xi^{(0)} = \mathbf{1}$. The exact and approximate solutions of the integral equation have been compared in Fig. 3. The error function (5.1) also can be seen in Fig. 4.

Example 5.3. In this example, a Volterra-Fredholm integral equation as follows:

$$x(s) = \cos(2\pi s) - \frac{3s}{4}\sin(4\pi s) + \int_0^s 2\pi s\cos(2\pi s)x(t)dt + \int_0^{0.5} s\sin(4\pi s + 2\pi t)x(t)dt, \ \ s \in [0, 0.5].$$

is considered where $x(s) = cos(2\pi s)$ is analytical solution of integral equation on $[0, 0.5]$, $\epsilon = 10^{-6}$, $h = \frac{1}{100}$ and $\xi^{(0)} = \mathbf{1}$. One can compare the exact and approximate solutions of the integral equation in Fig. 5. The error function (5.1) also can be seen in Fig. 6.

Example 5.4. Consider the nonlinear Volterran-Fredholm integral equation

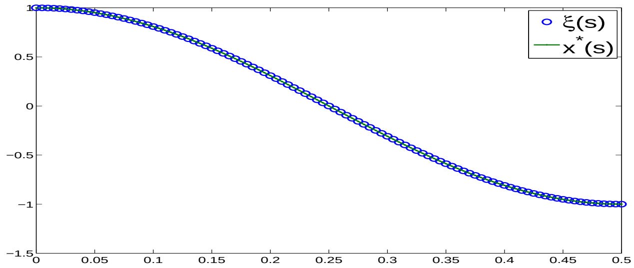

FIGURE 5. circle-wise curve shows approximate solution and dash-dotted curve shows exact solution.

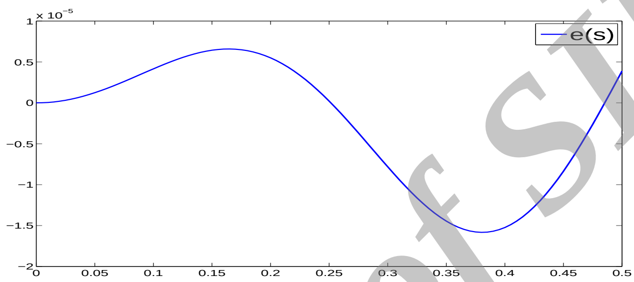

FIGURE 6. The error function of Example 5.3.

of Hammerstein type as follows [15]:

$$x(s) = -\frac{1}{30}s^6 + \frac{1}{3}s^4 - s^2 + \frac{5}{3}s - \frac{5}{4} + \int_0^s (s-t)x^2(t)dt + \int_0^1 (t+s)x(t)dt,\ s \in [0,1].$$

Considering the analytical solution $x(s) = s^2 - 2$ on $[0,1]$, $\epsilon = 10^{-6}$, $h = \frac{1}{100}$ and $\xi^{(0)} = \mathbf{1}$. One can observe comparison of the exact and approximate solutions of the integral equation in Fig. 7. The error function (5.1) also can be seen in Fig. 8.

Example 5.5. In this example, we apply our method to a Volterran-Fredholm integral equation as follows:

$$x(s) = e^s(1-s) + \frac{\pi}{4}s - stan^{-1}(e^s) + \int_0^s \frac{sx(t)}{1+x^2(t)}dt + \int_0^1 ste^s x(t)dt,\ s \in [0,1],$$

where the analytical solution is $x(s) = e^s$ on $[0,1]$. Taking $\epsilon = 10^{-6}$, $h = \frac{1}{100}$ and $\xi^{(0)} = \mathbf{1}$. One can compare the exact and approximate solutions of the integral equation in Fig. 9. Also the error function (5.1) has been shown in Fig. 10.

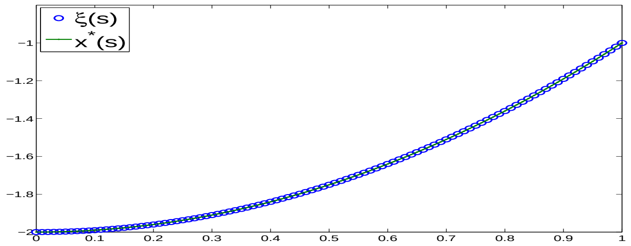

FIGURE 7. circle-wise curve shows approximate solution and dash-dotted curve shows exact solution.

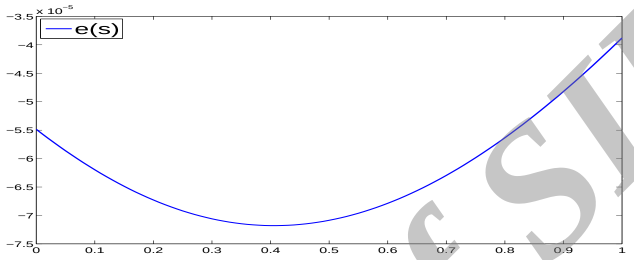

FIGURE 8. The error function of Example 5.4.

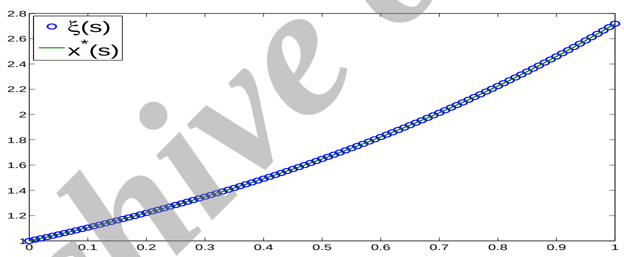

FIGURE 9. circle-wise curve shows approximate solution and dash-dotted curve shows exact solution.

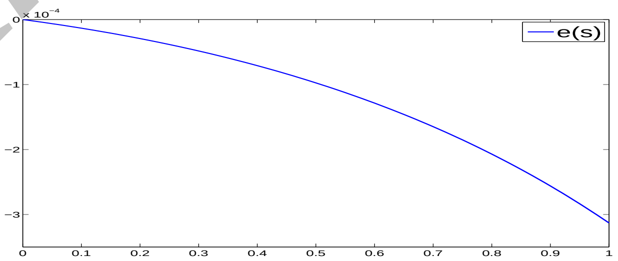

FIGURE 10. The error function of Example 5.5.

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