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p-Analog of the Semigroup Fourier-Steiltjes Algebras

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Abstract. In this paper we define the p-analog of the restricted representations and the p-analog of the Fourier–Stieltjes algebras on inverse semigroups and also we improve some results about Herz algebras on Clifford semigroups, and we give a necessary and sufficient condition for amenability of these algebras on Clifford semigroups.

Keywords: Restricted fourier–Stieltjes algebras, Restricted inverse semigroup, Restricted representations, QSL_p-spaces, p-Analog of the Fourier–Stieltjes algebras.

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1. Introduction and Preliminaries

An inverse semigroup S is a discrete semigroup such that for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s, s^*ss^* = s^*$. The set $E(S)$ of idempotents of S consists of elements of the form ss^*, $s \in S$. Actually for each abstract inverse semigroup S there is a $*$-semigroup homomorphism from S into the inverse semigroup of partial isometries on some Hilbert space[18].

Dunkl and Ramirez in [8] and T. M. Lau in [15] attempted to define a suitable substitution for Fourier and Fourier–Stieltjes algebras on semigroups. Each definition has its own difficulties. Amini and Medghalchi introduced and extensively studied the theory of restricted semigroups and restricted representations and restricted Fourier and Fourier–Stieltjes algebras, $A_{r,e}(S), B_{r,e}(S)$

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in [2] and [3]. Also they studied the spectrum of the Fourier Stieltjes algebra for a unital foundation topological $*$-semigroup in [4]. In this section we mention some of their results.

Throughout this paper S is an inverse semigroup. Given $x, y \in S$, the restricted product of x, y is xy if $x^*x = yy^*$, and undefined, otherwise. The set S with its restricted product forms a groupoid [16, 3.1.4] which is called the associated groupoid of S. If we adjoin a zero element 0 to this groupoid, and put $0^* = 0$, we will have an inverse semigroup S_r with the multiplication rule

$$x \bullet y = \begin{cases} xy & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in S \cup \{0\}$, which is called the restricted semigroup of S. A restricted representation $\{\pi, \mathcal{H}_\pi\}$ of S is a map $\pi : S \longrightarrow \mathcal{B}(\mathcal{H}_\pi)$ such that $\pi(x^*) = \pi(x)^*$ $(x \in S)$ and

$$\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in S$. Let $\Sigma_r = \Sigma_r(S)$ be the family of all restricted representations π of S with $\|\pi\| \leq 1$. Now it is clear that, via a canonical identification, $\Sigma_r(S) = \Sigma_0(S_r)$, consist of all $\pi \in \Sigma(S_r)$ with $\pi(0) = 0$, where the notation Σ has been used for all $*$-homomorphism from S into $\mathcal{B}(\mathcal{H})$ [2]. One of the central concepts in the analytic theory of inverse semigroups is the left regular representation $\lambda : S \longrightarrow \mathcal{B}(\ell^2(S))$ defined by

$$\lambda(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* \geq yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $\xi \in \ell^2(S), x, y \in S$. The restricted left regular representation $\lambda_r : S \longrightarrow \mathcal{B}(\ell^2(S))$ is defined in [2] by

$$\lambda_r(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $\xi \in \ell^2(S), x, y \in S$. The main objective of [2] is to change the convolution product on the semigroup algebra to restore the relation with the left regular representation.

For each $f, g \in \ell^1(S)$, define

$$(f \bullet g)(x) = \sum_{x^*x = yy^*} f(xy)g(y^*) \quad (x \in S),$$

and for all $x \in S$, $\tilde{f}(x) = \overline{f(x^*)}$. $\ell^1_r(S) := (\ell^1(S), \bullet, \tilde{ })$ is a Banach *-algebra with an approximate identity . The left regular representation λ_r lifts to a faithful representation $\tilde{\lambda}$ of $\ell^1_r(S)$. We call the completion $C^*_{\lambda_r}(S)$ of $\ell^1_r(S)$ with the norm $\|\cdot\|_{\lambda_r} := \|\tilde{\lambda_r}(\cdot)\|$ which is a C^*-norm on $\ell^1_r(S)$, the restricted reduced C^*-algebra and its completion with the norm $\|\cdot\|_{\Sigma_r} := \sup\{\|\tilde{\pi}(\cdot)\| \ , \pi \in \Sigma(S_r)\}$ the restricted full C^*-algebra and show it by $C^*_r(S)$. The dual space of C^*-algebra

$C_r^*(S)$ is a unital Banach algebra which is called the restricted Fourier-Stieltjes algebra and is denoted by $B_{r,e}(S)$. The closure of the set of finitely support functions in $B_{r,e}(S)$ is called the restricted Fourier algebra and is denoted by $A_{r,e}(S)$[2].

In [10], Figà-Talamanca introduced a natural generalization of the Fourier algebra, for a compact abelian group G, by replacing $L_2(G)$ by $L_p(G)$. In [11], Herz extended the notion to an arbitrary group, to get the commutative Banach algebra $A_p(G)$, called the Figà–Talamanca–Herz algebra. Figà–Talamanca–Herz algebra and Eymard's Fourier algebra have very similar behavior. For example, Leptin's theorem is valid: G is amenable if and only if $A_p(G)$ has a bounded approximate identity [12]. The p-analog, $B_p(G)$ of the Fourier–Stieltjes algebra is defined as the multiplier algebra of $A_p(G)$, by some authors, as mentioned in [5] and [19]. Runde in [20] defined and studied $B_p(G)$, the p-analog of the Fourier–Stieltjes algebra on the locally compact group G. He developed the theory of representations and defined the suitable coefficient functions on them.

For $p \in (1, \infty)$, Medghalchi and Pourmahmood Aghababa developed the theory of restricted representations on $\ell_p(S)$ and defined the Banach algebra of p-pseudomeasures $PM_p(S)$ and the Figa- Talamanca-Herz algebras $A_p(S)$. They showed that $A_q(S)^* = PM_p(S)$ for dual pairs p, q. They characterized $PM_p(S)$ and $A_p(S)$ for Clifford semigroups, in the sense of p-pseudomeasures and Figa–Talamanca–Herz algebras of maximal semigroups of S, respectively[17].

Amini also worked on quantum version of Fourier transforms in [1].

In this paper we will combine what Medghalchi–Pourmahmood Aghababa and Runde have done. We will define the restricted representations on QSL_p-spaces and the p-analog of the Fourier-Stieltjes algebra on the restricted inverse semigroup.

Section 2 is a review of the theory of QSL_p-spaces. In Section 3 we define the restricted representations on QSL_p-spaces and study their tensor product. In Sections 4 and 5 we construct the p-analog of the restricted Fourier–Stieltjes algebra and study its order structure. The last section will be about Clifford semigroups and the p-analog of their restricted Fourier–Stieltjes algebra. Some new results which improves the results of [17] and [22] will be given in Section 6.

2. Review of the Theory of QSL_p-spaces

This section is a review of the paper of Runde [20].

Definition 2.1. A Banach space $\mathcal{E}$ is called

(i) an L_p-space if it is of the form $L_p(X)$, for some measure space X.

(ii) a QSL_p-space if it is isometrically isomorphic to a quotient of a subspace of an L_p-space (or equivalently, a subspace of a quotient of an L_p-space [20, Section 1, Remark 1]).

If E is a QSL_p-space and if $p' \in (1, \infty)$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$, the dual space E^* is an $QSL_{p'}$-spaces. In particular, every QSL_p-space is reflexive.

By [14, Theorem 2], the QSL_p-spaces are precisely the p-spaces in the sense of [11], i.e. those Banach spaces E such that for any two measure spaces X and Y the amplification map

$$B(L_p(X), L_p(Y)) \to B(Lp(X,E), Lp(Y,E)), T \to T \otimes id_E$$

is an isometry. In particular, an L_q-space is a QSL_p-space if and only if $2 \leq q \leq p$ or $p \leq q \leq 2$. Consequently, if $2 \leq q \leq p$ or $p \leq q \leq 2$, then every QSL_q-space is a QSL_p-space.

Runde equipped the algebraic tensor product of two QSL_p-spaces with a suitable norm, which comes in the following.

Theorem 2.2. [20, Theorem 3.1] *Let E and F be QSL_p-spaces. Then there is a norm $\|\cdot\|_p$ on the algebraic tensor product $E \otimes F$ such that:*

(i) *$\|\cdot\|_p$ dominates the injective norm;*

(ii) *$\|\cdot\|_p$ is a cross norm;*

(iii) *the completion $E\tilde{\otimes}_p F$ of $E \otimes F$ with respect to $\|\cdot\|_p$ is a QSL_p-space. The Banach space $E\tilde{\otimes}_p F$ will be called p-projective tensor product of E and F.*

3. Restricted Representation on a Banach space

In this section we give an analog of the theory of group representations on a Hilbert space for the restricted representations for an inverse semigroup on a QSL_p-space.

Definition 3.1. A representation of a discrete inverse semigroup S on a Banach space E is a pair (π, E) consisting of a map $\pi : S \to B(E)$ satisfying $\pi(x)\pi(y) = \pi(xy)$, for $x, y \in S$ and $\|\pi(x)\| \leq 1$, for all $x \in S$.

Definition 3.2. A restricted representation of a discrete inverse semigroup S on a Banach space E is a pair (π, E) consisting of a map $\pi : S \to B(E)$ satisfying

$$\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in S$, and $\|\pi(x)\| \leq 1$, for all $x \in S$.

Definition 3.3. Let S be an inverse semigroup, and let (π, E) and (ρ, F) be restricted representations of S, then these restricted representations are said to be equivalent if there is a surjective isometry $T : E \to F$ such that

$$T\pi(x)T^{-1} = \rho(x), \quad (x \in S).$$

For any inverse semigroup S and $p \in (1,\infty)$, we denote by $\Sigma_{p,r}(S)$ the collection of all (equivalence classes) of restricted representations of S on a QSL_p-space.

Remark 3.4. By [17] for $p \in (1,\infty)$ the restricted left regular representation $\lambda_p : S \longrightarrow B(\ell^p(S))$

$$\lambda_p(s)(\delta_t) = \begin{cases} \delta_{st} & \text{if } s^*s = tt^*, \\ 0 & \text{otherwise} \end{cases}$$

for $s,t \in S$ is a restricted representation so it belongs to $\Sigma_{p,r}(S)$.

The following propositions are easy to check, similar to [2].

Proposition 3.5. *For an inverse semigroup S and its related restricted semigroup S_r, each restricted representation of S on a Banach space is a representation on S_r which is zero on $0 \in S_r$, i.e. it is multiplicative with respect to the restricted multiplication.*

Proposition 3.6. *For an inverse semigroup S, each restricted representation π of S on a Banach space lifts to a representation of $\ell^1_r(S)$, via*

$$\tilde{\pi}(f) = \sum_{x\in S} f(x)\pi(x).$$

4. Banach Algebra $B_{p,r}(S)$

In this section we define the p-analog of the Fourier–Stieltjes algebra on a inverse semigroup. We show that for $p = 2$ we get the known algebra $B_{r,e}(S)$, defined in [2].

Theorem 4.1. *Let $(\pi, E), (\rho, F) \in \Sigma_{p,r}(S)$ then $(\pi \otimes \rho, E\tilde{\otimes}_p F) \in \Sigma_{p,r}(S)$.*

Proof. By the definition of $\pi\otimes\rho$ we have $\pi\otimes\rho(x)(\xi\otimes\eta) = \pi(x)\xi\otimes\rho(x)\eta$. For $x,y \in S$, $x^*x = yy^*$,

$$\begin{aligned} \pi\otimes\rho(xy)(\xi\otimes\eta) &= \pi(xy)\xi\otimes\rho(xy)\eta \\ &= \pi(x)\pi(y)\xi\otimes\rho(x)\rho(y)\eta \\ &= \pi(x)(\pi(y)\xi)\otimes\rho(x)(\rho(y)\eta) \\ &= \pi\otimes\rho(x)(\pi(y)\xi\otimes\rho(y)\eta) \\ &= \pi\otimes\rho(x)\pi\otimes\rho(y)(\xi\otimes\eta) \end{aligned}$$

when $x^*x \neq yy^*$

$$\begin{aligned} \pi\otimes\rho(x)\pi\otimes\rho(y)(\xi\otimes\eta) &= \pi\otimes\rho(x)(\pi(y)\xi\otimes\rho(y)\eta) \\ &= \pi(x)\left(\pi(y)\xi\right)\otimes\rho(x)\left(\rho(y)\eta\right) \end{aligned}$$

which is equal to zero. Now it is enough to show that for $\pi(x) \in B(E)$ and $\rho(y) \in B(F)$, $\pi(x)\otimes\rho(y)$ could be extend to $E\tilde{\otimes}_p F$. This is shown as in the group case [20, Therem 3.1]. □

Definition 4.2. Let S be an invesre semigroup, and let $(\pi, E) \in \Sigma_{p,r}(S)$. A coefficient function of (π, E) is a function $f : S \longrightarrow \mathbb{C}$ of the form

$$f(x) = < \pi(x)\xi, \phi > \quad (x \in S),$$

where $\xi \in E$ and $\phi \in E^*$.

Definition 4.3. Let S be an inverse semigroup, let $p \in (1, \infty)$, and let $q \in (1, \infty)$ be the dual scalar to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. We define

$$B_{p,r}(S) := \{f : S \to \mathbb{C} : \, f \, \text{is a coefficient function of some} \, (\pi, E) \in \Sigma_{q,r}(S)\}$$

Proposition 4.4. *Let S be an inverse semigroup, let $p \in (1, \infty)$, and let $q \in (1, \infty)$ be the dual scalar to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, and let $f : S \longrightarrow \mathbb{C}$ defined by*

$$f(x) = \sum_{n=1}^{\infty} < \pi_n(x)\xi_n, \phi_n >, \quad (x \in S),$$

where $((\pi_n, E_n))_{n=1}^{\infty}$, $(\xi_n)_{n=1}^{\infty}$, and $(\phi_n)_{n=1}^{\infty}$ are sequences with $(\pi_n, E_n) \in \Sigma_{q,r}(S)$, $\xi_n \in E_n$, and $\phi_n \in E_n^*$, for $n \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < \infty.$$

Then f lies in $B_{p,r}(S)$.

Proof. The proof is similar to [20]. Without loss of generality, we may suppose that

$$\sum_{i=1}^{\infty} \|\xi_n\|_q < \infty, \quad \text{and} \sum_{i=1}^{\infty} \|\phi_n\|_p < \infty.$$

Then $E := \ell_q - \oplus_{n=1}^{\infty} E_n$ is a QSL_q-space and for $\xi := (\xi_1, \xi_2, ...)$ and $\phi := (\phi_1, \phi_2, ...)$, we have $\xi \in E$ and $\phi \in E^*$. Now the map $\pi : S \longrightarrow B(E)$ with $\pi(x)\eta = (\pi_1(x)\eta, \pi_2(x)\eta, ...)$ is a restricted representation of S on E, and f is the coefficient function of π. □

Definition 4.5. [17, Definition 3.1]. Let S be an inverse semigroup and let $p, q \in (1, \infty)$ be dual pairs. The space $A_q(S)$ consists of those $u \in c_0(S)$ such that there are sequences $(f_n)_{n=1}^{\infty} \subseteq \ell_q(S)$ and $(g_n)_{n=1}^{\infty} \subseteq \ell_p(S)$ with $\sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_q \leq \infty$ and $u = \sum_{n=1}^{\infty} f_n \bullet \check{g_n}$. For $u \in A_q(S)$, let

$$\|u\| = \inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_p : u = \sum_{n=1}^{\infty} f_n \bullet \check{g_n} \right\}$$

Proposition 4.6. [17, Proposition 3.2]. *Let S be an inverse semigroup and let $p \in (1, \infty)$, then $A_p(S)$ is a Banach space and is the closure of finite support functions on S.*

Proposition 4.7. *Let S be an inverse semigroup, let $p \in (1,\infty)$. Then $B_{p,r}(S)$ is a linear subspace of $c_b(S)$ containing $A_p(S)$. Moreover, if $2 \leq q \leq p$ or $p \leq q \leq 2$, we have $B_{q,r}(S) \subseteq B_{p,r}(S)$.*

Proof. Every thing is easy to check, and is similar to [20]. $\square$

Definition 4.8. Let S be an inverse semigroup, and let (π, E) be a restricted representation of S on the Banach space E. Then (π, E) is called cyclic if there is $x \in E$ such that $\pi(\ell_r^1(S))x$ is dense in E. For $p \in (1,\infty)$, we let $Cyc_{p,r}(S) := \{(\pi, E) : (\pi, E)$ is a cyclic restricted representation on a QSL_p- space $E\}$.

Definition 4.9. Let S be an inverse semigroup, let $p, q \in (1,\infty)$ the dual scalars, and let $f \in B_{p,r}(S)$. We define $\|f\|_{B_{p,r}(S)}$ as the infimum over all expressions $\sum_{n=1}^{\infty} \|\xi_n\|\|\phi_n\|$, where, for each $n \in \mathbb{N}$, there is $(\pi_n, E_n) \in Cyc_{q,r}(S)$ with $\xi_n \in E_n$ and $\phi_n \in E_n^*$ such that $\sum_{n=1}^{\infty} \|\xi_n\|\|\phi_n\| < \infty$ and

$$f(x) = \sum_{n=1}^{\infty} < \pi_n(x)\xi_n, \phi_n >, \quad (x \in S).$$

The proof of the following theorem is similar to the group case.

Theorem 4.10. *Let S be an inverse semigroup, let $p \in (1,\infty)$, and let $f, g : S \longrightarrow \mathbb{C}$ be coefficient function of (π, E) and (ρ, F) in $\Sigma_{p,r}(S)$, respectively. Then the pointwise product of f and g is a coefficient function of $(\pi\otimes\rho, E\tilde{\otimes}_p F)$.*

In the next theorem we give some result about our new constructed space and also the relation between semigroup restricted Herz algebra and it.

Theorem 4.11. *Let S be an inverse semigroup, let $p \in (1,\infty)$. Then:*

(i) *$B_{p,r}(S)$ is a commutative Banach algebra.*

(ii) *the inclusion $A_p(S) \subseteq B_{p,r}(S)$ is a contraction.*

(iii) *for $2 \leq p' \leq p$ or $p \leq p' \leq 2$, the inclusion $B_{p',r}(S) \subseteq B_{p,r}(S)$ is a contraction.*

(iv) *for $p = 2$, $B_{r,e}(S)$ is isometrically isomorphic to $B_{p,r}(S)$ as Banach algebras.*

Proof. (i) Let $\frac{1}{p} + \frac{1}{q} = 1$. The space $B_{p,r}(S)$ is the quotient space of complete q-projective tensor product of $E\tilde{\otimes}_q E^*$, for the universal restricted representation (π, E), on QSL_q-space E. Also Theorem 4.10 shows it is an algebra. The submultplicutive property for norm of $B_{p,r}(S)$ is similar to the group case in [20] and it is only based on characteristic property of infimum.

(ii) By the definition of semigroup Herz algebra in [17] for conjugate numbers p, q, each $f \in A_p(S)$ is a coefficient function of the restricted left regular representation on the ℓ_q-space, $\ell_q(S)$. So $A_p(S) \subseteq B_{p,r}(S)$. By the definition of the norm of $f \in B_{p,r}(S)$, the infimum is taken on all expressions of f as the coefficient function of some restricted representation on a QSL_q-space, and the norm on the $A_p(S)$ is the infimum only on expressions of f as the coefficient function of restricted left regular representation, so the inclusion map is a contraction.

(iii) For $2 \leq p' \leq p$ or $p \leq p' \leq 2$ and q, q' conjugate scalars to p and p' respectively. Then each restricted representation on a $QSL_{q'}$-space is a restricted representation on a QSL_q-space.

(iv) By the definition, each element of $B_{r,e}(S)$ is a coefficient function of a 2-restricted representation [3].

□

Remark 4.12. A very natural question is that when $A_p(S)$ is an ideal in $B_{p,r}(S)$. Even in $p = 2$ this question is not studied. If we want to go along the proof of the group case, a difficulty to prove this is that in general for $p \in (1, \infty)$, and $(\pi, E) \in \Sigma_{p,r}(S)$, the representations $(\lambda_p \otimes \pi, \ell_p(S, E))$ and $(\lambda_p \otimes id_E, \ell_p(S, E))$ are not equivalent. In fact we can not find a suitable substitution for representation $id : S \to B(E), \quad id(s) = id_E$ in the class of restricted representations. But in a special case, such as Clifford semigroups, we can give a better result.

5. Order Structure of the p-Analog of the Semigroup Fourier–Steiltjes Algebras $B_{p,r}(S)$

Studying the ordered spaces and order structures has a long history. The natural order structure of the Fourier-Stieltjes algebras was favorite in 80s. In [21] the authors studied the order structure of Figà–Talamanca– Herz algebra and generalized results on Fourier algebras. In this section, we consider the p-analog of the restricted Fourier–Stieltjes algebra, $B_{p,r}(S)$, introduced in Section 4, and study its order structure given by the p-analog of positive definite continuous functions.

A *compatible couple* of Banach spaces in the sense of interpolation theory (see [3]) is a pair $(\mathcal{E}_0, \mathcal{E}_1)$ of Banach spaces such that both $\mathcal{E}_0$ and $\mathcal{E}_1$ are embedded continuously in some (Hausdorff) topological vector space. In this case, the intersection $\mathcal{E}_0 \cap \mathcal{E}_1$ is again a Banach space under the norm $\| \cdot \|_{(\mathcal{E}_0, \mathcal{E}_1)} = \max\{\| \cdot \|_{\mathcal{E}_0}, \| \cdot \|_{\mathcal{E}_1}\}$. For example , for a locally compact group G, the pairs $(A_p(G), A_q(G))$ and $(L_p(G), L_q(G))$ are compatible couples.

Definition 5.1. Let (π, E) be a restricted representation of S on a Banach space E, such that $(\mathcal{E}, \mathcal{E}^*)$ is a compatible couple. We mean by a π_r-positive definite function on S, a function which has a representation as $f(x) = \langle \pi(x)\xi, \xi \rangle$, $(x \in S)$, where $\xi \in \mathcal{E} \cap \mathcal{E}^*$. For dual scalars $p, q \in (1, \infty)$, we call each element in the closure of the set of all π_r-positive definite functions on S in $B_{p,r}(S)$, where π is a restricted representation of S on an L_q-space, a restricted p-positive definite function on S and the set of all restricted p-positive definite functions on S, will be denoted by $P_{p,r}(S)$.

It follows from [21] and the definition of $P_{p,r}(S)$, that for each $f \in P_{p,r}(S)$, associated to a representation (π, E), for a QSL_q-space E, there exist a sequence $(\pi_n, \mathcal{E}_n)_{n=1}^{\infty}$ of cyclic restricted representations of S on closed subspaces E_n of $E \cap E^*$, and $\{\xi_n\}$ in $\mathcal{E}_n$, such that

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \xi_n \rangle \quad (x \in S).$$

Proposition 5.2. *The linear span of all finite support elements in $P_{p,r}(S)$ is dense in $A_p(S)$, and $A_p(S)$ is an ordered space.*

Proof. From [17, Proposition 3.2] $A_p(S)$ is a norm closure of the set of elements of the form $\sum_{i=1}^{n} f_i \bullet \check{g}_i$ where f_i, g_i are finite support functions on S, $i = 1, ...n$. Also $f_i \bullet \check{g}_i(x) = \langle \lambda_r(x^*) f_i, g_i \rangle$. Now by Polarization identity, we have the statement. □

Since $A_p(S)$ is the set of coefficient functions of the restricted left regular representation of S on $\ell_p(S)$, we define the positive cone of $A_p(S)$ as the closure in $A_p(S)$, of the set of all function of the form $f = \sum_{i=1}^{n} \xi_i \bullet \tilde{\xi}_i$, for a sequence (ξ_i) in $\ell_p(S) \cap \ell_q(S)$, and denote it by $A_p(S)_+$.

This order structure, in the case where $p = 2$, is the same as the order structure of $A_{r,e}(S)$, induced by the set $P_{r,e}(S) \cap A_{r,e}(S)$, as a positive cone. Because in the case $p = 2$, the extensible restricted positive definitive functions are exactly the closed linear span of $h \bullet \tilde{h}$, for $h \in \ell^2(S)$.

6. p-Analog of the Fourier–Stiletjes Algebras on Clifford Semigroups

Let S be a semigroup. Then, by [13, Chapter 2], there is an equivalence relation $\mathcal{D}$ on S by $s\mathcal{D}t$ if and only if there exists $x \in S$ such that

$$Ss \cup \{s\} = Sx \cup \{x\} \text{ and } \; tS \cup \{t\} = xS \cup \{x\}.$$

If S is an inverse semigroup, then by [13, Proposition 5.1.2(4)], $s\mathcal{D}t$ if and only if there exists $x \in S$ such that $s^*s = xx^*$ and $t^*t = x^*x$.

Proposition 6.1. [17, Proposition 4.1]. *Let S be an inverse semigroup,*

(i) *and let D be a $\mathcal{D}$-class of S. Then $\ell_p(D)$ is a closed $\ell_r^1(S)$-submodule of $\ell_p(S)$.*

(ii) *and let $\{D_\lambda; \lambda \in \Lambda\}$ be the family of $\mathcal{D}$-classes of S indexed by some set Λ. Then there is an isometric isomorphism of Banach $\ell_r^1(S)$-bimodules*

$$\ell^p(S) \cong \ell^p - \bigoplus_{\lambda \in \Lambda} \ell_p(D_\lambda). \tag{6.1}$$

Corollary 6.2. *Let S be an inverse semigroup, and let $\{D_\lambda; \lambda \in \Lambda\}$ be the family of $\mathcal{D}$-classes of S indexed by some set Λ. Then for a QSL_p-space E of functions on S, there is a family of QSL_p-spaces $\{E_\lambda\}_{\lambda \in \Lambda}$, where for each $\lambda \in \Lambda$, E_λ consists of functions on D_λ, and $E \cong \ell^p - \oplus_{\lambda \in \Lambda} E_\lambda$.*

Proof. This is clear by the definition of a QSL_p-space, and the fact that the isomorphism 6.1 is compatible with taking quotients and subspaces of $\ell_p(D_\lambda)$s. □

An inverse semigroup S is called a Clifford semigroup if $s^*s = ss^*$ for all $s \in S$. For $e \in E(S)$ define $G_e := \{s \in S | s^*s = ss^* = e\}$. Then G_e is a group with identity e. Here each $\mathcal{D}$-class D contains a single idempotent (say e) and we have $D = G_e$

We modified the isometrical isomorphism derived in [17, Section 5.3] in the following theorem.

Theorem 6.3. *Let S be a Clifford semigroup with the family of $\mathcal{D}$-classes $\{G_e\}_{e\in E(S)}$, and let $p \in (1,\infty)$. Then*

$$B_{p,r}(S) \cong \ell^1 - \bigoplus_{e\in E(S)} B_p(G_e)$$

Proof. Let p, q are conjugate scalars. Fix $e \in E(S)$, assume that $G_e = \{x \in S; x^*x = e\}$. Define $\pi : S \to B(\ell_q(G_e))$

$$\pi(s)(\delta_t) = \begin{cases} \delta_t & \text{if } s^*s = e, \\ 0 & \text{otherwise} \end{cases}$$

for $s \in S$. Then π is a restricted representation and $\chi_{G_e}(s) = \langle \pi(s)\delta_t, \delta_t\rangle$. Hence χ_{G_e} is in $B_{p,r}(S)$, and indeed χ_{G_e} is a restricted positive definite function. Now for each $u \in B_{p,r}(S)$, $u \cdot \chi_{G_e}$ is in $B_{p,r}(S)$. In fact the set $\{u \in B_{p,r}(S); u(s) = 0 \text{ for all } s \in S \setminus G_e\}$ is a closed subspace of $B_{p,r}(S)$ and it is isometrically isomorphic to $B_p(G_e)$. This follows from the fact that, each coefficient function of a restricted representation of S on a QSL_q-space that is zero on ${G_e}^c$, is a coefficient function of a representation on a QSL_q-space of G_e, using Corollary 6.2.

Let $u \in B_{p,r}(S)$, then we could decompose u to $(u_e)_{e\in E(S)}$, for some $u_e \in B_p(G_e)$, by the paragraph above. Now for all $e \in E(S)$ and all explanations of u_e as $u_e = \langle \pi_e(\cdot)\xi_e, \eta_e\rangle$, where $\pi_e \in \Sigma_q(G_e)$, ξ_e in some QSL_q and η_e in some QSL_p-space for dual scalars p, q we have $\|u_e\| \le \|\xi_e\|\|\eta_e\|$ and also $u = (u_e)_{e\in E(S)} = \langle \oplus\pi_e(\cdot) \oplus \xi_e, \oplus\eta_e\rangle$ and then $\oplus\pi_e$ is a restricted representation of S on a QSL_q-space and $\sum_{e\in E(S)} \|u_e\| \le \sum_{e\in E(S)} \|\xi_e\|\|\eta_e\| \le (\sum_{e\in E(S)} \|\xi_e\|^q)^{\frac{1}{q}} (\sum_{e\in E(S)} \|\eta_e\|^p)^{\frac{1}{p}} = \|(\xi_e)\|\|(\eta_e)\|$, the last equality comes from Proposition 6.1. Now we have

$$\sum_{e\in E(S)} \|u_e\| \le \|(u_e)_{e\in E(S)}\| = \|\sum_{e\in E(S)} u_e\|$$

by the definition of the norm in the Fourier–Stieltjes algebras. So we have an isometric isomorphism of Banach algebras. □

The following corollary improves [22, Proposition 2.6].

Corollary 6.4. *Let S be a Clifford semigroup with the family of $\mathcal{D}$-classes $\{G_e\}_{e\in E(S)}$, and let $p\in(1,\infty)$. Then $A_p(S)$ is an ideal of $B_{p,r}(S)$, and $B_{p,r}(S)$ is amenable if and only if $B_p(G_e)$ is amenable for all $e\in E(S)$.*

Theorem 6.5. *Let S be a Clifford semigroup with the family of amenable $\mathcal{D}$-classes $\{G_e\}_{e\in E(S)}$ and let $p\in(1,\infty)$. Then $A_p(S)$ is equal to the closure of $B_{p,r}(S)\cap F(S)$ in the norm of $A_p(S)$.*

Proof. Since for each $e\in E(S)$ the group G_e is amenable, the natural embedding $i_e : A_p(G_e)\to B_p(G_e)$ is an isometry by [20, Corollary 5.3]. Now let $f\in B_{p,r}(S)\cap F(S)$. Then by Theorem 6.3, $f=\sum_{e\in E(S)} f_e$, where $f_e\in B_p(G_e)\cap F(G_e)$, so f_e belongs to $A_p(G_e)$ by [20]. Now since

$$A_p(S)\cong \ell_1-\bigoplus_{e\in E(S)} A_p(G_e),$$

[17, Equation 5.1], we conclude that $f\in A_p(S)$. On the other hand let $f\in A_p(S)\cap F(S)$. Then for each $e\in E(S)$ the function f_e, which has been defined in Theorem 6.3, belongs to $A_p(G_e)\cap F(G_e)$ and so to $B_p(G_e)\cap F(G_e)$ with the same norm because of amenability of G_e. Now, by Theorem 6.3 we have $f\in B_{p,r}(S)\cap F(S)$. Since $F(S)$ is dense in $A_p(S)$ with norm of $A_p(S)$, [17, Proposition 3.2 vi], result follows.

□

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