Iranian Journal of Mathematical Sciences and Informatics
Vol. 10, No. 2 (2015), pp 67-75
DOI: 10.7508/ijmsi.2015.02.007

Generalized Douglas-Weyl Finsler Metrics

Mohammad Hosein Emamian, Akbar Tayebi*

Department of Mathematics, Faculty of Science University of Qom, Qom, Iran.

E-mail: hosein.emamian@gmail.com
E-mail: akbar.tayebi@gmail.com

Abstract. In this paper, we study generalized Douglas-Weyl Finsler metrics. We find some conditions under which the class of generalized Douglas-Weyl (α, β)-metric with vanishing S-curvature reduce to the class of Berwald metrics.

Keywords: Generalized Douglas-Weyl metrics, S-curvature.

2000 Mathematics subject classification: 53C60, 53C25.

1. Introduction

Let (M, F) be a Finsler manifold. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients [10]. F is called a Berwald metric, if G^i are quadratic in $y \in T_xM$ for any $x \in M$ or equivalently $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$. As a generalization of Berwald curvature, Bácsó-Matsumoto introduced the notion of Douglas metrics which are projective invariants in Finsler geometry [2]. F is called a Douglas metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k + P(x, y)y^i$.

A Finsler metric F is called generalized Douglas-Weyl metric (briefly, GDW-metric) if $D^i{}_{jkl||m}y^m = T_{jkl}y^i$ holds for some tensor T_{jkl}, where $D^i{}_{jkl||m}$ denotes the horizontal covariant derivatives of $D^i{}_{jkl}$ with respect to the Berwald connection of F [8][18]. For a manifold M, let $\mathcal{GDW}(M)$ denotes the class

*Corresponding Author

Received 21 December 2013; Accepted 13 November 2014

of all Finsler metrics satisfying in above relation for some tensor T_{jkl}. In [3], Bácsó-Papp showed that $\mathcal{G}DW(M)$ is closed under projective changes. Then, Najafi-Shen-Tayebi characterized generalized Douglas-Weyl Randers metrics [8]. In [18], it is proved that all generalized Douglas-Weyl spaces with vanishing Landsberg curvature have vanishing the quantity **H**. For other works, see [12] and [13].

The notion of S-curvature is originally introduced by Shen for the volume comparison theorem [9]. The Finsler metric F is said to be of isotropic S-curvature if $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on M. In [14], it is shown that every isotropic Berwald metric has isotropic S-curvature. In [4], Cheng-Shen show that every (α, β)-metric with constant Killing 1-form has vanishing S-curvature. Then, Bácsó-Cheng-Shen proved that a Finsler metric $F = \alpha \pm \beta^2/\alpha + \epsilon\beta$ has vanishing S-curvature if and only if β is a constant Killing 1-form [1]. Therefore, the Finsler metrics with vanishing S-curvature are of some important geometric structures which deserve to be studied deeply.

An (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta(y) = b_i(x)y^i$ is a 1-form on M [6]. In this paper, we are going to study generalized Douglas-Weyl (α, β)-metrics with vanishing S-curvature.

Theorem 1.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that*

$$F \neq c_3\alpha\left(\frac{\beta}{\alpha}\right)^{\frac{c_2}{1+c_2}}\left(c_1\frac{\beta}{\alpha} + c_2 + 1\right)^{\frac{1}{1+c_2}} \qquad \text{and} \qquad F \neq d_1\sqrt{\alpha^2 + d_2\beta^2} + d_3\beta.$$

where c_1, c_2 ,c_3 ,d_1, d_2 and d_3 are real constants. Let F has vanishing S-curvature. Then F is a GDW-metric if and only if it is a Berwald metric.

2. Preliminary

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4}g^{il}\Big\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\Big\}, \qquad y \in T_xM.$$

The $\mathbf{G}$ is called the spray associated to F.

Define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y)u^jv^k$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \qquad E_{jk} := \frac{1}{2}B^m{}_{jkm}.$$

B and **E** are called the Berwald curvature and mean Berwald curvature, respectively. F is called a Berwald and weakly Berwald if $\mathbf{B}=\mathbf{0}$ and $\mathbf{E}=0$, respectively [5][7].

Let

$$D^i_{j\ kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\big).$$

It is easy to verify that $\mathcal{D} := D^i_{j\ kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle TM_0. We call $\mathcal{D}$ the Douglas tensor. A Finsler metric with $\mathcal{D}=0$ is called a Douglas metric. The notion of Douglas metrics was proposed by B*á*cs*ó*-Matsumoto as a generalization of Berwald metrics [2]. The Douglas tensor $\mathcal{D}$ is a non-Riemannian projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x,y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is same as that of $\bar{F}$. Finsler metrics with vanishing Douglas tensor are called Douglas metrics [11].

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x) dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big[(y^i) \in R^n \ \Big|\ F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big]}.$$

Let G^i denote the geodesic coefficients of F in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. $\mathbf{S}$ said to be isotropic if there is a scalar functions $c = c(x)$ on M such that $\mathbf{S} = (n+1)cF$.

For an (α,β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, put

$$\Phi := -(q - sq')[n\Delta + 1 + sq] - (b^2 - s^2)(1+sq)q'',$$

where

$$q := \frac{\phi'}{\phi - s\phi'}, \qquad \Delta := 1 + sq + (b^2 - s^2)q'.$$

In [4], Cheng-Shen characterize (α,β)-metrics with isotropic S-curvature.

Lemma 2.1. ([4]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$. Suppose that $\phi \neq c_1\sqrt{1+c_2 s^2} + c_3 s$ for any constant $c_1 > 0$, c_2 and c_3. Then F is of isotropic S-curvature $\mathbf{S} = (n+1)cF$ if and only if one of the following holds
(a) β satisfies

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \tag{2.1}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, $b := \|\beta_x\|_\alpha$ and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{2.2}$$

where k is a constant. In this case, $\mathbf{S} = (n+1)cF$ with $c = k\varepsilon$.
(b) β satisfies

$$r_{ij} = 0, \quad s_j = 0 \tag{2.3}$$

In this case, $\mathbf{S} = 0$.

The characterization of Finsler metrics with isotropic S-curvature in Cheng-Shen's paper is not complete [4]. Their result is correct for dimension $n \geq 3$. For the case $dimension(M) = 2$, see [16].

3. Proof of Main Results

Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on a manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta(y) = b_i(x)y^i$. Define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j\theta_i{}^j$, where $\theta^i := dx^i$ and $\theta_i{}^j := \tilde{\Gamma}^j_{ik}dx^k$ denote the Levi-Civita connection forms of α. Let

$$\begin{aligned}
r_{ij} &:= \frac{1}{2}\Big[b_{i|j} + b_{j|i}\Big], \qquad s_{ij} := \frac{1}{2}\Big[b_{i|j} - b_{j|i}\Big],\\
r_{i0} &:= r_{ij}y^j, \quad r_{00} := r_{ij}y^iy^j, \quad r_j := b^ir_{ij}, \quad t^i_j := s^i{}_m s^m_j\\
s_{i0} &:= s_{ij}y^j, \quad s_j := b^is_{ij}, \quad r_0 := r_jy^j, \quad s_0 := s_jy^j.
\end{aligned}$$

Then $\beta = b_i(x)y^i$ is a constant Killing one-form on M if $r_{ij} = s_j = 0$ hold. By definition, we have

$$b_{i|j} = s_{ij} + r_{ij}.$$

Since $y^i{}_{|s} = 0$, then for a constant Killing 1-form β we have

$$r_{00} = 0, \quad r_i + s_i = 0.$$

For an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, the following hold.

Proposition 3.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M of dimension $n \geq 3$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form on M. Suppose that F is of vanishing S-curvature. Then F is a GDW-metric if and only if the following holds*

$$\begin{aligned}
C_1\, s_{j0|0}y^i - (C_2\, y_j + C_3\, b_j)y^it_{00} &= C_4\, y_js^i{}_{0|0} + C_5\big(b_js^i{}_{0|0} + s_{j0}s^i{}_0\big)\\
&\quad + C_6\, s^i{}_{j|0} + C_7(y_jt^i{}_0 + s_{j0}s^i{}_0) + C_8\, b_jt^i{}_0, \qquad (3.1)
\end{aligned}$$

where

$$
\begin{aligned}
C_1 &:= -\Big[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\Big]\alpha^{-3} - \Big[Q_{\alpha\alpha} + b^2 Q_{\beta\beta}\Big]\alpha^{-2},\\
C_2 &:= (n+1)\Big[Q_\alpha^2 + QQ_{\alpha\alpha} - \alpha^{-1}QQ_\alpha\Big]\alpha^{-4} - 2\Big[Q_\alpha Q_\beta + QQ_{\alpha\beta}\Big]\beta\alpha^{-5}\\
&+ \ 2\Big[2Q_\alpha Q_{\alpha\beta} + Q_{\alpha\alpha}Q_\beta + QQ_{\alpha\alpha\beta}\Big]\beta\alpha^{-4} + b^2\Big[2Q_{\alpha\beta}Q_\beta + Q_\alpha Q_{\beta\beta}\Big]\alpha^{-3}\\
&+ \ \Big[b^2 QQ_{\alpha\beta\beta} + 3Q_\alpha Q_{\alpha\alpha} + QQ_{\alpha\alpha\alpha}\Big]\alpha^{-3},\\
C_3 &:= (n+3)\Big[Q_\alpha Q_\beta + QQ_{\alpha\beta}\Big]\alpha^{-3} + 2\Big[Q_\alpha Q_{\beta\beta} + QQ_{\alpha\beta\beta}\Big]\beta\alpha^{-3}\\
&+ \ \Big[2Q_\alpha Q_{\alpha\beta} + Q_\beta Q_{\alpha\alpha} + QQ_{\alpha\alpha\beta} + 4\beta\alpha^{-1}Q_\beta Q_{\alpha\beta}\Big]\alpha^{-2}\\
&+ \ b^2\Big[3Q_\beta Q_{\beta\beta} + QQ_{\beta\beta\beta}\Big]\alpha^{-2},\\
C_4 &:= -\Big[(n+1)Q_\alpha + 2\beta Q_{\alpha\beta}\Big]\alpha^{-3} + 2\Big[\beta Q_{\alpha\alpha\beta} + Q_{\alpha\alpha}\Big]\alpha^{-2}\\
&+ \ \Big[b^2 Q_{\alpha\beta\beta} + Q_{\alpha\alpha\alpha}\Big]\alpha^{-1},\\
C_5 &:= (n+3)\alpha^{-1}Q_{\alpha\beta} + Q_{\alpha\alpha\beta} + 2\beta\alpha^{-1}Q_{\alpha\beta\beta} + b^2 Q_{\beta\beta\beta},\\
C_6 &:= (n+1)\alpha^{-1}Q_\alpha + Q_{\alpha\alpha} + 2\beta\alpha^{-1}Q_{\alpha\beta} + b^2 Q_{\beta\beta},\\
C_7 &:= (n+1)\alpha^{-3}QQ_\alpha - (n+1)\alpha^{-2}(Q_\alpha^2 + QQ_{\alpha\alpha}) - 2\beta\alpha^{-2}QQ_{\alpha\alpha\beta}\\
&+ \ 2\Big[QQ_{\alpha\beta} + Q_\alpha Q_\beta\Big]\beta\alpha^{-3} - b^2\Big[QQ_{\alpha\beta\beta} + 2Q_{\alpha\beta}Q_\beta\Big]\alpha^{-1}\\
&- \ 2\Big[2Q_\alpha Q_{\alpha\beta} + Q_\beta Q_{\alpha\alpha}\Big]\beta\alpha^{-2}\\
&- \ b^2\alpha^{-1}Q_\alpha Q_{\beta\beta} - 3\alpha^{-1}Q_\alpha Q_{\alpha\alpha} - 2\alpha^{-1}QQ_{\alpha\alpha\alpha},\\
C_8 &:= -(n+3)\Big[QQ_{\alpha\beta} + Q_\alpha Q_\beta\Big]\alpha^{-1} - 2\Big[2Q_\beta Q_{\alpha\beta} + QQ_{\alpha\beta\beta} + Q_\alpha Q_{\beta\beta}\Big]\beta\alpha^{-1}\\
&- \ b^2\Big[QQ_{\beta\beta\beta} + 3Q_\beta Q_{\beta\beta}\Big] - Q_\beta Q_{\alpha\alpha} - QQ_{\alpha\alpha\beta} - 2Q_\alpha Q_{\alpha\beta}.
\end{aligned}
$$

Proof. Let G^i and G^i_α denote the spray coefficients of F and α, respectively, in the same coordinate system. Then, we have

$$G^i = G^i_\alpha + Py^i + Q^i, \tag{3.2}$$

where

$$
\begin{aligned}
Q &:= \alpha q = \frac{\alpha\phi'}{\phi - s\phi'},\\
P &:= \alpha^{-1}\Theta(r_{00} - 2Qs_0), \quad Q^i := Qs^i{}_0 + \Psi(r_{00} - 2Qs_0)b^i,\\
\Theta &= \frac{q - sq'}{2\Delta} = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big[(\phi - s\phi') + (b^2 - s^2)\phi''\Big]}\\
\Psi &:= \frac{q'}{2\Delta} = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.
\end{aligned}
$$

By Lemma 2.1, we have $r_{00} = s_0 = 0$. Then (3.2) reduces to following

$$G^i = G^i_\alpha + Q s^i_0. \tag{3.3}$$

Let "$\|$" and "$|$" denote the covariant differentiations with respect to G^i and G^i_α respectively. Then by (3.3), we have

$$D^i_{jkl\|m} y^m = D^i_{jkl|m} y^m - 2Q s^p_0 \frac{\partial D^i_{jkl}}{\partial y^p} + D^p_{jkl} \tilde{N}^i_p - D^i_{pkl} \tilde{N}^p_j - D^i_{jpl} \tilde{N}^p_k - D^i_{jkp} \tilde{N}^p_l, \tag{3.4}$$

where

$$\begin{aligned} D^i_{jkl|m} y^m = \;& \alpha^{-4}(Q_{\alpha\alpha} - \alpha^{-1} Q_\alpha)(A_{jk} y_l + A_{kl} y_j + A_{jl} y_k) {s^i}_{0|0} \\ &+ \alpha^{-3} Q_\alpha (A_{jk} {s^i}_{l|0} + A_{kl} {s^i}_{j|0} + A_{jl} {s^i}_{k|0}) \\ &+ \alpha^{-3} Q_{\alpha\beta} \Big[(A_{jk} b_l + A_{kl} b_j + A_{jl} b_k) {s^i}_{0|0} \\ &+ (A_{jk} s_{l0} + A_{kl} s_{j0} + A_{jl} s_{k0}) {s^i}_0 \Big] \\ &+ \alpha^{-2} Q_{\alpha\alpha\beta} \Big[(y_j y_k b_l + y_k y_l b_j + y_j y_l b_k) {s^i}_{0|0} \\ &+ (y_j y_k s_{l0} + y_k y_l s_{j0} + y_j y_l s_{k0}) {s^i}_0 \Big] \\ &+ \alpha^{-1} Q_{\alpha\beta\beta} \Big[(y_j b_k b_l + y_k b_j b_l + y_l b_k b_j) {s^i}_{0|0} \\ &+ \big((y_j b_l + y_l b_j) s_{k0} + (y_j b_k + y_k b_j) s_{l0} \\ &+ (y_k b_l + y_l b_k) s_{j0} \big) {s^i}_0 \Big] + \alpha^{-2} Q_{\alpha\alpha} (y_j y_k {s^i}_{l|0} + y_k y_l {s^i}_{j|0} + y_j y_l {s^i}_{k|0}) \\ &+ Q_{\beta\beta\beta} (b_k b_l s_{j0} + b_j b_l s_{k0} + b_j b_k s_{l0}) s^i_0 + \alpha^{-3} Q_{\alpha\alpha\alpha} y_j y_k y_l {s^i}_{0|0} \\ &+ \alpha^{-1} Q_{\alpha\beta} \Big[(y_j b_k + y_k b_j) {s^i}_{l|0} + (y_k b_l + y_l b_k) {s^i}_{j|0} + (y_l b_j + y_j b_l) {s^i}_{k|0} \\ &+ (y_j s_{k0} + y_k s_{j0}) {s^i}_l + (y_k s_{l0} + y_l s_{k0}) {s^i}_j + (y_l s_{j0} + y_j s_{l0}) {s^i}_k \Big] \\ &+ Q_{\beta\beta} \Big[b_j b_k {s^i}_{l|0} + b_k b_l {s^i}_{j|0} + b_j b_l {s^i}_{k|0} + (s_{j0} b_k + b_j s_{k0}) {s^i}_l \\ &+ (s_{k0} b_l + b_k s_{l0}) {s^i}_j + (b_l s_{j0} + b_j s_{l0}) {s^i}_k \Big] + Q_{\beta\beta\beta} b_j b_k b_l {s^i}_{0|0} \end{aligned} \tag{3.5}$$

and

$$A_{ij} = \alpha^2 a_{ij} - y_i y_j, \tag{3.6}$$

$$\tilde{N}^i_p = Q {s^i}_p + \Big[\alpha^{-1} Q_\alpha y_p + Q_\beta b_p \Big] {s^i}_0, \tag{3.7}$$

$$\frac{\partial D^i_{jkl}}{\partial y^p} = Q_{jklp} {s^i}_0 + Q_{jkl} {s^i}_p + Q_{jkp} {s^i}_l + Q_{jlp} {s^i}_k + Q_{klp} {s^i}_j. \tag{3.8}$$

Let F is a GDW-metric. Then there is a tensor D_{jkl} such that

$$D^i_{jkl\|m} y^m = D_{jkl} y^i.$$

By (3.4), we have

$$D_{jkl}y^i = D^i_{jkl|m}y^m - 2Q\ \frac{\partial D^i_{jkl}}{\partial y^p}s^p{}_0 + D^p_{jkl}\tilde{N}^i_p - D^i_{pkl}\tilde{N}^p_j - D^i_{jpl}\tilde{N}^p_k - D^i_{jkp}\tilde{N}^p_l. \tag{3.9}$$

By contracting (3.9) with y_i and using (3.5), (3.7) and (3.8) we get the following

$$\begin{aligned}
D_{jkl} = \quad & D_1\Big[A_{jk}s_{l0|0} + A_{kl}s_{j0|0} + A_{jl}s_{k0|0}\Big]\\
&+ D_2\Big[y_jy_ks_{l0|0} + y_ky_ls_{j0|0} + y_jy_ls_{k0|0}\Big]\\
&+ D_3\Big[(y_jb_k + y_kb_j)s_{l0|0} + (y_kb_l + y_lb_k)s_{j0|0} + (y_jb_l + y_lb_j)s_{k0|0}\Big]\\
&+ D_4\Big[b_jb_ks_{l0|0} + b_kb_ls_{j0|0} + b_jb_ls_{k0|0}\Big]\\
&+ D_5\Big[A_{jk}y_l + A_{kl}y_j + A_{jl}y_k\Big]t_{00}\\
&+ D_6\Big[A_{jk}b_l + A_{kl}b_j + A_{jl}b_k\Big]t_{00}\\
&+ D_7\Big[y_jy_kb_l + y_ky_lb_j + y_jy_lb_k\Big]t_{00}\\
&+ D_8\Big[y_lb_jb_k + y_jb_kb_l + y_kb_jb_l\Big]t_{00}\\
&+ D_9\, y_jy_ky_lt_{00} + D_{10}\, b_jb_kb_lt_{00}\\
&+ D_{11}\Big[y_ls_{j0}s_{k0} + y_js_{k0}s_{l0} + y_ks_{j0}s_{l0}\Big]\\
&+ D_{12}\Big[b_ls_{j0}s_{k0} + b_js_{k0}s_{l0} + b_ks_{j0}s_{l0}\Big],
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
D_1 &:= -\alpha^{-5}Q_\alpha,\\
D_2 &:= -\alpha^{-4}Q_{\alpha\alpha},\\
D_3 &:= -\alpha^{-3}Q_{\alpha\beta},\\
D_4 &:= -\alpha^{-2}Q_{\beta\beta},\\
D_5 &:= -\alpha^{-6}Q^2_\alpha - \alpha^{-6}QQ_{\alpha\alpha} + \alpha^{-7}QQ_\alpha,\\
D_6 &:= -\alpha^{-5}Q_\alpha Q_\beta - \alpha^{-5}QQ_{\alpha\beta},\\
D_7 &:= -\alpha^{-4}Q_{\alpha\alpha}Q_\beta - 2\alpha^{-4}Q_{\alpha\beta}Q_\alpha - \alpha^{-4}QQ_{\alpha\alpha\beta},\\
D_8 &:= -\alpha^{-3}Q_{\beta\beta}Q_\alpha - 2\alpha^{-3}Q_{\alpha\beta}Q_\beta - \alpha^{-3}QQ_{\alpha\beta\beta},\\
D_9 &:= -3\alpha^{-5}Q_{\alpha\alpha}Q_\alpha - \alpha^{-5}QQ_{\alpha\alpha\alpha},\\
D_{10} &:= -3\alpha^{-2}Q_{\beta\beta}Q_\beta - \alpha^{-2}QQ_{\beta\beta\beta},\\
D_{11} &:= -2\alpha^{-3}Q_{\alpha\beta} + 2\alpha^{-4}Q^2_\alpha + 2\alpha^{-4}QQ_{\alpha\alpha} - 2\alpha^{-5}QQ_\alpha,\\
D_{12} &:= -2\alpha^{-2}Q_{\beta\beta} + 2\alpha^{-3}QQ_{\alpha\beta} + 2\alpha^{-3}Q_\alpha Q_\beta.
\end{aligned}$$

Now, by plugging (3.10) into (3.9), and contracting the obtained result with a^{kl}, we get (3.1). □

Proof of Theorem 1.1: Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M. By multiplying (3.1) with y_i and y^j, we get

$$-\alpha Q Q_{\alpha\alpha\alpha} t_{00} = 0. \tag{3.11}$$

If $Q_{\alpha\alpha\alpha} = 0$ then

$$Q = c_1\alpha + c_2\frac{\alpha^2}{\beta},$$

where c_1 and c_2 are real constants. Thus, we get

$$F = c_3\alpha\left(\frac{\beta}{\alpha}\right)^{\frac{c_2}{1+c_2}}\left(c_1\frac{\beta}{\alpha} + c_2 + 1\right)^{\frac{1}{1+c_2}},$$

where c_3 is a real constant. This is a contradiction with our assumption. Then by (3.11), we get $t_{00} = 0$ which results that $s_{i0} = 0$. This means that β is a closed one-form. By assumption, β is parallel one-form and then F reduces to a Berwald metric. □

Acknowledgments

The authors are very grateful to the anonymous referee for his or her comments and suggestions.

References

1. S. Bácsó, X. Cheng, Z. Shen, Curvature Properties of (α, β)-Metrics, *Adv. Stud. Pure. Math.*, **48**, (2007), 73-110.
2. S. Bácsó, M. Matsumoto, On Finsler Spaces of Douglas Type, A Generalization of Notion of Berwald Space, *Publ. Math. Debrecen.*, **51**, (1997), 385-406.
3. S. Bácsó, I. Papp, A Note on a Generalized Douglas Space, *Periodica. Math. Hungarica.*, **48**, (2004), 181-184.
4. X. Cheng, Z. Shen, A Class of Finsler Metrics with Isotropic S-curvature, *Israel. J. Math.*, **169**, (2009), 317-340.
5. P. Habibi, A. Razavi, Generalized Symmetric Berwald Spaces, *Iranian Journal of Mathematical Sciences and Informatics*, **9**(1), (2014), 63-69.
6. M. Matsumoto, Theory of Finsler Spaces with (α, β)-Metric, *Rep. Math. Phys.*, **31**, (1992), 43-84.
7. B. Najafi, A. Tayebi, Finsler Metrics of Scalar Flag Curvature and Projective Invariants, *Balkan J. Geom. Appl.*, **15**, (2010), 90-99.
8. B. Najafi, Z. Shen, A. Tayebi, On a Projective Class of Finsler Metrics, *Publ. Math. Debrecen.*, **70**, (2007), 211-219.
9. Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.
10. A. Tayebi, E. Peyghan, On C3-like Finsler Metrics, *Iranian Journal of Mathematical Sciences and Informatics*, **7**(1), (2012), 1-6.
11. A. Tayebi, E. Peyghan, On Douglas Surfaces, *Bull. Math. Soc. Science. Math. Roumanie*, Tome 55 (103), No 3, (2012), 327-335.

12. A. Tayebi, E. Peyghan, On a Subclass of Generalized Douglas-Weyl Metrics, *J. Cont. Math. Analysis*, **47**(2), (2012), 69-80.
13. A. Tayebi, E. Peyghan, On Special Generalized Douglas-Weyl Metrics, *J. Sci. Islamic Republic of Iran*, **23**(2), (2012), 179-184.
14. A. Tayebi, M. Rafie. Rad, S-curvature of Isotropic Berwald Metrics, *Sci. China. Series A: Mathematics.*, **51**, (2008), 2198-2204.
15. A. Tayebi, H. Sadeghi, E. Peyghan, On Generalized Douglas-Weyl Spaces, *Bull. Malays. Math. Sci. Soc.*, **36**(3), (2013), 587-594.
16. G. Yang, A Note on a Class of Finsler Metrics of Isotropic S-curvature, preprint. http://arxiv.org/abs/1310.3463
17. H. Shimada S. V. Sabau, An Introduction to Matsumoto Metric, *Nonlinear Analysis: RWA.*, **63**, (2005), 165-168.
18. A. Tayebi, H. Sadeghi and E. Peyghan, On Generalized Douglas-Weyl Spaces, *Bull. Malays. Math. Sci. Soc.*, **36**(3), (2013), 587-594.
19. C. H. Xiang X. Cheng, On a Class of Weakly-Berwald (α, β)-Metrics, *J. Math. Res. Expos.*, **29**, (2009), 227-236.