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A Numerical Method for Solving Ricatti Differential Equations

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Abstract. By adding a suitable real function on both sides of the quadratic Riccati differential equation, we propose a weighted type of Adams-Bashforth rules for solving it, in which moments are used instead of the constant coefficients of Adams-Bashforth rules. Numerical results reveal that the proposed method is efficient and can be applied for other nonlinear problems.

Keywords: Riccati differential equations, Adams-Bashforth rules, Weighting factor, Nonlinear differential equations, Stirling numbers.

2000 Mathematics subject classification: 65L05, 65L06.

1. Introduction

The Riccati differential equations indicated by

$$\begin{aligned} y'(x) &= p(x)y^2(x) + q(x)y(x) + r(x), \\ y(x_0) &= y_0,\ x_0 \le x \le x_f, \end{aligned} \tag{1.1}$$

play a significant role in many fields of applied science [10, 11, 18, 19]. For example, a one-dimensional static Schrödinger equation is closely related to (1.1) [13, 19, 20]. Solitary wave solutions of a nonlinear partial differential

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equation can be expressed as a polynomial in two elementary functions satisfying a projective Riccati equation [10, 11, 17, 19, 20]. Such types of equations also arise in optimal control problems. It is clear that Riccati differential equations with constant coefficients can be explicity solved by using various methods [10, 11, 17, 19]. In recent years, various types of these equations have been numerically solved by using different techniques such as variational iteration method [11], He's variational method [1], the cubic B-spline scaling functions and Chebyshev cardinal functions [14], the homotopy perturbation method [2, 3], the modified variational iteration method [10], the Taylor matrix method [12], the Adomian decomposition method [6] and a new form of homotopy perturbation method [4].

In this paper, by adding a suitable real function on both sides of equation (1.1) we propose a weighted kind of Adams-Beshforth rules for solving this type of nonlinear differential equations in which moments are used instead of the constant coefficients of Adams-Bashforth rules. In other words, for each Riccati differential equation we can obtain a new set of coefficients depending on a new weighting factor.

In Section 2, we formulate Adams-Bashforth methods [5, 7, 9] and weighted Adams-Bashforth methods in terms of Stirling numbers. Then, we show how to choose the suitable weight function in order to establish a weighted Adams-Bashforth method. Finally, in Section 4, some numerical examples are given to show the efficiency of the proposed methods for solving Riccati differential equations .

2. Explicit Forms of Weighted Adams-Bashforth Rules for Riccati Differential Equations

It is known that the first kind of Stirling numbers can be generated via the relation

$$\sum_{k=0}^{n} \mathbf{S}(n,k)x^k = (x)_n = \prod_{i=0}^{n-1}(x-i),$$

where $(x)_0 = 1$, while the second kind of Stirling numbers has the explicit form

$$S_2(n,k) = \frac{(-1)^k}{k!}\sum_{i=1}^{k}(-1)^i \binom{k}{i} i^n \; for \; \binom{k}{i} = \frac{k!}{i!(k-i)!}.$$

There is a direct connection between the first and second kind of Stirling numbers [8] as follows

$$S_2(n,m) = \sum_{k=0}^{n-m}(-1)^k \binom{n-1+k}{n-m+k}\binom{2n-m}{n-m-k}\mathbf{S}(k-m+n,k),$$

and conversely

$$\mathbf{S}(n,m)=\sum_{k=0}^{n-m}(-1)^k\binom{n-1+k}{n-m+k}\binom{2n-m}{n-m-k}S_2(k-m+n,k).$$

Now, consider equation (1.1) and let for convenience

$$F(x,y)=p(x)y^2(x)+q(x)y(x)+r(x).$$

If the main interval $[x_0,x_f]$ with the stepsize $h=\frac{x_f-x_0}{n}$ is divided, then by using the backward Newton interpolation formula [15, 16] we have for $F(x,y)$ that

$$F(x,y)\simeq\sum_{i=0}^{s}\frac{(-1)^i(-\lambda)_i}{i!}\nabla^i F_n. \tag{2.1}$$

By integrating from both sides of equation (1.1) over $[x_n,x_{n+1}]$ and then applying (2.1) we get

$$\begin{aligned}
y(x_{n+1})-y(x_n)&=\int_{x_n}^{x_{n+1}}F(x,y)dx\\
&=\int_{x_n}^{x_{n+1}}\left(\sum_{i=0}^{s}\frac{(-1)^i(-\lambda)_i}{i!}\nabla^iF_n\right)dx+\boldsymbol{E}\\
&=\int_0^1\left(\sum_{i=0}^{s}\frac{(-1)^i(-\lambda)_i}{i!}\nabla^iF_n\right)hd\lambda+\boldsymbol{E}\\
&=h\sum_{i=0}^{s}\frac{(-1)^i\nabla^iF_n}{i!}\int_0^1(-\lambda)_id\lambda+\boldsymbol{E}\\
&=h\sum_{i=0}^{s}\frac{(-1)^i\nabla^iF_n}{i!}\int_0^1\sum_{k=0}^{i}(-1)^k\mathbf{S}(i,k)\lambda^kd\lambda+\boldsymbol{E}\\
&=h\sum_{i=0}^{s}\frac{(-1)^i\nabla^iF_n}{i!}\sum_{k=0}^{i}(-1)^k\frac{\mathbf{S}(i,k)}{k+1}+\boldsymbol{E},
\end{aligned}\tag{2.2}$$

where $x=x_n+\lambda h$ and $\boldsymbol{E}$ is the truncation error denoted by

$$\boldsymbol{E}=\left|\int_{x_n}^{x_{n+1}}\frac{(x-x_n)(x-x_{n-1})\cdots(x-x_{n-s})}{(s+1)!}F^{(s+1)}(\xi_x,y(\xi_x))dx\right|, \tag{2.3}$$

where $\xi_x\in[x_{n-s},x_n]$. Since

$$\frac{\nabla^iF_n}{h^ii!}=\sum_{j=0}^{i}\frac{F(x_{n-j},y(x_{n-j}))}{\Phi'_{i+1}(x_{n-j})},$$

in which

$$\Phi_{i+1}(x)=\prod_{k=0}^{i}(x-x_{n-k})=\sum_{k=0}^{i+1}(-h)^{i+1-k}\mathbf{S}(i+1,k)(t-t_n)^k,$$

and

$$\Phi'_{i+1}(x_{n-j}) = (-h)^i \sum_{k=0}^{i} j^k (k+1)\mathbf{S}(i+1,k+1),$$

relation (2.2) is simplified as

$$\begin{aligned}
&y(x_{n+1}) - y(x_n)\\
&\simeq h\sum_{i=0}^{s}(-1)^i h^i \sum_{j=0}^{i} \frac{F(x_{n-i},y(x_{n-i}))}{(-h)^i \sum\limits_{k=0}^{i} j^k(k+1)\mathbf{S}(i+1,k+1)} \sum_{k=0}^{i}(-1)^k \frac{\mathbf{S}(i,k)}{k+1}\\
&= h\sum_{i=0}^{s}\sum_{j=0}^{i} \frac{F(x_{n-i},y(x_{n-i}))}{\sum\limits_{k=0}^{i} j^k(k+1)\mathbf{S}(i+1,k+1)} \sum_{k=0}^{i}(-1)^k \frac{\mathbf{S}(i,k)}{k+1}\\
&= h\sum_{i=0}^{s} F(x_{n-i},y(x_{n-i})) \sum_{j=i}^{s} \frac{\sum\limits_{k=0}^{j}(-1)^k \frac{\mathbf{S}(i,k)}{k+1}}{\sum\limits_{k=0}^{j} j^k(k+1)\mathbf{S}(i+1,k+1)}\\
&= h\sum_{i=0}^{s} F(x_{n-i},y(x_{n-i})) \sum_{j=i}^{s} \frac{\sum\limits_{k=0}^{j}(-1)^k \frac{\mathbf{S}(i,k)}{k+1}}{(-1)^j j! \sum\limits_{k=0}^{j} i^k(k+1)\mathbf{S}(j+1,k+1)}\\
&= \sum_{i=0}^{s} v_{n-i} F(x_{n-i},y(x_{n-i})),
\end{aligned}$$

where

$$v_{n-i} = h\sum_{j=i}^{s} \frac{\sum\limits_{k=0}^{j}(-1)^k \frac{\mathbf{S}(j,k)}{k+1}}{(-1)^j j! \sum\limits_{k=1}^{j} i^k(k+1)\mathbf{S}(j+1,k+1)}. \tag{2.4}$$

In other words, usual Adams-Bashforth rules for solving Riccati equation (1.1) take the general form

$$y(x_{n+1}) \simeq y(x_n) + \sum_{i=0}^{s} v_{n-i}\left(p(x_{n-i})y^2(x_{n-i}) + q(x_{n-i})y(x_{n-i}) + r(x_{n-i})\right), \tag{2.5}$$

where v_{n-i} is defined as (2.4).

To improve Adams-Bashforth methods in (2.5), we consider equation (1.1) again and add $a(x)y(x)$ to both side of (1.1) to get

$$\begin{aligned}
y'(x) + a(x)y(x) &= p(x)y^2(x) + (q(x)+a(x))y(x) + r(x),\\
y(x_0) &= y_0,\ x_0 \le x \le x_f.
\end{aligned} \tag{2.6}$$

Let

$$w(x) = \exp\left(\int a(x)dx\right), \tag{2.7}$$

and assume for convenience that

$$G(x,y) = F(x,y) + a(x)y(x) = p(x)y^2(x) + (q(x)+a(x))y(x) + r(x).$$

By applying the backward Newton interpolation formula for $G(x,y)$ as

$$G(x,y) \simeq \sum_{i=0}^{s} \frac{(-1)^i(-\lambda)_i}{i!}\nabla^i G_n,$$

and integrating from both sides of (2.6) over $[x_n, x_{n+1}]$ we obtain

$$\begin{aligned}
&w(x_{n+1})y(x_{n+1}) - w(x_n)y(x_n)\\
&= \int_{x_n}^{x_{n+1}} w(x)G(x,y)dx\\
&= \int_{x_n}^{x_{n+1}} w(x)\left(\sum_{i=0}^{s}\frac{(-1)^i(-\lambda)_i}{i!}\nabla^i G_n\right)dx + \boldsymbol{E}^*\\
&= \int_0^1 w(x_n+h\lambda)\left(\sum_{i=0}^{s}\frac{(-1)^i(-\lambda)_i}{i!}\nabla^i G_n\right)hd\lambda + \boldsymbol{E}^*\\
&= h\sum_{i=0}^{s}\frac{(-1)^i\nabla^i G_n}{i!}\int_0^1 w(x_n+h\lambda)\sum_{k=0}^{i}(-1)^k\mathbf{S}(i,k)\lambda^k d\lambda + \boldsymbol{E}^*\\
&= h\sum_{i=0}^{s}\frac{(-1)^i\nabla^i G_n}{i!}\sum_{k=0}^{i}(-1)^k\mathbf{S}(i,k)\int_0^1 w(x_n+h\lambda)\lambda^k d\lambda + \boldsymbol{E}^*,
\end{aligned} \tag{2.8}$$

where $x = x_n + \lambda h$ and $\boldsymbol{E}^*$ is the truncation error denoted by

$$\boldsymbol{E}^* = \left|\int_{x_n}^{x_{n+1}} w(x)\frac{(x-x_n)(x-x_{n-1})\cdots(x-x_{n-s})}{(s+1)!}G^{(s+1)}(\eta_x, y(\eta_x))dx\right|, \tag{2.9}$$

where $\eta_x \in [x_{n-s}, x_n]$. Since

$$\frac{\nabla^i G_n}{h^i i!} = \sum_{j=0}^{i}\frac{G(x_{n-j}, y(x_{n-j}))}{(-h)^i\sum_{k=0}^{i} j^k(k+1)\mathbf{S}(i+1,k+1)},$$

(2.8) is simplified as

$$\begin{aligned}
&w(x_{n+1})y(x_{n+1}) - w(x_n)y(x_n)\\
&\simeq h\sum_{i-0}^{s}\sum_{j=0}^{i}\frac{G(x_{n-j},y(x_{n-j}))}{\sum\limits_{k=0}^{i} j^k(k+1)\mathbf{S}(i+1,k+1)}\sum_{k=0}^{i}(-1)^k\mathbf{S}(i,k)\int_0^1 w(x_n+h\lambda)\lambda^k d\lambda\\
&= h\sum_{i-0}^{s}G(x_{n-j},y(x_{n-j}))\sum_{j=i}^{s}\frac{\sum\limits_{k=0}^{j}(-1)^k\mathbf{S}(j,k)\int_0^1 w(x_n+h\lambda)\lambda^k d\lambda}{\sum\limits_{k=0}^{j} i^k(k+1)\mathbf{S}(j+1,k+1)}\\
&= \sum_{i=0}^{s} w_{n-i}G(x_{n-i},y(x_{n-i}))
\end{aligned}$$

where

$$w_{n-i} = h\sum_{j=i}^{s}\frac{\sum_{k=0}^{j}(-1)^k\mathbf{S}(j,k)\mu_k^*(x_n)}{\sum_{k=0}^{j} i^k(k+1)\mathbf{S}(j+1,k+1)}, \tag{2.10}$$

in which

$$\mu_k^*(x_n) = \int_0^1 w(x_n+\lambda h)\lambda^k d\lambda = \frac{1}{h^{k+1}}\int_{x_n}^{x_{n+1}} w(x)(x-x_n)^k dx,$$

denotes the moment of order k with respect to the weight $w(x)$.

In conclusion, the weighted type of Adams-Bashforth rules for solving Riccati equation (1.1) takes the form

$$\begin{aligned}
&w(x_{n+1})y(x_{n+1}) \simeq w(x_n)y(x_n)\\
&+\sum_{i=0}^{s} w_{n-i}\left(p(x_{n-i})y^2(x_{n-i}) + \left(q(x_{n-i})+a(x_{n-i})\right)y(x_{n-i}) + r(x_{n-i})\right),
\end{aligned} \tag{2.11}$$

where w_{n-i} is defined as (2.10).

3. How to Choose the Weighting Factor *a(x)* ?

In this section, we show how to choose the appropriate $a(x)$ in (2.6) or equivalently $w(x)$ in (2.7). For this purpose, we should first simplify the truncation errors of usual and weighted Adams-Bashforth methods.

Lemma 3.1. *The truncation error of the formula (2.3) can be simplified as follows*

$$\boldsymbol{E} = \frac{h^{s+2}}{(s+1)!}\left|F^{(s+1)}(\xi_x,y(\xi_x))\left(\sum_{k=0}^{s+1}\mathbf{S}(s+1,k)\frac{(-1)^k}{k+1}\right)\right|,\ \xi_x\in(x_n,x_{n+1}). \tag{3.1}$$

Proof. According to (2.3), assume that $F^{(s+1)}(x,y)$ is a continuous function. Since $(x-x_n)(x-x_{n-1})\cdots(x-x_{n-s})$ is a non-negative polynomial on $[x_n, x_{n+1}]$, the mean value theorem can be applied for integrals and then change of the variable $x-x_n=\lambda h$ gives (3.1). □

Lemma 3.2. *The truncation error of the formula (2.9) can be denoted by*

$$\boldsymbol{E}^* = \frac{h^{s+2}}{(s+1)!}\left|G^{(s+1)}(\eta_x, y(\eta_x))\left(\sum_{k=0}^{s+1}(-1)^k\mu^*(x_n,k)\mathbf{S}(s+1,k)\right)\right|, \tag{3.2}$$

where $\eta_x \in (x_n, x_{n+1})$.

In order to minimize $\boldsymbol{E}^*$ in (3.2), the function $a(x)$ can be chosen in such a way that the condition

$$|G^{s+1}(x,y(x))| < \epsilon_0, \quad x \in (x_n, x_{n+1}), \tag{3.3}$$

is satisfied for any arbitrary small, but fixed $\epsilon_0 > 0$. For this purpose, we recall the Weierstrass approximation theorem, which says if $y(x)$ is a continuous function on $[\alpha,\beta]$ and $\epsilon > 0$ is given, then there exists a polynomial of free degree n such that

$$|y(x) - p_n(x)| < \epsilon, \ x \in [\alpha, \beta].$$

This means that any continuous function on a closed and bounded interval can be uniformly approximated by arbitrary polynomials. Hence, if we simultaneously suppose that the exact solution $y(x)$ tends to a polynomial of degree m, say $Q_m(x) = \sum_{k=0}^{m} q_k x^k$, and the function $G(x,y(x))$ tends to a polynomial of degree ℓ (where $\ell < s+1$), say $p_\ell(x) = \sum_{k=0}^{\ell} r_k x^k$, then our goal will be satisfied. For example, if $\ell = s$, then it is sufficient to choose $G(x,y(x)) = p_s(x) = \sum_{k=0}^{s} r_k x^k$ in order that the condition (3.3) is automatically satisfied. On the other hand , replacing the polynomial approximation $y(x) \cong Q_m(x)$ in the definition $G(x,y(x))$ yields

$$p_s(x) = \sum_{k=0}^{s} r_k x^k \cong p(x)(Q_m(x))^2 + (q(x)+a(x))\,Q_m(x) + r(x).$$

Therefore, the suitable approximate $a(x)$ for Riccati differential equations takes the form

$$a(x) \cong \frac{p_s(x) - p(x)(Q_m(x))^2 - q(x)Q_m(x) - r(x)}{Q_m(x)}. \tag{3.4}$$

It is clear that for $m = m_1$ the above choice would be better than e.g. the case $m = m_2$ if

$$|y(x) - Q_{m_1}(x)| < |y(x) - Q_{m_2}(x)|, \quad x \in [\alpha, \beta].$$

Moreover, if $\lim_{m\to\infty} Q_m(x) = y(x)$ then the approximation (3.4) will be transformed to the equality

$$a(x) = \frac{p_s(x) - p(x)y^2(x) - q(x)y(x) - r(x)}{y(x)} = \frac{p_s(x) - y'(x)}{y(x)}.$$

4. Numerical Results

To show the priority and efficiency of the weighted models (2.11) with respect to usual Adams-Bashforth methods, in this section we consider five numerical examples of the Riccati differential equation taken from [1, 10, 11, 17, 19, 20]. For each example, the step size, parameters s, m and the exact solution are given and also numerical results are shown in five tables in which y shows the exact solution at tested points and y_{AB}, $y^{(m)}_{wAB}$ and y_{wAB} are respectively the approximate solutions corresponding to usual Adams Bashfoth method, weighted Adams Bashfoth method of order m and weighted Adams Bashfoth method of order $m \to \infty$ (i.e. the exact solution).

Example 4.1. Consider the well-known equation taken from [10, 11, 17, 19]

$$\begin{aligned} y' &= 1 + x^2 - y^2, \ (0 \le x \le 1) \\ y(0) &= 1, \end{aligned} \tag{4.1}$$

with the exact solution $y(x) = x + \frac{e^{-x^2}}{1+\int_0^x e^{-t^2}dt}$.
Since $F(x, y) = 1 + x^2 - y^2$, the usual Adams-Bashforth method for e.g. $s = 2$ takes the form

$$\begin{aligned} &y_{AB}(x_{n+1}) = y_{AB}(x_n) \\ &+h\left(\frac{23}{12}F(x_n, y(x_n)) - \frac{4}{3}F(x_{n-1}, y(x_{n-1})) + \frac{5}{12}F(x_{n-2}, y(x_{n-2}))\right). \end{aligned} \tag{4.2}$$

For e.g. $m = 10$, the values $\{y^{(i)}(0)\}_{i=0}^{10}$ can be computed from (4.1) directly so that we have

$$\begin{aligned} &Q^{(1)}_{10}(x) = \sum_{i=0}^{10} \frac{y^{(i)}(0)}{i!} x^i \\ &= \frac{19}{3150}x^{10} - \frac{11}{945}x^9 + \frac{41}{2520}x^8 - \frac{1}{105}x^7 - \frac{1}{45}x^6 + \frac{1}{15}x^5 - \frac{1}{6}x^4 + \frac{1}{3}x^3 + 1. \end{aligned}$$

Since in this example

$$G(x, y) = 1 + x^2 - y^2 + \frac{p_2(x) - 1 - x^2 + (Q^{(1)}_{10}(x))^2}{Q^{(1)}_{10}(x)} y,$$

and

$$w^{(1)}_{10}(x) = \exp\left(\int \frac{p_2(x) - 1 - x^2 + (Q^{(1)}_{10}(x))^2}{Q^{(1)}_{10}(x)} dx\right),$$

if for example we assume that $p_2(x) = 0$, then the weighted model of Adams-Bashforth method of order $m = 10$ takes the form

$$\begin{aligned} & w_{10}^{(1)}(x_{n+1})y_{wAB}^{(m)}(x_{n+1}) = w_{10}^{(1)}(x_n)y_{wAB}^{(m)}(x_n) \\ & +h\Bigg((\mu_0^*(x_n) + \frac{3}{2}\mu_1^*(x_n) + \mu_2^*(x_n))G(x_n, y(x_n)) - (2\mu_1^*(x_n) + \mu_2^*(x_n))G(x_{n-1} \\ & , y(x_{n-1})) + \frac{1}{2}\mu_1^*(x_n) + \frac{1}{2}\mu_2^*(x_n))G(x_{n-2}, y(x_{n-2}))\Bigg), \end{aligned} \tag{4.3}$$

in which

$$\mu_k^*(x_n) = \frac{1}{h^{k+1}} \int_{x_n}^{x_{n+1}} \exp\Big(\int \frac{-1 - x^2 + (Q_{10}^{(1)}(x))^2}{Q_{10}^{(1)}(x)} dx\Big)(x - x_n)^k dx.$$

Similarly when $m \to \infty$ we have

$$\lim_{m\to\infty} Q_m^{(1)}(x) = y(x) = x + \frac{e^{-x^2}}{1 + \int_0^x e^{-t^2} dt}.$$

In this case $G(x, y) = p_2(x) = 0$, and the weight function takes the form

$$w_{exact}^{(1)}(x) = \exp\Big(\int \frac{-1 - x^2 + (x + \frac{e^{-x^2}}{1+\int_0^x e^{-t^2} dt})^2}{x + \frac{e^{-x^2}}{1+\int_0^x e^{-t^2} dt}} dx\Big) = \frac{1 + \int_0^x e^{-t^2} dt}{x + e^{-x^2} + x\int_0^x e^{-t^2} dt}.$$

Consequently, the weighted model (4.3) for the exact solution would be simplified as

$$w_{exact}^{(1)}(x_{n+1})y_{wAB}(x_{n+1}) = w_{exact}^{(1)}(x_n)y_{wAB}(x_n), \tag{4.4}$$

leading to the exact solution

$$y_{wAB}(x_{n+1}) = \frac{w_{exact}^{(1)}(x_0)(x_n + e^{-(x_n)^2} + x_n\int_0^{x_n} e^{-t^2} dt)}{1 + \int_0^{x_n} e^{-t^2} dt} y_{wAB}(x_0).$$

Table 1 shows the numerical results of the three introduced models (4.2), (4.3) and (4.4) for $h = 0.01$. As we observe in this table, the results for weighted Adams-Bashforth method of order $m = 10$ are better than usual method and the results for weighted Adams-Bashforth method corresponding to the exact solution are much better than usual and $m = 10$ cases respectively. In this sense, Figure 1 clearly shows the priority of weighted models with respect to the usual Adams-Bashforth method for Example 1.

Example 4.2. Consider the Riccati equation

$$\begin{aligned} y' &= -\sin x + (\cos^2 x)y - (\cos x)y^2, \ (0 \le x \le 1) \\ y(0) &= 1, \end{aligned} \tag{4.5}$$

x	y	y_{AB}	$y_{wAB}^{(m)}$	y_{wAB}	$\|y-y_{AB}\|$	$\|y-y_{wAB}^{(m)}\|$	$\|y-y_{wAB}\|$
0.04	1.0000209134	1.0000209416	1.0000209133	1.0000209133	2.8265×10^{-8}	4.4273×10^{-11}	4.3050×10^{-11}
0.12	1.000543029	1.0005431504	1.0005430297	1.0005430297	1.2066×10^{-7}	3.8798×10^{-11}	3.4260×10^{-11}
0.20	1.0024198254	1.0024200113	1.0024198254	1.0024198254	1.8591×10^{-7}	3.4236×10^{-11}	2.7247×10^{-11}
0.28	1.0063961567	1.0063963878	1.0063961566	1.0063961567	2.3108×10^{-7}	4.1938×10^{-11}	2.1623×10^{-11}
0.36	1.0131035173	1.0131037791	1.0131035170	1.0131035172	2.6185×10^{-7}	2.6319×10^{-10}	1.7097×10^{-11}
0.40	1.0176508788	1.0176511521	1.0176508780	1.0176508788	2.7324×10^{-7}	8.4804×10^{-10}	1.5173×10^{-11}
0.44	1.0230727184	1.0230730009	1.0230727158	1.0230727184	2.8258×10^{-7}	2.5558×10^{-9}	1.3446×10^{-11}
0.52	1.0367427877	1.0367430842	1.0367427698	1.0367427877	2.9645×10^{-7}	1.7907×10^{-8}	1.0498×10^{-11}
0.60	1.0544668099	1.0544671155	1.0544667170	1.0544668099	3.0560×10^{-7}	9.2834×10^{-8}	8.1197×10^{-12}
0.68	1.0765155093	1.0765158206	1.0765151329	1.0765155093	3.1127×10^{-7}	3.7645×10^{-7}	6.2045×10^{-12}
0.76	1.1030794073	1.1030797213	1.1030781740	1.1030794073	3.1396×10^{-7}	1.2333×10^{-6}	4.6724×10^{-12}
0.84	1.1342703768	1.1342706904	1.1342670903	1.1342703768	3.1361×10^{-7}	3.2864×10^{-6}	3.4567×10^{-12}
0.92	1.1701233713	1.1701236811	1.1701164943	1.1701233713	3.0976×10^{-7}	3.0178×10^{-6}	2.5055×10^{-12}
1.00	1.2105990146	1.2105993164	1.2105895716	1.2105990146	3.0178×10^{-7}	9.4430×10^{-6}	1.7730×10^{-12}

TABLE 1. Numerical results for the approximate solutions of Example 1 with $h = 0.01, s = 2$ and $m = 10$.

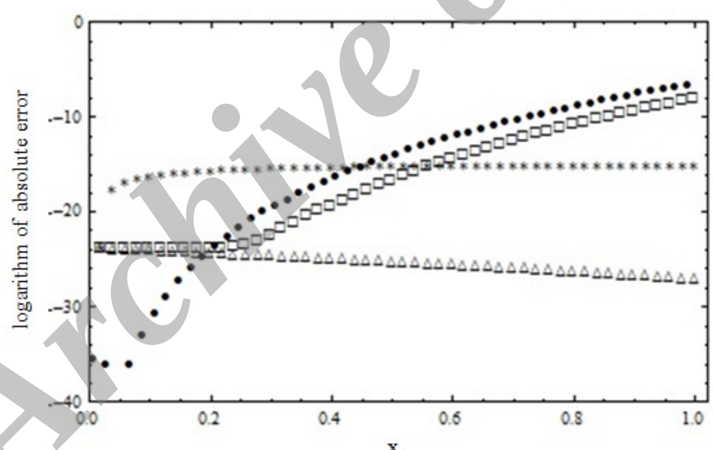

FIGURE 1. The logarithm of absolute error for three models introduced in Example 1, i.e. the usual Adams Bashforth method ($\star$), the weighted model of Adams Bashfoth for $m = 10$ ($\square$), the weighted model corresponding to the exact solution ($\triangle$) and finally $Q_{10}^{(1)}(x)$ ($\bullet$) with $h = 0.01, s = 2$ and $m = 10$.

with the exact solution $y(x) = \cos x$.
Since $F(x, y) = -\sin x + (\cos^2 x)y - (\cos x)y^2$, the usual Adams-Bashfoth

method for e.g. $s = 2$ would be in the form

$$y_{AB}(x_{n+1}) = y_{AB}(x_n) + h\left(\frac{23}{12}F(x_n, y(x_n)) - \frac{4}{3}F(x_{n-1}, y(x_{n-1})) + \frac{5}{12}F(x_{n-2}, y(x_{n-2}))\right). \tag{4.6}$$

For e.g. $m = 5$, the values $\{y^{(i)}(0)\}_{i=0}^{5}$ can be computed from (4.5) directly. Therefore

$$Q_5^{(2)}(x) = \sum_{i=0}^{5} \frac{y^{(i)}(0)}{i!}x^i = \frac{x^4}{24} - \frac{x^2}{2} + 1$$

Since in this example

$$G(x, y) = -(\cos x)y^2 + (\cos^2 x)y - \sin x + \Big(\frac{p_2(x) + (\cos x)\big(Q_5^{(2)}(x)\big)^2 - (\cos^2 x)Q_5^{(2)}(x) + \sin x}{Q_5^{(2)}(x)}\Big)y,$$

and

$$w_5^{(2)}(x) = \exp\Big(\int \Big(\frac{p_2(x) + (\cos x)(Q_5^{(2)}(x))^2 - (\cos^2 x)Q_5^{(2)}(x) + \sin x}{Q_5^{(2)}(x)}dx\Big),$$

if for example we assume that $p_2(x) = 0$, then the weighted model of Adams-Bashfoth method of order $m = 5$ takes the form

$$w_5^{(2)}(x_{n+1})y_{wAB}^{(m)}(x_{n+1}) = w_5^{(2)}(x_n)y_{wAB}^{(m)}(x_n) + h\Bigg((\mu_0^*(x_n) + \frac{3}{2}\mu_1^*(x_n) + \mu_2^*(x_n))G(x_n, y(x_n)) - (2\mu_1^*(x_n) + \mu_2^*(x_n))G(x_{n-1}, y(x_{n-1})) + \frac{1}{2}\mu_1^*(x_n) + \frac{1}{2}\mu_2^*(x_n))G(x_{n-2}, y(x_{n-2}))\Bigg), \tag{4.7}$$

in which

$$\mu_k^*(x_n) = \frac{1}{h^{k+1}} \times \int_{x_n}^{x_{n+1}} \exp\Big(\int \Big(\frac{p_2(x) + (\cos x)\big(Q_5^{(2)}(x)\big)^2 - (\cos^2 x)Q_5^{(2)}(x) + (\sin x)}{Q_5^{(2)}(x)}dx\Big)(x - x_n)^k dx.$$

Similarly when $m \to \infty$ we have

$$\lim_{m\to\infty} Q_m^{(2)}(x) = y(x) = \cos x.$$

In this case $G(x, y) = p_2(x) = 0$, and the weight function takes the form

$$w_{exact}^{(2)}(x) = \exp\Big(\int \tan x dx\Big) = \frac{1}{\cos x}.$$

Therefore, the weighted model (4.7) for the exact solution would be simplified as

$$w_{exact}^{(2)}(x_{n+1})y_{wAB}(x_{n+1}) = w_{exact}^{(2)}(x_n)y_{wAB}(x_n), \tag{4.8}$$

leading to the exact solution

$$y_{wAB}(x_{n+1}) = w^{(2)}_{exact}(x_0)\cos x_{n+1} y_{wAB}(x_0).$$

Table 2 shows the numerical results of the three introduced models (4.6), (4.7) and (4.8) for $h = 0.01$. Also, Figure 2 again shows the priority of weighted models with respect to the usual Adams-Bashforth method for Example 2.

x	y	y_{AB}	$y^{(m)}_{wAB}$	y_{wAB}	$\lvert y - y_{AB}\rvert$	$\lvert y - y^{(m)}_{wAB}\rvert$	$\lvert y - y_{wAB}\rvert$
0.04	0.9995500337	0.9995500307	0.9995500344	0.9995500344	3.0060×10^{-9}	7.3571×10^{-10}	7.3571×10^{-10}
0.12	0.9939560979	0.9939560664	0.9939560986	0.9939560985	3.1540×10^{-8}	6.6687×10^{-10}	6.4020×10^{-10}
0.20	0.9820042351	0.9820041773	0.9820042368	0.9820042356	5.7740×10^{-8}	1.6918×10^{-9}	5.5994×10^{-10}
0.28	0.9637708963	0.9637708146	0.9637709080	0.9637708968	8.1720×10^{-8}	1.1687×10^{-8}	4.9544×10^{-10}
0.36	0.9393727128	0.9393726091	0.9393727682	0.9393727132	1.0365×10^{-7}	5.5368×10^{-8}	4.4430×10^{-10}
0.44	0.9089657496	0.9089656259	0.9089659220	0.9089657500	1.2373×10^{-7}	1.7241×10^{-7}	4.0436×10^{-10}
0.52	0.8727445076	0.8727443654	0.8727448818	0.8727445080	1.4216×10^{-7}	3.7425×10^{-7}	3.7372×10^{-10}
0.60	0.8309406791	0.8309405199	0.8309412117	0.8309406794	1.5914×10^{-7}	5.3267×10^{-7}	3.5071×10^{-10}
0.68	0.7838216658	0.7838214910	0.7838218736	0.7838216662	1.7484×10^{-7}	2.0775×10^{-7}	3.3386×10^{-10}
0.76	0.7316888688	0.7316886794	0.7316873811	0.7316888691	1.8942×10^{-7}	1.4877×10^{-6}	3.218×10^{-10}
0.84	0.6748757600	0.6748755570	0.6748700495	0.6748757603	2.0299×10^{-7}	5.7105×10^{-6}	3.1329×10^{-10}
0.92	0.6137457494	0.6137455338	0.6137329517	0.6137457497	2.1561×10^{-7}	0.0000127977	3.0678×10^{-10}
1.00	0.5486898605	0.5486896332	0.5486707330	0.5486898608	2.2732×10^{-7}	0.0000191275	3.0067×10^{-10}

Table 2. Numerical results for the approximate solutions of Example 2 with $h = 0.01, s = 2$ and $m = 5$.

Example 4.3. Consider the following equation taken from [1, 10, 11, 17, 19, 20],

$$\begin{aligned} y' &= 1 + 2y - y^2, \ (0 \le x \le 1) \\ y(0) &= 0, \end{aligned} \tag{4.9}$$

with the exact solution $y(x) = 1 + \sqrt{2}\tanh(\sqrt{2}x + \frac{\log(\frac{-1+\sqrt{2}}{1+\sqrt{2}})}{2})$.

Since $F(x,y) = 1 + 2y - y^2$, the usual Adams-Bashforth method for e.g. $s = 2$ takes the form

$$\begin{aligned} &y_{AB}(x_{n+1}) = y_{AB}(x_n) \\ &+h\left(\frac{23}{12}F(x_n, y(x_n)) - \frac{4}{3}F(x_{n-1}, y(x_{n-1})) + \frac{5}{12}F(x_{n-2}, y(x_{n-2}))\right). \end{aligned} \tag{4.10}$$

For e.g. $m = 10$, the values $\{y^{(i)}(0)\}_{i=0}^{10}$ can be computed from (4.9) directly so that we have

$$Q^{(3)}_{10}(x) = \sum_{i=0}^{10} \frac{y^{(i)}(0)}{i!} x^i =$$

$$-\frac{1213}{14175}x^{10} + \frac{197}{2835}x^9 + \frac{71}{315}x^8 + \frac{53}{315}x^7 - \frac{7}{45}x^6 - \frac{7}{15}x^5 - \frac{1}{3}x^4 + \frac{1}{3}x^3 + x^2 + x.$$

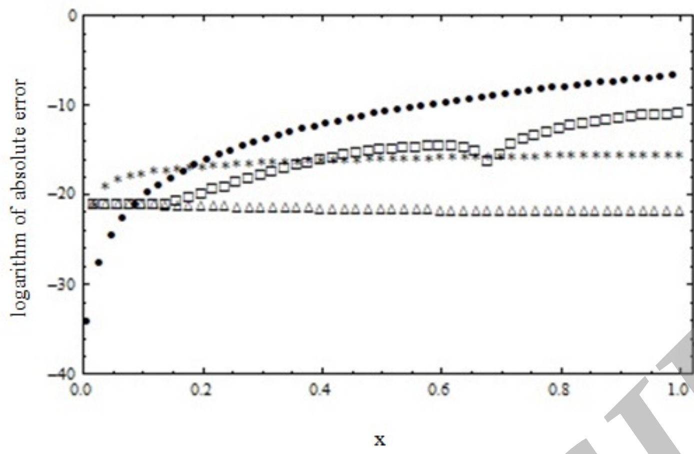

FIGURE 2. The logarithm of absolute error for three models introduced in Example 2, i.e. the usual Adams Bashforth method ($\star$), the weighted model of Adams Bashfoth for $m = 5$ ($\square$), the weighted model corresponding to the exact solution ($\triangle$) and finally $Q_5^{(2)}(x)$ ($\bullet$) with $h = 0.01, s = 2$ and $m = 5$.)

Since in this example

$$G(x,y) = 1 + 2y - y^2 + \frac{p_2(x) - 1 - 2Q_{10}^{(3)}(x) + (Q_{10}^{(3)}(x))^2}{Q_{10}^{(3)}(x)} y,$$

and

$$w_{10}^{(3)}(x) = \exp(\int \frac{p_2(x) - 1 - 2Q_{10}^{(3)}(x) + (Q_{10}^{(3)}(x))^2}{Q_{10}^{(3)}(x)} dx),$$

if for example we assume that $p_2(x) = 0$, then the weighted models of Adams-Bashforth method of order $m = 10$ takes the form

$$\begin{aligned} &w_{10}^{(3)}(x_{n+1}) y_{wAB}^{(m)}(x_{n+1}) = w_{10}^{(3)}(x_n) y_{wAB}^{(m)}(x_n) \\ &+h\left((\mu_0^*(x_n) + \frac{3}{2}\mu_1^*(x_n) + \mu_2^*(x_n)) G(x_n, y(x_n)) - (2\mu_1^*(x_n) + \mu_2^*(x_n)) \right. \\ &\left. G(x_{n-1}, y(x_{n-1})) + \frac{1}{2}\mu_1^*(x_n) + \frac{1}{2}\mu_2^*(x_n)) G(x_{n-2}, y(x_{n-2})) \right), \end{aligned} \tag{4.11}$$

in which

$$\mu_k^*(x_n) = \frac{1}{h^{k+1}} \int_{x_n}^{x_{n+1}} \exp(\int \frac{-1 - 2Q_{10}^{(3)}(x) + (Q_{10}^{(3)}(x))^2}{Q_{10}^{(3)}(x)} dx)(x - x_n)^k dx.$$

Similarly when $m \to \infty$, we have

$$\lim_{m\to\infty} Q_m^{(3)}(x) = y(x) = 1 + \sqrt{2}\tanh(\sqrt{2}x + \frac{\log(\frac{-1+\sqrt{2}}{1+\sqrt{2}})}{2}).$$

In this case $G(x,y) = p_2(x) = 0$, and the weight function takes the form

$$\mathrm{w}_{exact}^{(3)}(x) =$$

$$= \exp\Big(\int \frac{-1-2(1+\sqrt{2}\tanh(\sqrt{2}x+\frac{\log(\frac{-1+\sqrt{2}}{1+\sqrt{2}})}{2})) + (1+\sqrt{2}\tanh(\sqrt{2}x+\frac{\log(\frac{-1+\sqrt{2}}{1+\sqrt{2}})}{2}))^2}{1+\sqrt{2}\tanh(\sqrt{2}x+\frac{\log(\frac{-1+\sqrt{2}}{1+\sqrt{2}})}{2})}dx\Big)$$

$$= \frac{1}{1+\sqrt{2}\tanh(\sqrt{2}x+\frac{\log(\frac{-1+\sqrt{2}}{1+\sqrt{2}})}{2})}.$$

Consequently, the weighted model (4.11) for the exact solution is as

$$w_{exact}^{(3)}(x_{n+1})y_{wAB}(x_{n+1}) = w_{exact}^{(3)}(x_n)y_{wAB}(x_n), \tag{4.12}$$

leading to the exact solution

$$y_{wAB}(x_{n+1}) = w_{exact}^{(3)}(x_0)(1+\sqrt{2}\tanh(\sqrt{2}x_n+\frac{\log(\frac{-1+\sqrt{2}}{1+\sqrt{2}})}{2}))y_{wAB}(x_0).$$

Table 3 shows the numerical results of the three introduced models (4.10), (4.11) and (4.12) for $h = 0.01$. Also, Figure 3 again shows the priority of weighted models with respect to the usual Adams-Bashforth method for Example 3.

x	y	y_{AB}	$y_{wAB}^{(m)}$	y_{wAB}	$\lvert y - y_{AB}\rvert$	$\lvert y - y_{wAB}^{(m)}\rvert$	$\lvert y - y_{wAB}\rvert$
0.04	0.0416204316	0.0416204942	0.0416204180	0.0416204180	6.2600×10^{-8}	1.3519×10^{-8}	1.3600×10^{-8}
0.12	0.1348948736	0.1348953345	0.1348948298	0.1348948298	4.6090×10^{-7}	4.3781×10^{-8}	4.3800×10^{-8}
0.20	0.2419767996	0.2419778639	0.2419767239	0.2419767210	1.0643×10^{-6}	7.5677×10^{-8}	7.8600×10^{-8}
0.28	0.3628219749	0.3628238267	0.3628219059	0.3628218570	1.8518×10^{-6}	6.9023×10^{-8}	1.1790×10^{-7}
0.36	0.4965908125	0.4965935600	0.4965910069	0.4965906512	2.7475×10^{-6}	1.9433×10^{-7}	1.6130×10^{-7}
0.44	0.6415366029	0.6415402238	0.6415378018	0.6415363945	3.6209×10^{-6}	1.1989×10^{-6}	2.0840×10^{-7}
0.52	0.7950036737	0.7950079861	0.7950057378	0.7950034155	4.3124×10^{-6}	2.0640×10^{-6}	2.5820×10^{-7}
0.60	0.9535662164	0.9535708919	0.9535565210	0.9535659067	4.6755×10^{-6}	9.6954×10^{-6}	3.0970×10^{-7}
0.68	1.1133061389	1.1133107663	1.1132088582	1.1133057773	4.6274×10^{-6}	0.0000972807	3.616×10^{-7}
0.76	1.2701882271	1.2701924050	1.2696986013	1.2701878145	4.1779×10^{-6}	0.0004896257	4.1260×10^{-7}
0.84	1.4204593631	1.4204627898	1.4185980792	1.4204589017	3.4267×10^{-6}	0.0018612838	4.6140×10^{-7}
0.92	1.5609903824	1.5609929096	1.5550760962	1.5609898753	2.5272×10^{-6}	0.0059142861	5.0710×10^{-7}
1.00	1.6894983915	1.6895000288	1.6730753877	1.6894978428	1.6373×10^{-6}	0.0164230038	5.4870×10^{-7}

TABLE 3. Numerical results for the approximate solutions of Example 3 with $h = 0.01, s = 2$ and $m = 10$.

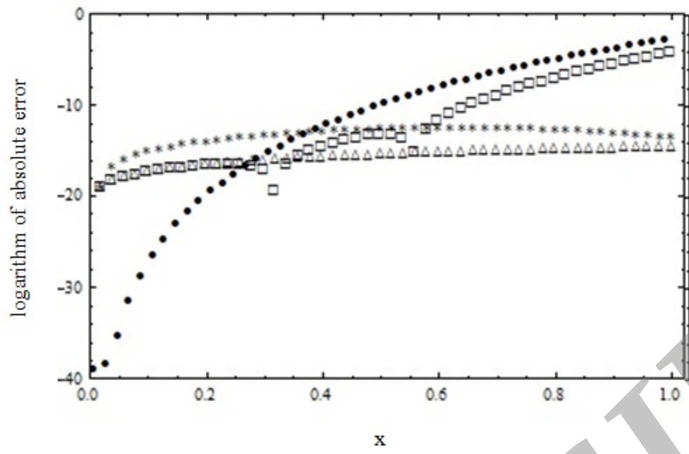

FIGURE 3. The logarithm of absolute error for three models introduced in Example 3, i.e. the usual Adams Bashforth method ($\star$), the weighted model of Adams Bashfoth for $m = 10$ ($\square$), the weighted model corresponding to the exact solution ($\triangle$) and finally $Q_{10}^{(3)}(x)$ ($\bullet$) with $h = 0.01, s = 2$ and $m = 10$.)

EXAMPLE 4.4. Consider the equation

$$\begin{aligned} &y' = -e^{2x} + y + y^2, (0 \leqslant x \leqslant 1) \\ &y(0) = 1, \end{aligned} \tag{4.13}$$

with the exact solution $y(x) = e^x$.

Since $F(x, y) = -e^{2x} + y + y^2$, the usual Adams-Bashforth method for e.g. $s = 2$ takes the form

$$\begin{aligned} &y_{AB}(x_{n+1}) = y_{AB}(x_n) \\ &+h\left(\frac{23}{12}F(x_n, y(x_n)) - \frac{4}{3}F(x_{n-1}, y(x_{n-1})) + \frac{5}{12}F(x_{n-2}, y(x_{n-2}))\right). \end{aligned} \tag{4.14}$$

For e.g. $m = 5$, the values $\{y^{(i)}(0)\}_{i=0}^{5}$ are computable from (4.13) so that we have

$$Q_5^{(4)}(x) = \sum_{i=0}^{5} \frac{y^{(i)}(0)}{i!}x^i = \frac{1}{120}x^5 + \frac{1}{24}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1.$$

Since in this example

$$G(x, y) = -e^{2x} + y + y^2 + \frac{p_2(x) + e^{2x} - Q_5^{(4)}(x) - (Q_5^{(4)}(x))^2}{Q_5^{(4)}(x)}y,$$

and

$$w_5^{(4)}(x) = \exp(\int \frac{p_2(x) + e^{2x} - Q_5^{(4)}(x) - (Q_5^{(4)}(x))^2}{Q_5^{(4)}(x)}dx),$$

if for example we assume that $p_2(x) = 0$, then the weighted models of Adams-Bashforth method of order $m = 5$ takes the form

$$\begin{aligned} & w_5^{(4)}(x_{n+1})y_{wAB}^{(m)}(x_{n+1}) = w_5^{(4)}(x_n)y_{wAB}^{(m)}(x_n) \\ & +h\left((\mu_0^*(x_n) + \frac{3}{2}\mu_1^*(x_n) + \mu_2^*(x_n))G(x_n, y(x_n)) - (2\mu_1^*(x_n) + \mu_2^*(x_n))G(x_{n-1}\right. \\ & \left., y(x_{n-1})) + \frac{1}{2}\mu_1^*(x_n) + \frac{1}{2}\mu_2^*(x_n))G(x_{n-2}, y(x_{n-2}))\right), \end{aligned} \tag{4.15}$$

in which

$$\mu_k^*(x_n) = \frac{1}{h^{k+1}} \int_{x_n}^{x_{n+1}} \exp(\int \frac{e^{2x} - Q_5^{(4)}(x) - (Q_5^{(4)}(x))^2}{Q_5^{(4)}(x)})(x - x_n)^k dx.$$

Also when $m \to \infty$

$$\lim_{m\to\infty} Q_m^{(4)}(x) = y(x) = e^x.$$

In this case $G(x, y) = p_2(x) = 0$, and the weight function takes the explicit form

$$w_{exact}^{(4)}(x) = e^{-x}.$$

Consequently, the weighted model (4.15) for the exact solution would be simplified as

$$w_{exact}^{(4)}(x_{n+1})y_{wAB}(x_{n+1}) = w_{exact}^{(4)}(x_n)y_{wAB}(x_n), \tag{4.16}$$

leading to the exact solution

$$y_{wAB}(x_{n+1}) = w_{exact}^{(4)}(x_0)e^{x_{n+1}}y_{wAB}(x_0).$$

Table 4 shows the numerical results of the three introduced models (4.14), (4.15) and (4.16) for $h = 0.01$. Also, Figure 4 again shows the priority of weighted models with respect to the usual Adams-Bashforth method for Example 4.

Example 4.5. Consider the equation

$$\begin{aligned} y' &= (\frac{1}{2(x+1)} - \sqrt{x+1})y + y^2, \ (0 \le x \le 1) \\ y(0) &= 1, \end{aligned} \tag{4.17}$$

with the exact solution $y(x) = \sqrt{x+1}$.

Since $F(x, y) = (\frac{1}{2(x+1)} - \sqrt{x+1})y + y^2$, the usual Adams-Bashforth method for e.g. $s = 2$ takes the form

$$\begin{aligned} & y_{AB}(x_{n+1}) = y_{AB}(x_n) \\ & +h\left(\frac{23}{12}F(x_n, y(x_n)) - \frac{4}{3}F(x_{n-1}, y(x_{n-1})) + \frac{5}{12}F(x_{n-2}, y(x_{n-2}))\right). \end{aligned} \tag{4.18}$$

x	y	y_{AB}	$y_{wAB}^{(m)}$	y_{wAB}	$\|y - y_{AB}\|$	$\|y - y_{wAB}^{(m)}\|$	$\|y - y_{wAB}\|$
0.04	1.0304545339	1.0304545226	1.0304545266	1.0304545266	1.1288×10^{-8}	7.3488×10^{-9}	7.3488×10^{-9}
0.12	1.1162780704	1.1162780197	1.1162780619	1.1162780619	5.0707×10^{-8}	8.5578×10^{-9}	8.5032×10^{-9}
0.20	1.2092495976	1.2092494920	1.2092495851	1.2092495876	1.0564×10^{-7}	1.2545×10^{-8}	9.9700×10^{-9}
0.28	1.3099644507	1.3099642676	1.3099644073	1.3099644388	1.8311×10^{-7}	4.3385×10^{-8}	1.1851×10^{-8}
0.36	1.4190675485	1.4190672547	1.4190673313	1.4190675342	2.9386×10^{-7}	2.1720×10^{-7}	1.4316×10^{-8}
0.44	1.5372575235	1.5372570687	1.5372566115	1.5372575059	4.5476×10^{-7}	9.1200×10^{-7}	1.7625×10^{-8}
0.52	1.6652911949	1.6652905021	1.6652881026	1.6652911727	6.9282×10^{-7}	3.0923×10^{-6}	2.2184×10^{-8}
0.60	1.8039884153	1.8039873631	1.8039795724	1.8039883867	1.0522×10^{-6}	8.8429×10^{-6}	2.8660×10^{-8}
0.68	1.9542373206	1.9542357134	1.9542151763	1.9542372824	1.6072×10^{-6}	0.0000221443	3.8175×10^{-8}
0.76	2.1170000166	2.1169975313	2.1169501588	2.1169999639	2.4852×10^{-6}	0.0000498577	5.2706×10^{-8}
0.84	2.2933187402	2.2933148280	2.2932160153	2.2933186643	3.9122×10^{-6}	0.0001027249	7.5893×10^{-8}
0.92	2.4843225333	2.4843162340	2.4841267670	2.4843224185	6.2993×10^{-6}	0.0001957663	1.1481×10^{-7}
1.00	2.6912344723	2.6912240520	2.6908881124	2.6912342882	0.0000104202	0.0003463598	1.8404×10^{-7}

TABLE 4. Numerical results for the approximate solutions of Example 4 with $h = 0.01, s = 2$ and $m = 5$.

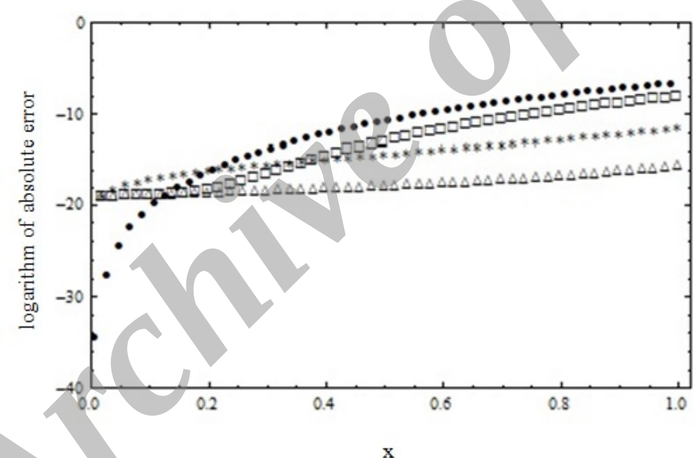

FIGURE 4. The logarithm of absolute error for three models introduced in Example 4, i.e. the usual Adams Bashforth method ($\star$), the weighted model of Adams Bashfoth for $m = 5$ ($\square$), the weighted model corresponding to the exact solution ($\triangle$) and finally $Q_5^{(4)}(x)$ ($\bullet$) with $h = 0.01, s = 2$ and $m = 5$.)

For e.g. $m = 5$, the values $\{y^{(i)}(0)\}_{i=0}^{5}$ are computable from (4.17) so that we have

$$Q_5^{(5)}(x) = \sum_{i=0}^{5} \frac{y^{(i)}(0)}{i!} x^i = \frac{7}{256}x^5 - \frac{5}{128}x^4 + \frac{1}{16}x^3 - \frac{1}{8}x^2 + \frac{1}{2}x + 1.$$

Since in this example

$$G(x,y) = (\frac{1}{2(x+1)} - \sqrt{x+1})y + y^2 + \frac{p_2(x) - (\frac{1}{2(x+1)} - \sqrt{x+1})Q_5^{(5)}(x) - \left(Q_5^{(5)}(x)\right)^2}{Q_5^{(5)}(x)}y,$$

and

$$w_5^{(5)}(x) = \exp(\int \frac{p_2(x) - (\frac{1}{2(x+1)} - \sqrt{x+1})Q_5^{(5)}(x) - (Q_5^{(5)}(x))^2}{Q_5^{(5)}(x)}dx),$$

if for example we assume that $p_2(x) = 0$, then the weighted models of Adams-Bashforth method of order $m = 5$ takes the form

$$\begin{aligned} &w_5^{(5)}(x_{n+1})y_{wAB}^{(m)}(x_{n+1}) = w_5^{(5)}(x_n)y_{wAB}^{(m)}(x_n) + h\,((\mu_0^*(x_n)+ \\ &\frac{3}{2}\mu_1^*(x_n) + \mu_2^*(x_n))G(x_n, y(x_n)) - (2\mu_1^*(x_n) + \mu_2^*(x_n))G(x_{n-1} \\ &, y(x_{n-1})) + \frac{1}{2}\mu_1^*(x_n) + \frac{1}{2}\mu_2^*(x_n))G(x_{n-2}, y(x_{n-2}))\Big), \end{aligned} \tag{4.19}$$

in which

$$\mu_k^*(x_n) = \frac{1}{h^{k+1}}\int_{x_n}^{x_{n+1}} \exp(\int -(\frac{1}{2(x+1)} - \sqrt{x+1}) - Q_5^{(5)}(x))(x-x_n)^k dx.$$

Also when $m \to \infty$

$$\lim_{m\to\infty} Q_m^{(5)}(x) = y(x) = \sqrt{x+1}.$$

In this case $G(x,y) = p_2(x) = 0$, and the weight function takes the form

$$w_{exact}^{(5)}(x) = \exp(\int \frac{-1}{2(x+1)}dx) = \frac{1}{\sqrt{x+1}}.$$

Consequently, the weighted model (4.19) for the exact solution would be simplified as

$$w_{exact}^{(5)}(x_{n+1})y_{wAB}(x_{n+1}) = w_{exact}^{(5)}(x_n)y_{wAB}(x_n), \tag{4.20}$$

leading to the exact solution

$$y_{wAB}(x_{n+1}) = w_{exact}^{(5)}(x_0)\sqrt{x_{n+1}+1}\,y_{wAB}(x_0).$$

Table 5 shows the numerical results of the three introduced models (4.18), (4.19) and (4.20) for $h = 0.01$. Also, Figure 5 again shows the priority of weighted models with respect to the usual Adams-Bashforth method for Example 5.

Remark 4.6. It is important to mention that the proposed weighted method can also be applied for solving two points boundary value problems. In other words, by using the shooting method one can reduce the solution of a two points boundary value problem to the solution of an initial value problem. Moreover,

x	y	y_{AB}	$y_{wAB}^{(m)}$	y_{wAB}	$\lvert y - y_{AB}\rvert$	$\lvert y - y_{wAB}^{(m)}\rvert$	$\lvert y - y_{wAB}\rvert$
0.04	1.0148891565	1.0148891604	1.0148891570	1.0148891570	3.9261×10^{-9}	5.7763×10^{-10}	5.7760×10^{-10}
0.12	1.0535653752	1.0535654043	1.0535653760	1.0535653759	2.9036×10^{-8}	8.0047×10^{-10}	6.3606×10^{-10}
0.20	1.0908712114	1.0908712632	1.0908712185	1.0908712121	5.1804×10^{-8}	7.0431×10^{-9}	7.0280×10^{-10}
0.28	1.1269427669	1.1269428404	1.1269428336	1.1269427677	7.3500×10^{-8}	6.6718×10^{-8}	7.7828×10^{-10}
0.36	1.1618950038	1.1618950989	1.1618953670	1.1618950047	9.5068×10^{-8}	3.6319×10^{-7}	8.6407×10^{-10}
0.44	1.1958260743	1.1958261915	1.1958274375	1.1958260752	1.1727×10^{-7}	1.3632×10^{-6}	9.6195×10^{-10}
0.52	1.2288205727	1.2288207135	1.2288245452	1.2288205738	1.4079×10^{-7}	3.9725×10^{-6}	1.0741×10^{-9}
0.60	1.2609520212	1.2609521875	1.2609616405	1.2609520224	1.6625×10^{-7}	9.6192×10^{-6}	1.2030×10^{-9}
0.68	1.2922847983	1.2922849926	1.2923049467	1.2922847996	1.9427×10^{-7}	0.0000201484	1.3518×10^{-9}
0.76	1.3228756555	1.3228758810	1.3229130248	1.3228756570	2.2551×10^{-7}	0.0000373693	1.5242×10^{-9}
0.84	1.3527749258	1.3527751865	1.3528369722	1.3527749275	2.6067×10^{-7}	0.0000620463	1.7245×10^{-9}
0.92	1.3820274961	1.3820277966	1.38211954341	1.3820274980	3.0053×10^{-7}	0.0000920473	1.9582×10^{-9}
1.00	1.4106735979	1.4106739439	1.4107928520	1.4106736001	3.4599×10^{-7}	0.0001192540	2.2316×10^{-9}

TABLE 5. Numerical results for the approximate solutions of Example 5 with $h = 0.01, s = 2$ and $m = 5$.

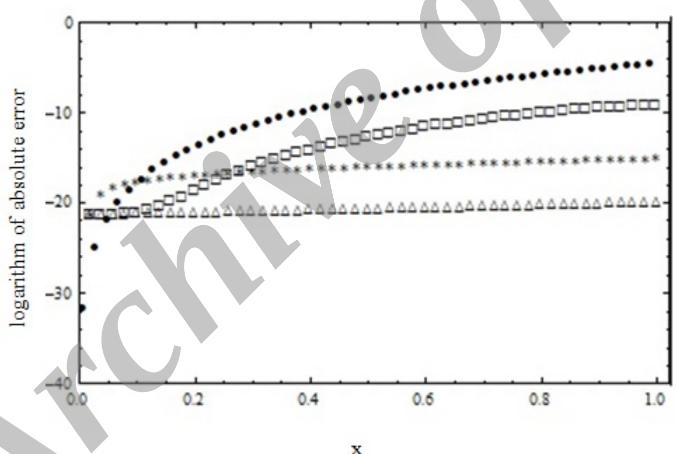

FIGURE 5. The logarithm of absolute error for three models introduced in Example 5, i.e. the usual Adams Bashforth method ($\star$), the weighted model of Adams Bashfoth for $m = 5$ ($\square$), the weighted model corresponding to the exact solution ($\triangle$) and finally $Q_5^{(5)}(x)$ ($\bullet$) with $h = 0.01, s = 2$ and $m = 5$.)

since matrix Riccati differential equations can be transformed to a system of usual Riccati differential equations, the aforesaid problem may be extended to a matrix form [in preparation].

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References

1. S. Abbasbandy, A New Application of He's Variational Iteration Method for Quadratic Riccati Differential Equation by Using Adomians Polynomials, *Journal of Computational and Applied Mathematics*, **207**, (2007), 59-63.
2. S. Abbasbandy, Iterated He's Homotopy Perturbation Method for Quadratic Riccati Differential Equation, *Applied Mathematics and Computation*, **175**, (2006), 581-589.
3. S. Abbasbandy, Homotopy Perturbation Method for Quadratic Riccati Differential Equationan and Comparison with Adomians Decomposition Method, *Applied Mathematics and Computation*, **172**, (2006), 485-490.
4. H. Aminikhah, M.Hemmatnezhad, An Efficient Method for Quadratic Riccati Differential Equation, *Communications in Nonlinear Science and Numerical Simulation*, **15**, (2010), 835-839.
5. K. Atkinson, W. Ham, D. Stewart, *Numerical Solution of Ordinary Differential Equations*, Wiley, 2009.
6. H. Bulut, D.J.Evans, On the Solution of the Riccati Equation by the Decomposition Method, *International Journal of Computer Mathematics*, **79**, (2002), 103-109.
7. J. C. Butcher, *Numerical Methods for Ordinary Differential Equations*, John Wiley & Sons, Ltd, 2008.
8. P. L. Butzer, M. Hauss, On Stirling Functions of the Second Kind, *Studies in Applied Mathematics*, **84**, (1991), 71-91.
9. G. Dahlquist, Convergence and Stability in the Numerical Integration of Ordinary Differential Equations, *Mathematica Scandinavica*, **4**, (1956), 33-53.
10. F. Geng , A Modified Variational Iteration Method for Solving Riccati Differential Equations, *Computers & Mathematics with Applications*, **60**, (2010), 1868-1872.
11. F. Geng, Y. Lin, M. Cui, A Piecewise Variational Iteration Method for Riccati Differential Equations, *Computers & Mathematics with Applications*, **58**, (2009), 2518-2522.
12. M. Gulsu, M. Sezer, On the Solution of the Riccati Equation by the Taylor Matrix Method, *Applied Mathematics and Computation*, **176**, (2006), 414-421.
13. B. Jazbi, M. Moini, Application of Hes Homotopy Perturbation Method for SchröDinger Equation, *Iranian Journal of Mathematical Sciences and Informatics*, **3**(2), (2008), 13-19.
14. M. Lakestani, M. Dehghan, Numerical Solution of Riccati Equation Using the Cubic B-spline Scaling Functions and Chebyshev Cardinal Functions, *Computer Physics Communications*, **181**, (2010), 957-966.
15. M. Masjed-Jamei, On Constructing New Interpolation Formulas Using Linear Operators and an Operator Type of Quadratur Rules, *Journal of Computational and Applied Mathematics*, **216**, (2008), 307-318.
16. M. Masjed-Jamei, On Constructing New Expansions of Functions Using Linear Operators, *Journal of Computational and Applied Mathematics*, **234**, (2010), 365-374.
17. F. Mohammadi, M. M. Hosseini, A Comparative Study of Numerical Methods for Solving Quadratic Riccati Differential Equations, *Journal of the Franklin Institute*, **348**, (2011), 156-164.

18. P. Mokhtary, S. M. Hosseini, Some Implementation Aspects of the General Linear Methods Withinherent Runge-Kutta Stability, *Iranian Journal of Mathematical Sciences and Informatics*, **3**(1), (2008), 63-76.
19. B. Tang, X. Li, A New Method for Determining the Solution of Riccati Dierential Equations, *Applied Mathematics and Computation*, **194**, (2007), 431-440.
20. P. Tsai, C. Chen, An Approximate Analytic Solution of the Nonlinear Riccati Differential equation, *Journal of the Franklin Institute*, **347**, (2010), 1850-1862.