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Integral Inequalities for Riemann-Liouville Fractional Integrals of a Function With Respect to Another Function

Ergun Kacar[a,*], Zeynep Kacar[b] , Huseyin Yildirim[c]

[a,c]University of Kahramanmaras Sütçü Imam, Department of Mathematics, Avsar 46100 Kahramanmaras, Turkey.
[b] University of Maryland, Department of Statistics, 4176 Campus drive-William E Kirwan Hall, College Park, MD, 20742-4015, USA.

E-mail: kacarergun@gmail.com
E-mail: zkacar@math.umd.edu
E-mail: hyildir@ksu.edu.tr

Abstract. In this article, we obtain generalizations for Gruss type integral inequality by using h(x)-Riemann-Liouville fractional integrals.

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1. Introduction

If f and g are two continuous functions on $[a,b]$ satisfying $m \leq f(t) \leq M$ and $p \leq g(t) \leq P$ for all $t \in [a,b]$, $m, M, p, P \in \mathbb{R}$,

$$\left| \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt \int_a^b g(t)dt \right| \leq \frac{1}{4}(M-m)(P-p). \tag{1.1}$$

(1.1) inequality is well-known in literature as Grüss Inequality. It is defined as the integral inequality that establishes as a connection between the product of two functions and the product of the integrals [1].

*Corresponding Author

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Grüss type inequalities are now vast and many extensions of the classical inequalities were intensively studied by many authors. Integral inequalities and applications have been addressed extensively by several researchers. For example, we refer the reader to [4, 5, 6, 7, 8, 9] and the references cited therein. Also, there are many extensions of Grüss type inequalities by using Riemann-Liouville fractional integrals [2, 13, 14, 16].

2. Preliminaries

In this section, we will give some definitions, lemmas and theorems which we use later in this article.

Definition 2.1. [12] Let $f \in L_1[0,\infty)$.The Riemann- Liouville fractional integral of order $\alpha \geq 0$ is defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)\, dx, \qquad (2.1)$$
$$I^0 f(t) = f(t),$$

where Γ is the gamma function.

Definition 2.2. [10, 11] A function $f(t)$ is said to be in the $L_{p,k}[0,\infty)$ space if

$$L_{p,k}[0,\infty) = \left\{ f : \|f\|_{L_{p,k[0,\infty)}} = \left(\int_a^b |f(t)|^p t^k dt \right)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty,\ k \geq 0 \right\}. \qquad (2.2)$$

For $k = 0$,

$$L_p[0,\infty) = \left\{ f : \|f\|_{L_{p[0,\infty)}} = \left(\int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty,\ 1 \leq p < \infty \right\}.$$

Definition 2.3. [10, 11] Let $f \in L_{1,k}[0,\infty)$. The Generalized Riemann-Liouville fractional integral $I^{\alpha,k} f(x)$ of order $\alpha \geq 0$ and $k \geq 0$ is defined by

$$I^{\alpha,k} f(x) = \frac{(k+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^x (x^{k+1} - t^{k+1})^{\alpha-1} t^k f(t) dt, \qquad (2.3)$$
$$I^{0,k} f(x) = f(x),$$

where Γ is the gamma function.

Definition 2.4. Let $f \in L_1[0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$ and also derivative $h'(x)$ is continuous on $[0,\infty)$ and $h(0) = 0$. The space $X_h^p(0,\infty)$ $(1 \leq p < \infty)$ of those real-valued Lebesque measurable functions f on $[0,\infty)$ for which

$$\|f\|_{X_h^p} = \left(\int_0^{\infty} |f(t)|^p h'(t) dt \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty \qquad (2.4)$$

and for the case $p = \infty$

$$\|f\|_{X_h^\infty} = ess \sup_{0 \le t < \infty} [h'(t)f(t)].$$

In particular, when $h(x) = x \quad (1 \le p < \infty)$ the space $X_h^p(0,\infty)$ coincides with the $L_p[0,\infty)$-space and also if we take $h(x) = \frac{x^{k+1}}{k+1} \quad (1 \le p < \infty,\ k \ge 0)$ the space $X_h^p(0,\infty)$ coincides with the $L_{p,k}[0,\infty)$-space.

Definition 2.5. [10, 11] Let $f \in X_h^p(0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$ and also derivative $h'(x)$ is continuous on $[0,\infty)$ and $h(0) = 0$. The Riemann-Liouville fractional integral of a function $f(x)$ with respect to another function $h(x)$ is defined by

$$I_h^\alpha f(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t (h(t) - h(x))^{\alpha-1} f(x)\, h'(x) dx. \tag{2.5}$$

For the convenience of establishing our results, we give the semi-group property:

$$I_h^\alpha I_h^\beta f(t) = I_h^{\alpha+\beta} f(t), \quad \alpha \ge 0,\ \beta \ge 0, \tag{2.6}$$

which implies the commutative property

$$I_h^\alpha I_h^\beta f(t) = I_h^\beta I_h^\alpha f(t). \tag{2.7}$$

From Definition 5, if $f(t) = h^\gamma(t)$, then we have

$$I_h^\alpha \left[h^\gamma(t)\right] = \frac{h^{\gamma+\alpha}(t)\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)},\ \alpha \ge 0,\ \beta \ge 0. \tag{2.8}$$

$(2.5) - (2.8)$ results for (2.5) Riemann-Liouville fractional integral are reduced (2.1) fractional integral and its properties when $h(x) = x$.

In [15]some special applications and convergency cases of $(2,5)$ are studied.

Dahmani et al. [2] gave the following fractional integral inequalities for the integral (2.1) in Definition 1.

Theorem 2.6. *Let f,$g \in L[0,\infty)$ and satisfy the following conditions:*

$$m \le f(t) \le M, \quad p \le g(t) \le P, \quad t \in [0,\infty), \quad m, M, p, P \in \mathbb{R},$$

then for all $t > 0$, $\alpha > 0$, $\beta > 0$,

$$i)\ \left|\tfrac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha(fg)(t) - I^\alpha f(t) I^\alpha g(t)\right| \le (\tfrac{t^\alpha}{\Gamma(\alpha+1)})^2 (M-m)(P-p), \tag{2.9}$$

$$
\begin{aligned}
ii)\ &\left(\frac{t^\alpha}{\Gamma(\alpha+1)}I^\beta(fg)(t)+\frac{t^\beta}{\Gamma(\beta+1)}I^\alpha(fg)(t)-I^\alpha f(t)I^\beta g(t)-I^\beta f(t)I^\alpha g(t)\right)^2\\
&\leq\left\{\left(M\frac{t^\alpha}{\Gamma(\alpha+1)}-I^\alpha f(t)\right)\left(I^\beta f(t)-m\frac{t^\beta}{\Gamma(\beta+1)}\right)\right.\\
&\qquad\left.+\left(I^\alpha f(t)-m\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\left(M\frac{t^\beta}{\Gamma(\beta+1)}-I^\beta f(t)\right)\right\}\\
&\qquad\times\left\{\left(P\frac{t^\alpha}{\Gamma(\alpha+1)}-I^\alpha g(t)\right)\left(I^\beta g(t)-p\frac{t^\beta}{\Gamma(\beta+1)}\right)\right.\\
&\qquad\left.+\left(I^\alpha g(t)-p\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\left(P\frac{t^\beta}{\Gamma(\beta+1)}-I^\beta g(t)\right)\right\}.
\end{aligned}
\tag{2.10}
$$

Jessada Tariboon et al. [3] gave the following fractional integral inequalities for functions in Theorem 2, Theorem 3, Theorem 4, and Lemma 1 which are bounded with integrable functions.

Theorem 2.7. *Let $f\in L[0,\infty)$. Suppose that*
(H_1) there exist two integrable functions φ_1, φ_2 on $[0,\infty)$ such that

$$\varphi_1(t)\leq f(t)\leq\varphi_2(t),\ \forall t\in[0,\infty).$$

Then for $t>0$, $\alpha,\beta>0$, one has

$$I^\beta\varphi_1(t)I^\alpha f(t)+I^\alpha\varphi_2(t)I^\beta f(t)\geq I^\alpha\varphi_2(t)I^\beta\varphi_1(t)+I^\alpha f(t)I^\beta f(t). \tag{2.11}$$

Theorem 2.8. *Let $f,g\in L[0,\infty)$. Assume that*
(H_1) holds and moreover one assumes that
(H_2)there exist two integrable functions ψ_1, ψ_2 on $[0,\infty)$ such that

$$\psi_1(t)\leq g(t)\leq\psi_2(t),\ \forall t\in[0,\infty).$$

Then for $t>0$, $\alpha,\beta>0$, the following inequalities hold:

$$
\begin{aligned}
&(a)\quad I^\beta\psi_1(t)I^\alpha f(t)+I^\alpha\varphi_2(t)I^\beta g(t)\geq I^\beta\psi_1(t)I^\alpha\varphi_2(t)+I^\alpha f(t)I^\beta g(t),\\
&(b)\quad I^\beta\varphi_1(t)I^\alpha g(t)+I^\alpha\psi_2(t)I^\beta f(t)\geq I^\beta\varphi_1(t)I^\alpha\psi_2(t)+I^\beta f(t)I^\alpha g(t),\\
&(c)\quad I^\alpha\varphi_2(t)I^\beta\psi_2(t)+I^\alpha f(t)I^\beta g(t)\geq I^\alpha\varphi_2(t)I^\beta g(t)+I^\beta\psi_2(t)I^\alpha f(t),\\
&(d)\quad I^\alpha\varphi_1(t)I^\beta\psi_1(t)+I^\alpha f(t)I^\beta g(t)\geq I^\alpha\varphi_1(t)I^\beta g(t)+I^\beta\psi_1(t)I^\alpha f(t).
\end{aligned}
\tag{2.12}
$$

Lemma 2.9. *Let $f,\varphi_1,\varphi_2\in L[0,\infty)$. Assume that the condition (H_1)holds. Then, for $t>0$, $\alpha>0$, we have*

$$
\begin{aligned}
\frac{t^\alpha}{\Gamma(\alpha+1)}I^\alpha f^2(t)-(I^\alpha f(t))^2=\ &\{I^\alpha\varphi_2(t)-I^\alpha f(t))(I^\alpha f(t)-I^\alpha\varphi_1(t))\\
&-\frac{t^\alpha}{\Gamma(\alpha+1)}I^\alpha((\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\Big\}\\
&+\frac{t^\alpha}{\Gamma(\alpha+1)}I^\alpha\varphi_1(t)f(t)-I^\alpha\varphi_1(t)I^\alpha f(t)\\
&+\frac{t^\alpha}{\Gamma(\alpha+1)}I^\alpha\varphi_2(t)f(t)-I^\alpha\varphi_2(t)I^\alpha f(t)\\
&+I^\alpha\varphi_1(t)I^\alpha\varphi_2(t)-\frac{t^\alpha}{\Gamma(\alpha+1)}I^\alpha\varphi_1(t)\varphi_2(t).
\end{aligned}
\tag{2.13}
$$

Theorem 2.10. *Let* f,$g,\varphi_1,\ \varphi_2,\ \psi_{1,}\ \psi_2 \in L[0,\infty)$ *and satisfy the conditions* (H_1) *and* (H_2) *on* $[0,\infty)$. *Then for all* $t>0,\ \alpha>0,$ *one has*

$$\left|\tfrac{t^\alpha}{\Gamma(\alpha+1)}I^\alpha(fg)(t)-I^\alpha f(t)I^\alpha g(t)\right| \leq \sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_{1,}\ \psi_2)}, \tag{2.14}$$

where $T(u,v,w)$ is defined by

$$\begin{aligned} T(u,v,w) = \quad & (I^\alpha w(t)-I^\alpha u(t))(I^\alpha u(t)-I^\alpha v(t)) \\ & +\tfrac{t^\alpha}{\Gamma(\alpha+1)}I^\alpha(v(t)u(t))-I^\alpha v(t)I^\alpha u(t) \\ & +\tfrac{t^\alpha}{\Gamma(\alpha+1)}I^\alpha(w(t)u(t))-I^\alpha w(t)I^\alpha u(t) \\ & +I^\alpha v(t)I^\alpha w(t)-\tfrac{t^\alpha}{\Gamma(\alpha+1)}I^\alpha(v(t)w(t)). \end{aligned} \tag{2.15}$$

Main Results: In this section, we will obtain Grüss type inequalities by using (2.5) $h(x)$-Riemann-Liouville fractional integral. Our first result is the following theorem.

Theorem 2.11. *Let* $f\in X_h^p(0,\infty)$ *and* $h(x)$ *be an increasing and positive monotone function on* $[0,\infty)$, *and also derivative* $h'(x)$ *is continuous on* $[0,\infty)$ *and also* $h(0)=0$. *Suppose that there exist two integrable functions* $\varphi_1,\ \varphi_2$ *on* $[0,\infty)$ $t>0,\ \alpha,\beta>0$ *such that*

$$\varphi_1(t)\leq f(t)\leq\varphi_2(t)\quad \forall t\in[0,\infty). \tag{2.16}$$

Then we obtain the following inequality

$$I_h^\beta\varphi_1(t)I_h^\alpha f(t)+I_h^\alpha\varphi_2(t)I_h^\beta f(t)\geq I_h^\alpha\varphi_2(t)I_h^\beta\varphi_1(t)+I_h^\alpha f(t)I_h^\beta f(t). \tag{2.17}$$

Proof. From (3.1), for all $x\geq0,\ y\geq0$, we have

$$(\varphi_2(x)-f(x))(f(y)-\varphi_1(y))\geq0,$$

$$\varphi_2(x)f(y)+\varphi_1(y)f(x)\geq\varphi_1(y)\varphi_2(x)+f(x)f(y). \tag{2.18}$$

If we multiply both sides of (3.3) by $\dfrac{(h(t)-h(x))^{\alpha-1}h'(x)}{\Gamma(\alpha)}$, and integrate with respect to x on $(0,t)$, we obtain

$$f(y)I_h^\alpha\varphi_2(t)+\varphi_1(y)I_h^\alpha f(t)\geq\varphi_1(y)I_h^\alpha\varphi_2(t)+f(y)I_h^\alpha f(t). \tag{2.19}$$

If we multiply both sides of (3.4) by $\dfrac{(h(t)-h(y))^{\beta-1}h'(y)}{\Gamma(\beta)}$, and integrate with respect to y on $(0,t)$, we get

$$I_h^\beta\varphi_1(t)I_h^\alpha f(t)+I_h^\alpha\varphi_2(t)I_h^\beta f(t)\geq I_h^\alpha\varphi_2(t)I_h^\beta\varphi_1(t)+I_h^\alpha f(t)I_h^\beta f(t). \tag{2.20}$$

This proves the theorem. ∎

Corollary 2.12. *Let $h(x) = x$ in Theorem 5, then we have the inequality* (2.11) *in Theorem* 2.

Corollary 2.13. *Let $h(x) = \frac{x^{k+1}}{k+1}$, $k \geq 0$ in Theorem 5, then we have the results in* [13].

Corollary 2.14. *Let $f \in X_h^p(0,\infty)$. Suppose that $m \leq f(t) \leq M$, $\forall t \in [0,\infty)$ and $m, M \in \mathbb{R}$. Then for $t > 0$, $\alpha > 0$, $\beta > 0$, we have*

$$\begin{aligned} &m\frac{h^{\beta}(t)}{\Gamma(\beta+1)}I_h^{\alpha}f(t) + M\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\beta}f(t) \\ &\geq Mm\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}\frac{h^{\beta}(t)}{\Gamma(\beta+1)} + I_h^{\alpha}f(t)I_h^{\beta}f(t). \end{aligned} \tag{2.21}$$

Theorem 2.15. *Let f and g be two integrable functions on $[0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$, derivative $h'(x)$ is continuous on $[0,\infty)$ and also $h(0) = 0$, $t > 0$, $\alpha, \beta > 0$. Suppose that* (3.1) *holds and moreover assume that there exist ψ_1 and ψ_2 integrable functions on $[0,\infty)$ such that*

$$\psi_1(t) \leq g(t) \leq \psi_2(t), \quad \forall t \in [0,\infty). \tag{2.22}$$

Then the following inequalities hold:

$$\begin{aligned} &(a) \quad I_h^{\beta}\psi_1(t)I_h^{\alpha}f(t) + I_h^{\alpha}\varphi_2(t)I_h^{\beta}g(t) \geq I_h^{\beta}\psi_1(t)I_h^{\alpha}\varphi_2(t) + I_h^{\alpha}f(t)I_h^{\beta}g(t), \\ &(b) \quad I_h^{\beta}\varphi_1(t)I_h^{\alpha}g(t) + I_h^{\alpha}\psi_2(t)I_h^{\beta}f(t) \geq I_h^{\beta}\varphi_1(t)I_h^{\alpha}\psi_2(t) + I_h^{\beta}f(t)I_h^{\alpha}g(t), \\ &(c) \quad I_h^{\alpha}\varphi_2(t)I_h^{\beta}\psi_2(t) + I_h^{\alpha}f(t)I_h^{\beta}g(t) \geq I_h^{\alpha}\varphi_2(t)I_h^{\beta}g(t) + I_h^{\beta}\psi_2(t)I_h^{\alpha}f(t), \\ &(d) \quad I_h^{\alpha}\varphi_1(t)I_h^{\beta}\psi_1(t) + I_h^{\alpha}f(t)I_h^{\beta}g(t) \geq I_h^{\alpha}\varphi_1(t)I_h^{\beta}g(t) + I_h^{\beta}\psi_1(t)I_h^{\alpha}f(t). \end{aligned} \tag{2.23}$$

Proof. From (3.1) and (3.7) for $\forall t \in [0,\infty)$, we have

$$(\varphi_2(x) - f(x))(g(y) - \psi_1(y)) \geq 0,$$

then

$$\varphi_2(x)g(y) + \psi_1(y)f(x) \geq \psi_1(y)\varphi_2(x) + f(x)g(y). \tag{2.24}$$

If we multiply both sides of (3.9) by $\dfrac{(h(t)-h(x))^{\alpha-1}h'(x)}{\Gamma(\alpha)}$ and integrate with respect to x on $(0,t)$, we get

$$g(y)I_h^{\alpha}\varphi_2(t) + \psi_1(y)I_h^{\alpha}f(t) \geq \psi_1(y)I_h^{\alpha}\varphi_2(t) + g(y)I_h^{\alpha}f(t). \tag{2.25}$$

If we multiply both sides of (3.10) by $\dfrac{(h(t)-h(y))^{\beta-1}h'(y)}{\Gamma(\beta)}$ and integrate with respect to y on $(0,t)$, we get

$$I_h^{\beta}\psi_1(t)I_h^{\alpha}f(t)+I_h^{\alpha}\varphi_2(t)I_h^{\beta}g(t)\geq I_h^{\alpha}\varphi_2(t)I_h^{\beta}\psi_1(t)+I_h^{\alpha}f(t)I_h^{\beta}g(t). \tag{2.26}$$

This proves (a).

To prove $(b)-(d)$, we use the following inequalities:

$$(b)\ (\psi_2(x)-g(x))(f(y)-\varphi_1(y))\geq 0,$$

$$(c)\ (\varphi_2(x)-f(x))(g(y)-\psi_2(y))\leq 0,$$

$$(d)\ (\varphi_1(x)-f(x))(g(y)-\psi_1(y))\leq 0.$$

∎

The following inequalities are the special case of Theorem 6.

Corollary 2.16. *Let $f,g\in X_h^p(0,\infty)$, $t>0$, $\alpha>0$, $\beta>0$. Suppose that there exist real constants m,M,n,N, such that*

$$m\leq f(t)\leq M,\quad n\leq g(t)\leq N,\quad \forall t\in[0,\infty).$$

Then we have

$$\begin{aligned}
&(a^*)\ n\frac{h^{\beta}(t)}{\Gamma(\beta+1)}I_h^{\alpha}f(t)+M\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\beta}g(t)\\
&\qquad\geq nM\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}\frac{h^{\beta}(t)}{\Gamma(\beta+1)}+I_h^{\alpha}f(t)I_h^{\beta}g(t),\\
&(b^*)\ m\frac{h^{\beta}(t)}{\Gamma(\beta+1)}I_h^{\alpha}g(t)+N\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\beta}f(t)\\
&\qquad\geq mN\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}\frac{h^{\beta}(t)}{\Gamma(\beta+1)}+I_h^{\beta}f(t)I_h^{\alpha}g(t),\\
&(c^*)\ NM\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}\frac{h^{\beta}(t)}{\Gamma(\beta+1)}+I_h^{\alpha}f(t)I_h^{\beta}g(t)\\
&\qquad\geq M\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\beta}g(t)+N\frac{h^{\beta}(t)}{\Gamma(\beta+1)}I_h^{\alpha}f(t),\\
&(d^*)\ \ mn\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}\frac{h^{\beta}(t)}{\Gamma(\beta+1)}+I_h^{\alpha}f(t)I_h^{\beta}g(t)\\
&\qquad\geq m\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\beta}g(t)+n\frac{h^{\beta}(t)}{\Gamma(\beta+1)}I_h^{\alpha}f(t).
\end{aligned} \tag{2.27}$$

Corollary 2.17. *Let $f,g\in L_1[0,\infty)$ and $h(x)=x$, $t>0$, $\alpha>0$, $\beta>0$. Suppose that there exist real constants m,M,n,N, such that*

$$m\leq f(t)\leq M,\quad n\leq g(t)\leq N,\quad \forall t\in[0,\infty).$$

Then we have the results in [3].

Corollary 2.18. *Let $f, g \in L_{1,k}[0,\infty)$ and $h(x) = \frac{x^{k+1}}{k+1}$, $k \geq 0$, $t > 0$, $\alpha > 0$, $\beta > 0$. Suppose that there exist real constants m, M, n, N such that*

$$m \leq f(t) \leq M, \quad n \leq g(t) \leq N, \quad \forall t \in [0,\infty).$$

Then we have the results in [13].

Lemma 2.19. *Let $f \in X_h^p(0,\infty)$ and assume φ_1, φ_2 be two integrable functions on $[0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$, derivative $h'(x)$ is continuous on $[0,\infty)$ and also $h(0) = 0$, $t > 0$, $\alpha > 0$. Suppose that the condition* (3.1) *holds. Then*

$$\begin{aligned} &\frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\alpha f^2(t) - (I_h^\alpha f(t))^2 \\ &\qquad = (I_h^\alpha \varphi_2(t) - I_h^\alpha f(t))(I_h^\alpha f(t) - I_h^\alpha \varphi_1(t)) \\ &\qquad\quad - \frac{h^\alpha(t)}{\Gamma(\alpha+1)} (I_h^\alpha \varphi_2(t) - I_h^\alpha f(t))(I_h^\alpha f(t) - I_h^\alpha \varphi_1(t)) \\ &\qquad\quad + \frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\alpha (\varphi_1(t) f(t)) - I_h^\alpha \varphi_1(t) I_h^\alpha f(t) \\ &\qquad\quad + \frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\alpha (\varphi_2(t) f(t)) - I_h^\alpha \varphi_2(t) I_h^\alpha f(t) \\ &\qquad\quad + I_h^\alpha (\varphi_1(t)\varphi_2(t)) - \frac{h^\alpha(t)}{\Gamma(\alpha+1)} I_h^\alpha (\varphi_1(t)\varphi_2(t)) \end{aligned} \tag{2.28}$$

Proof. For any $x, y > 0$, we have

$$\begin{aligned} &(\varphi_2(y) - f(y))(f(x) - \varphi_1(x)) + (\varphi_2(x) - f(x))(f(y) - \varphi_1(y)) \\ &-(\varphi_2(x) - f(x))(f(x) - \varphi_1(x)) - (\varphi_2(y) - f(y))(f(y) - \varphi_1(y)) \\ &\qquad = f^2(x) + f^2(y) - 2f(x)f(y) + \varphi_2(y)f(x) + \varphi_1(x)f(y) \\ &\qquad\quad - \varphi_1(x)\varphi_2(y) + \varphi_2(x)f(y) + \varphi_1(y)f(x) - \varphi_1(y)\varphi_2(x) \\ &\qquad\quad - \varphi_2(x)f(x) + \varphi_1(x)\varphi_2(x) - \varphi_1(x)f(x) - \varphi_2(y)f(y) \\ &\qquad\quad + \varphi_1(y)\varphi_2(y) - \varphi_1(y)f(y). \end{aligned} \tag{2.29}$$

If we multiply both sides of (3.14) by $\dfrac{(h(t)-h(x))^{\alpha-1}h'(x)}{\Gamma(\alpha)}$ and integrate with respect to x on $(0,t)$, we get

$$\begin{aligned} &(\varphi_2(y) - f(y))(I_h^\alpha f(t) - I_h^\alpha \varphi_1(t)) + (I_h^\alpha \varphi_2(t) - I_h^\alpha f(t))(f(y) - \varphi_1(y)) \\ &- I_h^\alpha (\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) - (\varphi_2(y) - f(y))(f(y) - \varphi_1(y)) \frac{h^\alpha(t)}{\Gamma(\alpha+1)} \\ &\quad = I_h^\alpha f^2(t) + f^2(y) \frac{h^\alpha(t)}{\Gamma(\alpha+1)} - 2f(y) I_h^\alpha f(t) + \varphi_2(y) I_h^\alpha f(t) + f(y) I_h^\alpha \varphi_1(t) \\ &\qquad - \varphi_2(y) I_h^\alpha \varphi_1(t) + f(y) I_h^\alpha \varphi_2(t) + \varphi_1(y) I_h^\alpha f(t) - \varphi_1(y) I_h^\alpha \varphi_2(t) \\ &\qquad - I_h^\alpha (\varphi_2(t) f(t)) + I_h^\alpha (\varphi_1(t)\varphi_2(t)) - I_h^\alpha (\varphi_1(t) f(t)) - \varphi_2(y) f(y) \frac{h^\alpha(t)}{\Gamma(\alpha+1)} \\ &\qquad + \varphi_1(y)\varphi_2(y) \frac{h^\alpha(t)}{\Gamma(\alpha+1)} - \varphi_1(y) f(y) \frac{h^\alpha(t)}{\Gamma(\alpha+1)}. \end{aligned} \tag{2.30}$$

If we multiply both sides of (3.15) by $\dfrac{(h(t)-h(y))^{\alpha-1}h'(y)}{\Gamma(\alpha)}$ and integrate with respect to y on $(0,t)$, we get

$$\begin{aligned}
&(I_h^\alpha\varphi_2(t)-I_h^\alpha f(t))(I_h^\alpha f(t)-I_h^\alpha\varphi_1(t))\\
&+(I_h^\alpha\varphi_2(t)-I_h^\alpha f(t))(I_h^\alpha f(t)-I_h^\alpha\varphi_1(t))\\
&-(I_h^\alpha\varphi_2(t)-I_h^\alpha f(t))(I_h^\alpha f(t)-I_h^\alpha\varphi_1(t))\frac{h^\alpha(t)}{\Gamma(\alpha+1)}\\
&-(I_h^\alpha\varphi_2(t)-I_h^\alpha f(t))(I_h^\alpha f(t)-I_h^\alpha\varphi_1(t))\frac{h^\alpha(t)}{\Gamma(\alpha+1)}\\
&\qquad=\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha f^2(t)+\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha f^2(t)\\
&\qquad-2I_h^\alpha f(t)I_h^\alpha f(t)+I_h^\alpha\varphi_2(t)I_h^\alpha f(t)+I_h^\alpha\varphi_1(t)I_h^\alpha f(t)\\
&\qquad-I_h^\alpha\varphi_1(t)I_h^\alpha\varphi_2(t)+I_h^\alpha\varphi_2(t)I_h^\alpha f(t)+I_h^\alpha\varphi_1(t)I_h^\alpha f(t)\\
&\qquad-I_h^\alpha\varphi_1(t)I_h^\alpha\varphi_2(t)-\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_2(t)f(t))\\
&\qquad+\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_1(t)\varphi_2(t))-\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_1(t)f(t))\\
&\qquad-\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_2(t)f(t))+\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_1(t)\varphi_2(t))\\
&\qquad-\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_1(t)f(t)).
\end{aligned}\tag{2.31}$$

This proves lemma. ∎

Corollary 2.20. *Let $h(x)=x$ in Lemma 2, then we have inequality (2.13) in Lemma 1.*

Corollary 2.21. *Let $f\in X_h^p(0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$, derivative $h'(x)$ is continuous on $[0,\infty)$ and also $h(0)=0$. Suppose that $m\le f(t)\le M, \forall t\in[0,\infty)$ and $m,M\in\mathbb{R}$. Then for, $t>0$, $\alpha>0$, $\beta>0$, we have*

$$\begin{aligned}
&\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha f^2(t)-(I_h^\alpha f(t))^2\\
&=\left(M\frac{h^\alpha(t)}{\Gamma(\alpha+1)}-I_h^\alpha f(t)\right)\left(I_h^\alpha f(t)-m\frac{h^\alpha(t)}{\Gamma(\alpha+1)}\right)\\
&-\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha((M-f(t))(f(t)-m)).
\end{aligned}\tag{2.32}$$

Corollary 2.22. *Let $f\in L_{1,k}[0,\infty)$ and $h(x)=\frac{x^{k+1}}{k+1}$. Suppose that $m\le f(t)\le M$, $\forall t\in[0,\infty)$ and $m,M\in\mathbb{R}$.Then for $k\ge 0$, $t>0$, $\alpha>0$, $\beta>0$, we have the results in [13].*

Theorem 2.23. *Let $f,g,\varphi_1,\varphi_2,\psi_1$ and ψ_2 be six integrable functions on $[0,\infty)$ and $h(x)$ be an increasing and positive monotone function on $[0,\infty)$, derivative*

$h'(x)$ is continuous on $[0,\infty)$ and also $h(0)=0$. Satisfying the conditions (3.1) *and* (3.7) *on* $[0,\infty)$. *Then for all* $t>0$, $\alpha>0$, *one has*

$$\left|\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(f(t)g(t))-I_h^{\alpha}f(t)I_h^{\alpha}g(t)\right| \leq \sqrt{T(f,\varphi_1,\varphi_2)T(g,\psi_1,\psi_2)}\ , \tag{2.33}$$

where $T(u,v,w)$ is defined by

$$\begin{aligned}T(u,v,w)&=(I_h^{\alpha}w(t)-I_h^{\alpha}u(t))(I_h^{\alpha}u(t)-I_h^{\alpha}v(t))\\&+\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(v(t)u(t))-I_h^{\alpha}v(t)I_h^{\alpha}u(t)+\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(w(t)u(t))\\&-I_h^{\alpha}w(t)I_h^{\alpha}u(t)+I_h^{\alpha}v(t)I_h^{\alpha}w(t)-\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(v(t)w(t)).\end{aligned} \tag{2.34}$$

Proof. Let f and g be two integrable functions defined $[0,\infty)$ satisfying (3.1) and (3.7). Define

$$H(x,y)=(f(x)-f(y))(g(x)-g(y)),\quad x,y\in(0,t),\ t>0. \tag{2.35}$$

Multiplying both sides of (3.20) by $\dfrac{(h(t)-h(x))^{\alpha-1}h'(x)(h(t)-h(y))^{\alpha-1}h'(y)}{2\Gamma^2(\alpha)}$, $x,y\in(0,t)$ and integrating the resulting identity with respect to x and y, from 0 to t, we can state that

$$\begin{aligned}&\frac{1}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(h(t)-h(x))^{\alpha-1}(h(t)-h(y))^{\alpha-1}H(x,y)h'(y)h'(x)dxdy\\&\qquad=\frac{h^{\alpha}(t)}{\Gamma(\alpha+1)}I_h^{\alpha}(f(t)g(t))-I_h^{\alpha}f(t)I_h^{\alpha}g(t).\end{aligned} \tag{2.36}$$

Applying the Cauchy-Schwarz inequality to (3.21), we have

$$\begin{aligned}&\left(\frac{1}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(h(t)-h(x))^{\alpha-1}(h(t)-h(y))^{\alpha-1}(f(x)-f(y))(g(x)-g(y))h'(y)h'(x)dxdy\right)^2\\&\leq\frac{1}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(h(t)-h(x))^{\alpha-1}(h(t)-h(y))^{\alpha-1}(f(x)-f(y))^2h'(y)h'(x)dxdy\\&\times\frac{1}{2\Gamma^2(\alpha)}\int_0^t\int_0^t(h(t)-h(x))^{\alpha-1}(h(t)-h(y))^{\alpha-1}(g(x)-g(y))^2h'(y)h'(x)dxdy.\end{aligned} \tag{2.37}$$

From (3.21) and (3.22) we obtain

$$\left(\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(f(t)g(t)) - I_h^\alpha f(t) I_h^\alpha g(t)\right)^2 \\ \leq \left(\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha f^2(t) - (I_h^\alpha f(t))^2\right) \times \left(\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha g^2(t) - (I_h^\alpha g(t))^2\right). \tag{2.38}$$

Since $(\varphi_2(t)-f(t))(f(t)-\varphi_1(t)) \geq 0$ and $(\psi_2(t)-g(t))(g(t)-\psi_1(y)) \geq 0$, for $t \in [0,\infty)$, we have

$$\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha((\varphi_2(t)-f(t))(f(t)-\varphi_1(t))) \geq 0, \tag{2.39}$$

$$\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha((\psi_2(t)-g(t))(g(t)-\psi_1(y))) \geq 0. \tag{2.40}$$

Thus, from Lemma 2, we obtain

$$\begin{aligned}\frac{h^\alpha(t)}{\Gamma(\alpha+1)}&I_h^\alpha f^2(t) - (I_h^\alpha f(t))^2 \\ &\leq (I_h^\alpha\varphi_2(t) - I_h^\alpha f(t))(I_h^\alpha f(t) - I_h^\alpha\varphi_1(t)) \\ &\quad+\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_1(t)f(t)) - I_h^\alpha\varphi_1(t)I_h^\alpha f(t) \\ &\quad+\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha\varphi_2(t) - I_h^\alpha\varphi_2(t)I_h^\alpha f(t) \\ &\quad+I_h^\alpha\varphi_1(t)I_h^\alpha\varphi_2(t) - \frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\varphi_1(t)\varphi_2(t)) \\ &= T(f,\varphi_1,\varphi_2),\end{aligned} \tag{2.41}$$

$$\begin{aligned}\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha g^2(t) - (I_h^\alpha g(t))^2 &\leq (I_h^\alpha\psi_2(t) - I_h^\alpha g(t))(I_h^\alpha g(t) - I_h^\alpha\psi_1(t)) \\ &\quad+\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\psi_1(t)g(t)) - I_h^\alpha\psi_1(t)I_h^\alpha g(t) \\ &\quad+\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha\psi_2(t) - I_h^\alpha\psi_2(t)I_h^\alpha g(t) \\ &\quad+I_h^\alpha\psi_1(t)I_h^\alpha\psi_2(t) - \frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(\psi_1(t)\psi_2(t)) \\ &= T(g,\psi_1,\psi_2).\end{aligned} \tag{2.42}$$

From (3.22), (3.26), and (3.27), we get (3.18). ∎

Corollary 2.24. *If* $T(f,\varphi_1,\varphi_2) = T(f,m,M)$, $T(g,\psi_1,\psi_2) = T(g,p,P)$ *and* $m,M,p,P \in \mathbb{R}$, *then inequality* (3.18) *reduces to*

$$\left|\frac{h^\alpha(t)}{\Gamma(\alpha+1)}I_h^\alpha(f(t)g(t)) - I_h^\alpha f(t)I_h^\alpha g(t)\right| \\ \leq \left(\frac{h^\alpha(t)}{2\Gamma(\alpha+1)}\right)^2(M-m)(P-p). \tag{2.43}$$

Corollary 2.25. *Let* $h(x) = \frac{x^{k+1}}{k+1}$*,* $k \geq 0$ *if* $T(f, \varphi_{1,} \varphi_2) = T(f, m, M)$*,* $T(g, \psi_{1,} \psi_2) = T(g, p, P)$ *and* $m, M, p, P \in \mathbb{R}$*, then we have the results in* [13].

Corollary 2.26. *Let* $h(x) = x$*, if* $T(f, \varphi_{1,} \varphi_2) = T(f, m, M)$*,* $T(g, \psi_{1,} \psi_2) = T(g, p, P)$ *and* $m, M, p, P \in \mathbb{R}$*, then we have the results in* [3].

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