

## A Graphical Characterization for *SPAP*-Rings

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**ABSTRACT.** Let  $R$  be a commutative ring and  $I$  an ideal of  $R$ . The zero-divisor graph of  $R$  with respect to  $I$ , denoted by  $\Gamma_I(R)$ , is the simple graph whose vertex set is  $\{x \in R \setminus I \mid xy \in I, \text{ for some } y \in R \setminus I\}$ , with two distinct vertices  $x$  and  $y$  are adjacent if and only if  $xy \in I$ . In this paper, we state a relation between zero-divisor graph of  $R$  with respect to an ideal and almost prime ideals of  $R$ . We then use this result to give a graphical characterization for *SPAP*-rings.

**Keywords:** *SPAP*-ring, Almost prime ideal, Zero-divisor graph with respect to an ideal.

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### 1. INTRODUCTION

Throughout this paper, all rings are assumed to be commutative with identity. A graph (simple graph)  $G$  is an ordered pair of disjoint sets  $(V, E)$  such that  $V = V(G)$  is the vertex set of  $G$  and  $E = E(G)$  is its edge set. A graph  $F$  is called a subgraph of a graph  $G$  if  $V(F) \subseteq V(G)$  and  $E(F) \subseteq E(G)$ . A subgraph  $F$  of  $G$  is said to be an induced subgraph of  $G$  if each edge of  $G$  having its ends in  $V(F)$  is also an edge of  $F$ . A graph in which each pair of distinct vertices is joined by an edge is called complete.

There have been several studies concerning the assignment a graph to a ring, a group, a semigroup or a module, for more information see [1], [8] and [12]. The

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concept of the zero-divisor graph of a commutative ring  $R$  was first introduced by Beck [6]. The zero-divisor graph of a commutative ring  $R$  is defined to be the graph  $\Gamma(R)$ , whose vertices are the non-zero zero-divisors of  $R$ , and where  $x$  is adjacent to  $y$  if  $xy = 0$ . In [10] Redmond has generalized the notion of the zero-divisor graph. For a given ideal  $I$  of a commutative ring  $R$ , he defined the zero-divisor graph of  $R$  with respect to  $I$ , denoted by  $\Gamma_I(R)$ , is the simple graph whose vertex set is  $\{x \in R \setminus I \mid xy \in I, \text{ for some } y \in R \setminus I\}$ , with two distinct vertices  $x$  and  $y$  joined by an edge when  $xy \in I$ . Clearly  $\Gamma_0(R) = \Gamma(R)$ . Bhatwadekar and Sharma [7] defined a proper ideal  $I$  of an integral domain  $R$  to be almost prime if for  $a, b \in R$ ,  $ab \in I \setminus I^2$ , then either  $a \in I$  or  $b \in I$ . Anderson and Bataineh [3], use this definition for an arbitrary commutative ring and stated a necessary and sufficient condition for a commutative Noetherian ring under which every proper ideal of  $R$  is a product of almost prime ideals. Then Rostami and Nekooei [11], considered *SPAP*-rings and characterized the structure of *SPAP*-rings, in special cases. Also, they showed that *SPAP*-rings are *quasi-Frobenius* (a Noetherian self-injective ring), and *SPAP*-rings are an applicative class of rings in Coding Theory, for more information see [11]. In the next section, we state a relation between zero-divisor graph with respect to an ideal of  $R$  and almost prime ideals of  $R$ . Then we state the concept of the intersection graph of ideals of  $R$ , and we give a graphical characterization for *SPAP*-rings.

## 2. MAIN RESULTS

A proper ideal  $I$  in a ring  $R$  is called almost prime if for all  $a, b \in R$ ,  $ab \in I \setminus I^2$  either  $a \in I$  or  $b \in I$ . Also, a proper ideal  $I$  of a ring  $R$  is called weakly prime if for all  $a, b \in R$  with  $0 \neq ab \in I$ , either  $a \in I$  or  $b \in I$ . Clearly, every weakly prime ideal is almost prime. The following lemma which plays an important role in this paper gives a graphical characterization for almost prime ideals.

**Lemma 2.1.** *Let  $I$  be a proper ideal of  $R$ . Then  $I$  is an almost prime ideal of  $R$  if and only if  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ .*

*Proof.* Let  $I$  be an almost prime ideal of  $R$  and  $x \in V(\Gamma_I(R))$ , then  $x \in R \setminus I$ , and there exists  $y \in R \setminus I$  such that  $xy \in I$ . Thus  $x, y \notin I^2$ . Now if  $xy \notin I^2$ , then we have  $xy \in I \setminus I^2$ , this gives  $x \in I$  or  $y \in I$ , a contradiction. Thus  $xy \in I^2$ , and so  $x \in V(\Gamma_{I^2}(R))$ . Now let  $x, y \in V(\Gamma_I(R))$  be adjacent in  $\Gamma_I(R)$ , so  $xy \in I$ , if  $xy \notin I^2$ , then we have  $xy \in I \setminus I^2$ , this gives  $x \in I$  or  $y \in I$ , a contradiction. Therefore,  $x$  and  $y$  are adjacent in  $\Gamma_{I^2}(R)$ . Thus  $E(\Gamma_I(R)) \subseteq E(\Gamma_{I^2}(R))$ . Clearly, each edge of  $\Gamma_{I^2}(R)$  having its ends in  $\Gamma_I(R)$  is also an edge of  $\Gamma_I(R)$ . Therefore,  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ . Conversely, let  $\Gamma_I(R)$  be an induced subgraph of  $\Gamma_{I^2}(R)$  and  $ab \in I \setminus I^2$ , if

$a, b \notin I$  then,  $a$  and  $b$  are adjacent in  $\Gamma_I(R)$  and so  $a$  and  $b$  are adjacent in  $\Gamma_{I^2}(R)$ , thus  $ab \in I^2$ , a contradiction. Therefore, either  $a \in I$  or  $b \in I$ .  $\square$

The following lemma is a similar result for weakly prime ideals.

**Lemma 2.2.** *Let  $I$  be a proper ideal of  $R$ . Then  $I$  is a weakly prime ideal of  $R$  if and only if  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma(R)$ .*

*Proof.* Let  $I$  be a weakly prime ideal of  $R$  and  $x \in V(\Gamma_I(R))$ . Then  $x \in R \setminus I$ , and there exists  $y \in R \setminus I$  such that  $xy \in I$ . If  $xy \neq 0$ , we have  $0 \neq xy \in I$ , this gives  $x \in I$  or  $y \in I$ , a contradiction. Thus  $xy = 0$ , and so  $x \in V(\Gamma(R))$ . Now, let  $x, y \in V(\Gamma_I(R))$  be adjacent in  $\Gamma_I(R)$ , thus  $xy \in I$ . Repeating the previous argument leads to  $xy = 0$ . Hence  $x, y$  are adjacent in  $\Gamma(R)$ . Clearly, each edge of  $\Gamma(R)$  having its ends in  $\Gamma_I(R)$  is also an edge of  $\Gamma_I(R)$ . Therefore  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma(R)$ . Conversely, let  $\Gamma_I(R)$  be an induced subgraph of  $\Gamma(R)$  and  $0 \neq ab \in I$  for  $a, b \in R$ , if  $a \notin I$  and  $b \notin I$  then,  $a$  and  $b$  are adjacent in  $\Gamma_I(R)$ , thus  $a$  and  $b$  are adjacent in  $\Gamma(R)$ . This gives  $ab = 0$ , a contradiction. Thus either  $a \in I$  or  $b \in I$ .  $\square$

**Lemma 2.3.** *Let  $I$  be a proper ideal of  $R$ . Then  $I$  is a prime ideal of  $R$  if and only if  $\Gamma_I(R) = \emptyset$ .*

*Proof.* The proof is straightforward.  $\square$

Now let  $I$  be a prime ideal of  $R$ . Thus  $\Gamma_I(R) = \emptyset$  and so  $\Gamma_I(R) = \emptyset$  is an induced subgraph of  $\Gamma_{I^2}(R)$  and  $\Gamma(R)$ , this is a graphical verification for the fact that “prime ideals are almost prime and weakly prime”.

**Definition 2.4.** A local ring  $(R, m)$  is called special product of almost prime ideals ring (*SPAP-ring*), if for each  $x \in m \setminus m^2$ ,  $\langle x^2 \rangle = m^2$  and  $m^3 = 0$ .

*SPAP-rings* were first introduced in [3] by D. D. Anderson and M. Bataineh. In [3], D. D. Anderson and M. Bataineh used *SPAP-rings* to characterize Noetherian rings whose proper ideals are a product of almost prime ideals. In general, an *SPAP-ring* is not Noetherian, see [3, Example 20]. For an *SPAP-ring*  $(R, m)$ ,  $m$  is the unique prime ideal of  $R$ , thus  $R$  is a Noetherian ring if and only if  $R$  is an Artinian ring if and only if  $m$  is a finitely generated ideal of  $R$ .

Before proceeding, we mention the definition of the intersection graph of ideals of a ring which helps us to give a characterization for *SPAP-rings*.

**Definition 2.5.** Let  $R$  be a ring, the intersection graph of ideals of  $R$ , denoted by  $G(R)$ , is the graph whose vertices are proper non-trivial ideals of  $R$  and two distinct vertices are adjacent if and only if the corresponding ideals of  $R$  have a non-trivial (non-zero) intersection.

**Lemma 2.6.** [5, Theorem 2.11.] *Let  $(R, m)$  be an Artinian local ring. Then the intersection graph of ideals of  $R$  is complete if and only if  $R$  has a unique minimal ideal.*

For more information about intersection graph of ideals of  $R$ , see [2, 5]. In the remainder of this section, we characterize Artinian local rings which  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$  for all non-minimal ideals  $I$  of  $R$ , and the intersection graph of ideal of  $R$  is complete.

**Lemma 2.7.** *Let  $(R, m)$  be an Artinian local ring and  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal  $I$  of  $R$ . Then  $m^2$  is a minimal ideal of  $R$  or  $m^2 = 0$ .*

*Proof.* Let  $m^2$  be a non-minimal ideal of  $R$ . Then by Lemma 2.1,  $m^2$  is an almost prime ideal of  $R$ . We show that  $m^2$  must be zero in this case. For this purpose, we show that  $m^2 = m^3 = m^4$  and the Nakayama's Lemma gives  $m^2 = 0$ . If for all  $x, y \in m$ ,  $xy \in m^4$ , we have  $m^2 \subseteq m^4$ , thus  $m^2 = m^3 = m^4$ . Now let there exist  $x, y \in m$  such that  $xy \notin m^4$ , so  $xy \in m^2 \setminus m^4 = m^2 \setminus (m^2)^2$ , since  $m^2$  is almost prime, only one of the following cases happens;

$x \in m^2$  and  $y \notin m^2$  or  $x \notin m^2$  and  $y \in m^2$ . Suppose  $x \in m^2$  and  $y \notin m^2$ . Since  $y^2 \in m^2$ ,  $y \notin m^2$  and  $m^2$  is almost prime, we must have  $y^2 \in m^4$ . Repeating the previous argument and  $y, x+y \notin m^2$  and  $y(x+y) \in m^2$  leads to  $y(x+y) \in m^4$ . Thus  $xy + y^2 = y(x+y)$ ,  $y^2 \in m^4$ , so  $xy \in m^4$ , a contradiction. Thus  $m^2$  is zero or a minimal ideal.  $\square$

Now we mention the definition of a class of rings which are important in the rest of this paper.

**Definition 2.8.** A commutative ring  $R$  is called special principal ideal ring (*SPIR*), if it is a principal ideal ring with unique prime ideal and that prime ideal is nilpotent.

Mori [9] has shown that a ring has the property that every ideal is a product of prime ideals if and only if it is a finite direct product of Dedekind domains and special principal ideal rings (*SPIRs*) (For more information about special principal ideal ring see [9]). In the next lemma, we state a relation between *SPAP*-rings and *SPIR* rings.

**Lemma 2.9.** *Let  $(R, m)$  be an *SPIR* ring such that  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal  $I$  and  $m^2$  is the unique minimal ideal of  $R$ . Then  $(R, m)$  is an *SPAP*-ring.*

*Proof.* Since  $R$  is an *SPIR* ring,  $m = \langle x \rangle$  for some  $x \in m$ . Now let  $0 \neq J \neq m^2$  be an ideal of  $R$ . If  $J = J^2$ , Nakayama's Lemma gives  $J = 0$ , a contradiction. So  $J \neq J^2$ , thus we can select  $y \in J \setminus J^2 \subseteq m$  such that  $J = \langle y \rangle$ . Thus  $y = rx \in J \setminus J^2$ , for some  $r \in R$ . Since  $J \neq m^2$ , Lemma

2.1 gives  $J$  is an almost prime ideal of  $R$  and since  $y = rx \in J \setminus J^2$ , we have  $x \in J$  or  $r \in J$ . If  $x \in J$ , then  $J = m$  and if  $r \in J \subseteq m$ , then we have  $J = \langle y \rangle = \langle rx \rangle \subseteq m^2$  and since  $m^2$  is the unique minimal ideal of  $R$ ,  $J = 0$  or  $J = m^2$ , a contradiction. This means, the set of all ideals of  $R$  is  $\{0, m^2, m, R\}$ .

Now if  $m = m^2$ , we have  $m = 0$ , a contradiction. Thus  $m \neq m^2$ . If  $a \in m \setminus m^2$ , since the set of all ideals of  $R$  is  $\{0, m^2, m, R\}$ , we have  $m = \langle a \rangle$ , so  $m^2 = \langle a^2 \rangle$ . Now if  $m^3 \neq 0$ , we have  $m^2 = m^3$ , and Nakayama's Lemma gives  $m = 0$ , a contradiction. Thus  $m^3 = 0$ . This completes the proof.  $\square$

D. D. Anderson and M. Bataineh in [3], by using *SPAP*-rings, characterized Noetherian rings whose proper ideals are a product of almost prime ideals. Actually, they stated the following theorem.

**Theorem 2.10.** [3, Theorem 22]. *Let  $R$  be a Noetherian ring. Then every proper ideal of  $R$  is a product of almost prime ideals if and only if  $R$  is a finite direct product of Dedekind domains, SPIRs, and (Noetherian) SPAP-rings.*

**Proposition 2.11.** *Let  $(R, m)$  be an Artinian local ring such that  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal  $I$  of  $R$  and the intersection graph of ideal of  $R$  is complete. If  $m^2 \neq 0$  then  $R$  is an SPAP-ring.*

*Proof.* Since the intersection graph of ideals of  $R$  is complete, by Lemma 2.6,  $R$  has a unique minimal ideal. Since  $m^2 \neq 0$ , Lemma 2.7 gives,  $m^2$  is the unique minimal ideal of  $R$ . Now let  $I$  be an arbitrary proper ideal of  $R$ , if  $I$  is a non-minimal ideal of  $R$ , then  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , so  $I$  is an almost prime ideal of  $R$ , by Lemma 2.1, and if  $I$  is a minimal ideal of  $R$ , then  $I = m^2$ . Therefore, in all cases  $I$  is finite product of almost prime ideals (note that  $m$  is prime and so is almost prime), thus by Theorem 2.10,  $R$  is a finite direct product of Dedekind domains, SPIR rings, and SPAP-rings. Since  $R$  is a local ring, this direct product must have a single ring.

Let  $R$  be a Dedekind domain. Since  $m^2$  is a minimal ideal of  $R$ , we have  $m^3 = 0$  or  $m^2 = m^3$ ; in both cases, we have  $m^2 = 0$ . Thus  $R$  is not a Dedekind domain and Lemma 2.9, completes the proof.  $\square$

An  $R$ -module  $M$  is said to be a multiplication  $R$ -module if for each submodule  $N$  of  $M$  there exists an ideal  $I$  of  $R$  such that  $N = IM$ . Clearly, every cyclic module is multiplication module, see [4] for more information. After stating the main result, we require the following three lemmas.

**Lemma 2.12.** *Let  $(R, m)$  be an SPAP-ring. If  $m^2 \neq 0$ , then  $m^2$  is a minimal ideal of  $R$ .*

*Proof.* If  $m = m^2$ , then  $m^2 = m^3 = 0$ , a contradiction. Therefore  $m \neq m^2$ , thus there exists  $y \in m \setminus m^2$ . So  $m^2 = \langle y^2 \rangle$ . Therefore,  $m^2$  is a cyclic  $R$ -module and so it is a multiplication  $R$ -module. Now if  $J$  is a submodule (ideal of  $R$ ) of  $m^2$ , there exists an ideal  $K$  of  $R$ , such that  $J = Km^2$ . If  $K = R$ , then  $J = m^2$  and if  $K \neq R$  then  $J = Km^2 \subseteq m^3 = 0$ , hence  $J = 0$ . Therefore  $m^2$  is a minimal ideal of  $R$ .  $\square$

**Lemma 2.13.** *Let  $(R, m)$  be an SPAP-ring. If  $m^2 \neq 0$  and  $I$  is a proper ideal of  $R$ , then  $I = 0$  or  $I = m^2$  or  $I^2 = m^2$ .*

*Proof.* Since  $m^2 \neq 0$ , by Lemma 2.12,  $m^2$  is a minimal ideal of  $R$ . Now let  $I$  be a proper ideal of  $(R, m)$ . If  $I \subseteq m^2$ , then  $I = 0$  or  $I = m^2$ . If  $I \not\subseteq m^2$ , then there exists  $y \in I \setminus m^2$ . So  $m^2 = \langle y^2 \rangle$ , hence  $m^2 = \langle y^2 \rangle \subseteq I^2 \subseteq m^2$ . Thus  $I^2 = m^2$ .  $\square$

By combining the above two lemmas, we have the following lemma.

**Lemma 2.14.** *Let  $(R, m)$  be an SPAP-ring. If  $m^2 \neq 0$ , then  $m^2$  is the unique minimal ideal of  $R$ .*

Now we can state the main result of this paper.

**Theorem 2.15.** *Let  $(R, m)$  be an Artinian local ring with  $m^2 \neq 0$ . Then  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal  $I$  of  $R$  and the intersection graph of ideals of  $R$  is complete if and only if  $R$  is an SPAP-ring.*

*Proof.* Let  $R$  be an SPAP-ring by Lemma 2.14,  $m^2$  is the unique minimal ideal of  $R$ , so by Lemma 2.6, the intersection graph of ideals of  $R$  is complete. Now let  $I$  be a proper ideal of  $R$ , Lemma 2.13 gives  $I = 0$  or  $I = m^2$  or  $I^2 = m^2$ . If  $I$  is a non-zero non-minimal ideal of  $R$  and  $ab \in I \setminus I^2$ , for  $a, b \in R$ , then  $ab \notin I^2 = m^2$ , so  $a$  or  $b$  is not in  $m$ , thus  $a$  or  $b$  is unit. Thus  $a$  or  $b$  must be in  $I$ . This shows that  $I$  is an almost prime ideal of  $R$ . Hence, by Lemma 2.1,  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ . In general, the zero ideal is an almost prime of  $R$ . Thus every non-minimal ideal of  $R$  is almost prime and so  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal  $I$  of  $R$ .

The converse of theorem is valid by Proposition 2.11.  $\square$

**EXAMPLE 2.16.** Let  $k$  be an ordered field. Then for a non-empty set  $\{x_\alpha\}_{\alpha \in \Delta}$  of indeterminates. Define  $R = k[[\{x_\alpha\}_{\alpha \in \Delta}]]$ ,  $m = \langle \{x_\alpha\}_{\alpha \in \Delta} \rangle$ , and  $J = \langle \{x_\alpha x_\beta, x_\alpha^2 - x_\beta^2\}_{\alpha \neq \beta}, \{x_\alpha^3\}_\alpha \rangle$ . Let  $\bar{R} = \frac{R}{J}$ . Then  $\bar{R}$  is an SPAP-ring with  $\bar{m}^2 \neq 0$  and  $\bar{m}$  is not principal for  $|\Delta| > 1$ , see [3, Example 20]. If  $\Delta$  is a finite set, then  $\bar{R}$  is a Noethrian SPAP-ring with  $\bar{m}^2 \neq 0$ , and thus  $\Gamma_{\bar{I}}(\bar{R})$  is an induced subgraph of  $\Gamma_{\bar{I}^2}(\bar{R})$ , for every non-minimal ideal  $\bar{I}$  of  $\bar{R}$  and the intersection graph of ideals of  $\bar{R}$  is complete.

**Theorem 2.17.** *Let  $(R, m)$  be an Artinian local ring with  $m^2 \neq 0$ , such that  $\Gamma_I(R)$  is an induced subgraph of  $\Gamma_{I^2}(R)$ , for every non-minimal ideal  $I$  of  $R$  and the intersection graph of ideals of  $R$  is complete. If  $\text{char}(R) \neq p^2$ , for any prime number  $p$  and  $\text{char}(\frac{R}{m}) \neq 2$ , then there exists a regular local ring  $(S, n)$ , a positive integer number  $h$ , and subset  $\{x_a\}_{a=1, \dots, h}$  of  $n$  such that  $R \cong \frac{S}{K}$  in which  $K$  is minimally generated by the elements  $\{x_i x_j\}_{1 \leq i < j \leq h}$ ,  $\{x_j^2\}_{2 \leq j \leq \tau}$  and  $\{x_i^2 u_i x_1^2\}_{\tau+1 \leq i \leq h}$ , where the  $u_i$  are unit in  $R$  and  $\tau$  is the Cohen- Macaulay type of  $R$ .*

*Proof.* By Theorem 2.15 and [11, Proposition 6.3].  $\square$

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