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A Shorter and Simple Approach to Study Fixed Point Results via b-Simulation Functions

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Abstract. The purpose of this short note is to consider much shorter and nicer proofs about fixed point results on b-metric spaces via b-simulation function introduced very recently by Demma et al. [M. Demma, R. Saadati, P. Vetro, Fixed point results on b-metric space via Picard sequences and b-simulation functions, Iranian J. Math. Sci. Infor. 11 (1) (2016) 123-136].

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1. Introduction and Preliminaries

In 2015, Khojasteh et al. [4] gave a new approach to study fixed point results in the framework of metric spaces via simulation function as follows:

A mapping $\zeta : [0,+\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following:

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$;

(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0,+\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\overline{\lim_{n\to\infty}} \zeta(t_n, s_n) < 0$.

Also, they denoted the set of all simulation functions by $\mathcal{Z}$.

It is worth noticing that Argoubi et al. [1] revised the above definition by withdrawing the condition (ζ_1) (also, see [7]). Also, Roldan et al. [8] revised (ζ_3) by taking $t_n < s_n$. Hence, we can say that a mapping $\zeta : [0,+\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies:

(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$;

(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0,+\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\overline{\lim_{n\to\infty}} \zeta(t_n, s_n) < 0$.

For several examples of simulation functions, see [1, 2, 4, 6, 7, 8].

Definition 1.1. [4] Let (X,d) be a metric space and $\zeta \in \mathcal{Z}$. Then a mapping $T : X \to X$ is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied:

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0 \qquad \forall x, y \in X. \tag{1.1}$$

Now, it is clear that $\zeta(t,t) < 0$ when $t > 0$; further (1.1) implies that $d(Tx,Ty) < d(x,y)$ when $x \neq y$ for each $x, y \in X$. This means that each $\mathcal{Z}$-contraction with respect to ζ is continuous.

Theorem 1.2. [4] *Let (X,d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to ζ. Then T has a unique fixed point in X and for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to the fixed point of T.*

One very important and significant kind of generalized (standard) metric spaces are so-called b-metric spaces (or metric type spaces). Namely, (X,d) is b-metric space if $X \neq \emptyset$ and $d : X \times X \to [0,+\infty)$ be a mapping such that for all $x, y, z \in X$ hold: $d(x,y) = 0 \Leftrightarrow x = y; d(x,y) = d(y,x)$ and $d(x,y) \leq b(d(x,y) + d(y,z))$ for $b \geq 1$. Then d is called $b-$metric. For more details on b-metric spaces, see [2, 3, 5] and the references contained therein.

Recently, Demma et al. [2] introduced the b-simulation function in the framework of b-metric spaces as follows.

Definition 1.3. Let (X,d) be a b-metric space. A b-simulation function is a function $\zeta : [0,+\infty)^2 \to \mathbb{R}$ satisfying the following:

(ξ_1) $\xi(t,s) < s - t$ for all $t, s > 0$;
(ξ_2) if $\{t_n\}, \{s_n\}$ are sequences in $(0,+\infty)$ such that

$$0 < \lim_{n\to+\infty} t_n \le \underline{\lim}_{n\to+\infty} s_n \le \overline{\lim}_{n\to\infty} s_n \le b \lim_{n\to+\infty} t_n < +\infty, \tag{1.2}$$

then $\overline{\lim_{n\to\infty}} \xi(bt_n, s_n) < 0$.

It is clear if $b = 1$, then b-simulation function is in the fact the simulation function in the framework of (standard) metric spaces.

Example 1.4. [2] Let $\xi : [0,+\infty)^2 \to \mathbb{R}$ be defined by
(i) $\xi(t,s) = \lambda s - t$ for all $t, s \in [0,+\infty)$, where $\lambda \in [0,1)$.
(ii) $\xi(t,s) = \psi(s) - \varphi(t)$ for all $t, s \in [0,+\infty)$, where $\varphi, \psi : [0,+\infty) \to [0,+\infty)$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \le \varphi(t)$ for all $t > 0$.
(iii) $\xi(t,s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t, s \in [0,+\infty)$, where $f, g : [0,+\infty)^2 \to (0,+\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$.
(iv) $\xi(t,s) = s - \varphi(s) - t$ for all $t, s \in [0,+\infty)$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.
(v) $\xi(t,s) = s\varphi(s) - t$ for all $t, s \in [0,+\infty)$, where $\varphi : [0,+\infty) \to [0,1)$ is such that $\lim_{t\to r^+} \varphi(t) < 1$ for all $r > 0$.
Each of the function considered in (i)-(v) is a b-simulation function.

The following important and very interesting results are proved in [2].

Lemma 1.5. *Let (X,d) be a b-metric space and $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that following condition holds.*

$$\xi(bd(fx,fy), d(x,y)) \ge 0 \qquad \forall x, y \in X. \tag{1.3}$$

Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \ne x_n$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0.$$

Lemma 1.6. *Let (X,d) be a b-metric space and $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that (1.3) holds. Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \ne x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a bounded sequence.*

Lemma 1.7. *Let (X,d) be a b-metric space and $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that (1.3) holds. Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \ne x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.*

Theorem 1.8. *Let (X,d) be a complete b-metric space and let $f: X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that* (1.3) *holds; that is,*

$$\xi\left(bd\left(fx,fy\right),d\left(x,y\right)\right) \geq 0 \qquad \forall x,y \in X.$$

Then f has a unique fixed point.

For the proof of Theorem 1.8, Demma et al. [2] used Lemmas 1.5-1.7.

2. Main results

In this section we improve the main result from [2]; that is, we prove Theorem 1.8 without using all three lemmas 1.5-1.7. At the first, we quote some well known results from b-metric spaces. The following lemma was used (and proved) in the course of proofs of several fixed point results in the framework of b-metric spaces in [3].

Lemma 2.1. *Let $\{y_n\}$ be a sequence in a b-metric space (X,d) such that*

$$d\left(y_n,y_{n+1}\right) \leq \lambda d\left(y_{n-1},y_n\right) \tag{2.1}$$

for some λ, $0 \leq \lambda < \frac{1}{b}$ and each $n = 1,2,\cdots$. Then $\{y_n\}$ is a Cauchy sequence in (X,d).

By utilizing Lemma 2.1, Jovanović et al. [3] proved following result.

Theorem 2.2. *Let (X,d) be a complete b-metric space and $f: X \to X$ be a map such that*

$$d\left(fx,fy\right) \leq \lambda d\left(x,y\right) \tag{2.2}$$

holds for all $x,y \in X$, where $0 \leq \lambda < \frac{1}{b}$. Then f has a unique fixed point z and for every $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to z.

Now we formulate and prove Theorem 1.8 via a shorter and simple approach.

Theorem 2.3. *Let (X,d) be a complete b-metric space and $f: X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that* (1.3) *holds; that is,*

$$\xi\left(bd\left(fx,fy\right),d\left(x,y\right)\right) \geq 0 \qquad \forall x,y \in X. \tag{2.3}$$

Then f has a unique fixed point.

Proof. It is enough clear that (2.3) implies

$$bd\left(fx,fy\right) \leq d\left(x,y\right) \qquad \forall x,y \in X. \tag{2.4}$$

Indeed, (2.4) holds if $x = y$. In the case that $x \neq y$ there are two possibilities, either $fx = fy$ or $fx \neq fy$. In the first case we have that $b \cdot d\left(fx,fy\right) = 0 < d\left(x,y\right)$, while in second case the result follows from (ξ_1). This means that (2.3) implies (2.4) for all $x,y \in X$. Further, obviously, (2.4) implies that

$$d\left(f^2x,f^2y\right) \leq \frac{1}{b^2} d\left(x,y\right) = \lambda d\left(x,y\right). \tag{2.5}$$

Since $\lambda = \frac{1}{b^2} \in [0, \frac{1}{b})$, then according to Theorem 2.2, f^2 has a unique fixed point (say z) in X. This further means that f has a unique fixed point z in X. Now, the proof of this theorem is complete. □

Obviously, our proof is much shorter than the corresponding ones from Demma et al.'s work [2]. It is very interesting that all four Corollaries 4.1-4.4 from [2] follows immediately according to our easy approach. Thus we have following corollary.

Corollary 2.4. *Let (X, d) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that*

(i) $\lambda \in [0, 1)$ *such that* $bd(fx, fy) \le \lambda d(x, y)$*;*
(ii) *a lower semi-continuous function* $\varphi : [0, +\infty) \to [0, \infty)$ *with* $\varphi^{-1}(0) = \{0\}$ *such that* $bd(fx, fy) \le d(x, y) - \varphi(d(x, y))$*;*
(iii) $\varphi : [0, +\infty) \to [0, 1)$ *with* $\lim_{t \to r^+} \varphi(t) < 1$ *for all* $r > 0$ *such that* $bd(fx, fy) \le \varphi(d(x, y))\, d(x, y)$*;*
(iv) $\eta : [0, +\infty) \to [0, \infty)$ *with* $\eta(t) < t$ *for all* $t > 0$ *and* $\eta(0) = 0$ *such that* $bd(fx, fy) \le \eta(d(x, y))$

for all $x, y \in X$. Then f has a unique fixed point in each one of above condition.

Proof. Obviously, each one of mentioned conditions implies the condition (2.4) by selecting the appropriate b-simulation function in Example 1.4. Hence, we obtain that $bd(fx, fy) \le d(x, y)$ for all $x, y \in X$. The result then follows according to Theorem 2.3. □

Example 2.5. Now, we consider Example 4.5 from [2]. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|^2$. Then (X, d) is a complete b-metric space with $b = 2$. Consider a mapping $f : X \to X$ by

$$fx = \frac{ax}{1 + x}$$

for all $x \in X$, where $a \in [0, \frac{1}{\sqrt{2}}]$. Now, we have

$$2d(fx, fy) = 2\left|\frac{ax}{1+x} - \frac{ay}{1+y}\right|^2 = 2a^2 \frac{|x-y|^2}{(1+x)^2(1+y)^2} \le |x-y|^2 = d(x, y) \tag{2.6}$$

for all $x, y \in X$. Further, (2.6) implies that

$$d\left(f^2x, f^2y\right) \le \frac{1}{4} d(x, y);$$

that is, f^2 has a unique fixed point according to Theorem 2.2. This means that f has a unique fixed point. Here it is $z = 0$.

The next result is probably known, but our proof is very condensed.

Theorem 2.6. *Let (X, d) be a complete b-metric space and let $f : X \to X$ be a mapping such that*

$$d(fx, fy) \leq \lambda d(x, y) \tag{2.7}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then f has a unique fixed point (say z) in X and for $x_0 \in X$ the sequence $\{f^n x_0\}_{n\in\mathbb{N}}$ converges to z.

Proof. The condition (2.7) implies that

$$d(f^n x, f^n y) \leq \lambda d\left(f^{n-1}x, f^{n-1}y\right) \leq \cdots \leq \lambda^n d(x, y)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Since $\lambda^n \to 0$ as $n \to \infty$, there is $k \in \mathbb{N}$ such that $\lambda^k < \frac{1}{b}$. Therefore, we have

$$d\left(f^{k+1}x, f^{k+1}y\right) \leq \frac{1}{b^2} d(x, y).$$

The result now follows by Theorem 2.2. □

Question 1. Does Theorem 2.3 holds if $\xi(d(fx, fy), d(x, y)) \geq 0$ for all $x, y \in X$, where (X, d) is a given complete b-metric space and $f : X \to X$ be a mapping and ξ a given b-simulation function?

Question 2. Can you obtain this results by considering ordered b-metric spaces or cone b-metric spaces instead of b-metric spaces?

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