Iranian Journal of Mathematical Sciences and Informatics
Vol. 13, No. 1 (2018), pp 131-138
DOI: 10.7508/ijmsi.2018.1.012

Some Algebraic and Combinatorial Properties of the Complete T-Partite Graphs

Seyyede Masoome Seyyedi, Farhad Rahmati*

Department of Mathematics and Computer Science, Amirkabir University of Technology, P. O. Box 15875-4413, Tehran, Iran.

E-mail: mseyyedi@aut.ac.ir
E-mail: frahmati@aut.ac.ir

Abstract. In this paper, we characterize the shellable complete t-partite graphs. It is also shown that for these types of graphs the concepts vertex decomposable, shellable and sequentially Cohen-Macaulay are equivalent. Furthermore, we give a combinatorial condition for the Cohen-Macaulay complete t-partite graphs.

Keywords: Cohen-Macaulay, shellable, Vertex decomposable, Edge ideal.

2000 Mathematics subject classification: 13C14, 05E99, 13F55

1. Introduction

Let G be a finite simple graph on n vertices. Let V_G and E_G denote, respectively, the vertex set and the edge set of G. An independent set in G is a subset of V_G which none of elements are adjacent and the independence complex Δ_G of a graph G is defined by

$$\Delta_G = \{A \subseteq V_G : A \, \text{is an independent set in} \, A\}.$$

We denote by $MIS(G)$ the set of all maximal independent sets in G (the set of facets of Δ_G); see [16]. In this paper, we obtain a lower bound for $|MIS(G)|$. Recently, researchers studied the algebraic properties of a commutative ring by

*Corresponding Author

Received 12 October 2015; Accepted 11 June 2016

its associated combinatorial structure, like for instance zero divisor graph; see [1, 8, 13]. Now let $R = k[x_1, \dots, x_n]$ be the polynomial ring over a field k in n variable $x_1, \dots, x_n$. Identifying $i \in V_G$ with the variable x_i in R, the edge ideal $I(G)$ of G will be defined as the monomial ideal generated by all of monomials $x_i x_j$ such that $\{x_i, x_j\} \in E_G$.

In recent years, researchers tried to identify Cohen-Macaulay graphs in terms of their combinatorial properties. Estrada and Villarreal in [3] showed that Cohen-Macaulayness and shellability of a bipartite graph G are the same. Herzog and Hibi in [6] proved that a bipartite graph G is Cohen-Macaulay if and only if $|V_1| = |V_2|$ and there is an order on vertices of V_1 and V_2 as $x_1, \dots, x_n$ and $y_1, \dots, y_n$, respectively, such that:

i) $x_i \sim y_i$ for $i = 1, \dots, n$,

ii) if $x_i \sim y_j$, then $i \leq j$,

iii) if $x_i \sim y_j$ and $x_j \sim y_k$, then $x_i \sim y_k$.

Here, we present a necessary and sufficient condition under which shellability of a complete t-partite graph is equivalent to Cohen-Macaulayness.

It is known that any vertex decomposable graph is shellable (hence sequentially Cohen-Macaulay), but the converse is not valid in general. So, it is interesting to know a family of graphs in which the property of shellability, vertex decomposability and sequentially Cohen-Macaulayness are the same. Francisco and Van Tuyl in [4] showed that n- cycles for $n = 3, 5$ belong to this family of graphs. F. Mohammadi and D. Kiani in [10] proved that in $\theta_{n_1,\dots,n_k}$ for $\{n_1, \dots, n_k\} \neq \{2, 5\}$, vertex decomposability, shellability and sequentially Cohen-Macaulayness are coincide. Van Tuyl in [14] showed that in bipartite graphs, three concepts are equivalent. In this paper, attempts have been particularly made to introduce another member of this family, that is, complete t-partite graphs.

Herzog and Hibi, in [6], showed that a bipartite graph without isolated vertices G is unmixed if and only if there exists a bipartition $V_1 = \{x_1, \dots, x_g\}$ and $V_2 = \{y_1, \dots, y_g\}$ of V_G such that:

i) $\{x_i, y_i\} \in E_G$ for all i, and

ii) if $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are in E_G and i, j, k are distinct, then $\{x_i, y_k\} \in E_G$.

In the current paper, among other results, we provide a condition for identifying all the unmixed complete t-partite graphs.

2. Main Results

A graph G is *t-partite* if its vertex set can be partitioned into disjoint independent subsets $V_1, \dots, V_t$. Moreover, in this paper, we consider t as the smallest number that has this property. The graph G is called *complete t-partite graph* if its vertex set can be partitioned into disjoint independent subsets $V_1, \dots, V_t$ such that for all u and v in different partition sets, $uv \in E_G$. A

k-coloring of a graph G is a labeling $f : V(G) \to S$ where $|S| = k$. The labels are considered as colors and the set of vertices of one given color form a color class. A k-coloring is said to be proper if adjacent vertices have different labels. A graph is *k-colorable* if it has a proper k-coloring. The chromatic number of graph G, $\chi(G)$, is the least k such that G is k-colorable [11].

The following proposition gives a lower bound for cardinality of the set of all maximal independent sets in $G, |MIS(G)|$, in terms of chromatic number.

Proposition 2.1. *Let G be a graph. If $\chi(G) = t$, then $|MIS(G)| \geq t$.*

Proof. Since $\chi(G) = t$, there exist t color classes for G. Hence, the graph G can be considered as a t-partite graph. Suppose that V_i is the set of elements of i-th color class. Then V_i is an independent set of G and there exists a maximal independent set F_i in G such that $V_i \subseteq F_i$ for all $1 \leq i \leq t$. Thus, there exists at least t maximal independent set for G. □

Remark 2.2. Using the definition of a complete t-partite graph, it follows that $|MIS(G)| = t$ for any complete t-partite graph G.

Definition 2.3. A simplicial complex Δ is called *shellable* if the facets (maximal faces) of Δ can be ordered $F_1, \ldots, F_s$ such that for all $1 \leq i < j \leq s$, there exist some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$. We call $F_1, \ldots, F_s$ to be a *shelling* of a shellable complex, Δ, when the facets are ordered as in the definition.

Now by the next theorem, all shellable complete t-partite graphs can be classified.

Theorem 2.4. *Let G be a complete t-partite graph. G is shellable if and only if G is t-colorable such that exactly one of color classes has arbitrary elements and other classes have only one element.*

Proof. $\Leftarrow$) Assume that $V_G = \{x_1, \ldots, x_n\}$ is the set of vertices. To prove that G is shellable, we have to find a shelling $F_1, \ldots, F_t$ for Δ_G. Since proper t-vertex coloring gives a partition of V_G into t color classes, we suppose that the set of elements of i-th color class is V_i. By assumption, $V_1 = \{x_1, \ldots, x_m\}$ where $m = n - t + 1$ and $V_i = \{x_{m+i-1}\}$ for all $2 \leq i \leq t$. We know that each V_i is an independent set. Now, if $x_{m+i-1} \in V_1$ for $2 \leq i \leq t$, then we can replace V_1 by $V_1 \cup V_i$ and obtain $(t-1)$-partition for G that is a contradiction. Therefore, V_1 is a maximal independent set of G and hence a facet of Δ_G. By the same argument, each V_i is a maximal independent set. We put $F_i = V_i$. Thus, we find an ordering on the facets of Δ_G as follows:

$$F_1 = \{x_1, \ldots, x_m\}, F_2 = \{x_{m+1}\}, \ldots, F_t = \{x_{m+t-1}\}.$$

Since $F_i \setminus F_1 = \{x_{m+i-1}\}$ for all $2 \leq i \leq t$, $F_1, \ldots, F_t$ is a shelling of Δ_G.
$\Rightarrow$) Suppose that $V_1, \ldots, V_t$ is a partition of V_G. According to Remark 2.2,

we have $|MIS(G)| = t$, so Δ_G has exactly t facets. On the other hand, G is shellable and we can consider $F_1, \ldots, F_t$ as a shelling of Δ_G. Since V_i's are maximal independent sets, we obtain $\{V_1, \ldots, V_t\} = \{F_1, \ldots, F_t\}$. Without loss of generality, put $F_i = V_i$ for all $1 \leq i \leq t$. There exists $x_2 \in F_2 \setminus F_1$ such that $F_2 \setminus F_1 = \{x_2\}$ because $F_1, \ldots, F_t$ is a shelling. Thus $F_2 = (F_2 \setminus F_1) \cup (F_2 \cap F_1) = \{x_2\}$.
Now, suppose by induction that $F_1 = \{x_1, \ldots, x_m\}$, $F_2 = \{x_{m+1}\}$, $\ldots$, $F_i = \{x_{m+i-1}\}$. Since $F_1, \ldots, F_t$ is a shelling of Δ_G, there exists $x_{i+1} \in F_{i+1} \setminus F_1$ and $l \in \{1, \ldots, i\}$ such that $F_{i+1} \setminus F_l = \{x_{i+1}\}$, then $F_{i+1} = (F_{i+1} \setminus F_l) \cup (F_{i+1} \cap F_l) = \{x_{i+1}\}$. Hence, one of the color classes has arbitrary elements and the other classes have exactly one element. □

Definition 2.5. A simplicial complex Δ is recursively defined to be *vertex decomposable* if it is either a simplex, or else has some vertex v so that
i) both $\Delta \setminus v$ and $link_\Delta^v$ are vertex decomposable and
ii) no face of $link_\Delta^v$ is a facet of $\Delta \setminus v$, where

$$link_\Delta^F = \{G : G \cap F = \emptyset, G \cup F \in \Delta\}.$$

A graph G is called vertex decomposable if the simplicial complex Δ_G is vertex decomposable.

The following theorem is one of the main results of this paper which characterizes all vertex decomposable complete t-partite graphs.

Theorem 2.6. *Let G be a complete t-partite graph. Then, G is vertex decomposable if and only if G is t-colorable such that exactly one of the color classes has arbitrary elements and the other classes have only one element.*

Proof. $\Rightarrow$) Assume that for any proper t-vertex coloring of G, there exists at least two classes with at least two elements. By Theorem 2.4, G is not shellable and hence is not vertex decomposable, by ([17], Corollary 7).
$\Leftarrow$) By assumption, it follows that G is a chordal graph, so G is vertex decomposable by ([17], Corollary 7). □

Definition 2.7. A subset $C \subset V_G$ is a *minimal vertex cover* of G if:
i) every edge of G is incident with one vertex in C, and
ii) there is no proper subset of C with the first property.
If C satisfies only condition (i), it is called a vertex cover of G. A graph G is said to be *unmixed* if all the minimal vertex covers of G have the same number of elements.

By the next theorem, we present a combinatorial characterization of all the unmixed complete t-partite graphs.

Theorem 2.8. *Let G be a complete t-partite graph. G is unmixed if and only if G is t-colorable such that all color classes have the same number of elements.*

Proof. Since any minimal vertex cover of a complete t-partite graph G contains all the elements of $(t-1)$ classes, then each selected $(t-1)$ of color classes has the same number of elements if and only if all the classes have the same cardinality. □

Definition 2.9. ([15], Definition 3.3.8) A pure d-dimensional complex Δ is called *strongly connected* if each pair of facets F, G can be connected by a sequence of facets $F = F_0, F_1, \ldots, F_s = G$ such that $dim(F_i \cap F_{i-1}) = d-1$ for $1 \leq i \leq s$.

Lemma 2.10. ([2], Proposition 11.7) *Every Cohen-Macaulay complex is strongly connected.*

Lemma 2.11. ([15], Corollary 3.3.7) *A Cohen-Macaulay simplicial complex Δ is pure, that is, all its maximal faces have the same dimension.*

The following theorem is an effective combinatorial criterion for Cohen-Macaulayness of the complete t-partite graphs.

Theorem 2.12. *Let G be a complete t-partite graph. G is Cohen-Macaulay graph if and only if G is t-colorable such that all color classes have exactly one element.*

Proof. $\Leftarrow$) By assumption, we have $G = K_t$, so G is a chordal graph. By [7], G is Cohen-Macaulay if and only if G is unmixed. Thus, the assertion follows from Theorem 2.8.
$\Rightarrow$) Suppose that for any proper t-vertex coloring of G, there exists at least two classes with at least two elements. Since G is Cohen-Macaulay, Δ_G is pure, by Lemma 2.11. Therefore, all the color classes have the same number of elements. Assume that their size is $d+1$, hence all facets of Δ_G are of dimension d and then $dim(\Delta_G) = d$. We will show that it is not possible that $d+1 \geq 2$. For any two facets F and E, ($F \neq E$), we have $F \cap E = \phi$, then $dim(F \cap E) = -1 \neq d-1$, because of $d+1 \geq 2$. Therefore, Δ_G is not strongly connected, a contradiction to Lemma 2.10. It follows that all the color classes have exactly one element. □

Now, we give a special condition for complete t-partite graphs under which shellability is equal to Cohen-Macaulayness.

Corollary 2.13. *Let G be a complete t-partite graph. The property of being Cohen-Macaulay for G is equivalent to being shellable if and only if G is t-colorable such that all color classes have exactly one element.*

Definition 2.14. Let k be a field and $R = k[x_1, \ldots, x_n]$ be the polynomial ring over k. A graded R-module M is called *sequentially Cohen-Macaulay* (over k) if there exists finite filtration of graded R-modules

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$dim(M_1/M_0) < dim(M_2/M_1) < \ldots < dim(M_r/M_{r-1}).$$

We call a graph G sequentially Cohen-Macaulay over the field k if $R/I(G)$ is sequentially Cohen-Macaulay.

Suppose that I is a homogeneous ideal of R. The ideal generated by all homogeneous degree d elements of I is denoted by (I_d). The concept of componentwise linear was introduced by Herzog and Hibi [5]. A homogeneous ideal I is componentwise linear if (I_d) has a linear resolution for all d. Let I be a square-free monomial ideal of R and $I_{[d]}$ be the ideal generated by the square-free monomials of degree d of I. Herzog and Hibi ([5], Proposition 1.5) have shown that the square-free monomial ideal I is componentwise linear if and only if $I_{[d]}$ has a linear resolution for all d. In [5], it is also shown that:

Theorem 2.15. *Let I be a square-free monomial ideal in a polynomial ring. Then $I^\vee$ is componentwise linear if and only if R/I is sequentially Cohen-Macaulay.*

In [12], Stanley showed that shellability implies the sequentially Cohen-Macaulayness.

Theorem 2.16. *Let Δ be a simplicial complex, and suppose that R/I_Δ is the associated Stanley Reisner ring. If Δ is shellable, then R/I_Δ is sequentially Cohen-Macaulay.*

In the following theorem, we show that being sequentially Cohen-Macaulay of the complete t-partite graph is really a combinatorial property.

Theorem 2.17. *Let G be a complete t-partite graph. G is sequentially Cohen-Macaulay if and only if G is t-colorable such that exactly one of color classes has arbitrary elements and other classes have only one element.*

Proof. $\Leftarrow$) It follows from Theorems 2.4 and 2.16.
$\Rightarrow$) Assume that $V_G = \{x_1, \ldots, x_n\}$ and $V_1, \ldots, V_t$ is a partition of V_G where V_i is the set of elements in i-th color class. We proceed by contradiction. One may consider the following cases:
Case(1) : Suppose that there exist at least two parts V_i and V_j with $|V_i| \geq 2$ and $|V_j| \geq 3$ and r is the maximum cardinality of parts of G which is at least 3. Let $J = I(G)^\vee_{[d]}$ where $d = n - r + 1$. Using Theorem 2.15, to show that G is not sequentially Cohen-Macaulay, it suffices to prove that J does not have a linear resolution.

We use simplicial homology to compute the Betti numbers of J. A square-free vector is a vector that its entries are in $\{0, 1\}$. For a monomial ideal I and a degree $b \in \mathbb{N}^n$, define

$$K^b(I) = \{squarefree\ vectors\ c \in \{0,1\}^n\ such\ that\ \frac{x^b}{x^c} \in I\}$$

to be the upper Koszul simplicial complex of I in degree b ([9], Definition 1.33). For a vector $b \in \mathbb{N}^n$, the Betti numbers of I in degree b can be expressed as $\beta_{i,b}(I) = dim_k H^{\sim}_{i-1}(K^b(I), k)$ ([9], Theorem 1.34). The sum of $\beta_{i,b}(I)$ over all square-free vectors b of degree j is equal to $\beta_{i,j}(I)$.
To prove that J does not have a linear resolution, we will show that $\beta_{1,n}(J) \neq 0$. We associate to the monomial $m = x_1 \dots x_n$ a unique square-free vector $b = (1, \dots, 1)$. We have a chain complex

$$\dots \to C_2(K^b(J)) \xrightarrow{\partial_2} C_1(K^b(J)) \xrightarrow{\partial_1} C_0(K^b(J)) \xrightarrow{\partial_0} C_{-1}(K^b(J)) \to 0.$$

The s-dimensional faces $[x_{i_0}, \dots, x_{i_s}]$ of $K^b(J)$ are the basis of $C_s(K^b(J))$ and

$$\partial_s([x_{i_0}, \dots, x_{i_s}]) = \Sigma_{t=0}^{s} (-1)^t [x_{i_0}, \dots, \hat{x_{i_t}}, \dots, x_{i_s}].$$

The above chain complex is introduced in ([4], Proposition 4.1). To obtain $\beta_{1,n}(J)$, we need to compute $dim_k H^{\sim}_0(K^b(M), k) = dim_k(ker\partial_0 / im\partial_1)$. If we can find an element in $ker\partial_0$ that is not in $im\partial_1$, we have shown that $\beta_{1,n}(J) > 0$. We suppose that x_1, x_2, x_3 belong to the part with maximum cardinality. Put $f = [x_1]$.Then, $\partial_0(f) = 0$ and hence f is in the ker of ∂_0. We claim that f is not in the image of ∂_1. To prove, assume that $\partial_1([x_l, x_s]) = [x_1]$. Then, $[x_s] - [x_l] = [x_1]$, hence $[x_s] - [x_l] - [x_1] = 0$ that is a contradiction because $[x_l], [x_s], [x_1]$ are linear independent.
Case(2) : Assume that for any $1 \leq i \leq t$, we have $|V_i| = 2$. The proof of this case is similar to that of case 1. It suffices to consider $J = I(G)^{\vee}_{[d]}$ where $d = n - 2$.
Case(3) : Suppose that there exist at least two parts V_i and V_j with $|V_i| = |V_j| = 2$ and at least one part V_r with $|V_r| = 1$. One can apply the same argument to case 2. □

Corollary 2.18. *Let G be a complete t-partite graph. The followings are equivalent:*

(1) *G is shellable.*
(2) *G is vertex decomposable.*
(3) *G is sequentially Cohen-Macaulay.*

Acknowledgments

The authors are highly grateful to referees for their valuable comments and suggestions for improving the paper.

References

1. A. Abbasi, H. Roshan-Shekalgourabi, D. Hassanzadeh-Lelekaami, Associated graphs of Modules over commutative rings, *Iranian Journal of Mathematical Sciences and Informatics*, **10**(1), (2015), 45-58.
2. A. Björner, Topological methods, Handbook of combinatorics, *Elsevier, Amsterdam*,**2**, (1995), 1819-1872.
3. M. Estrada, R. H. Villarreal, Cohen-Macaulay bipartite graphs, *Arch. Math.*, **68**, (1997), 124-128.
4. C. A. Francisco, A. Van Tuyl, Sequentially Cohen-Macaulay edge ideals, *Proc. Amer. Math. Soc.*, **135**(8), (2007), 2327-2337.
5. J. Herzog, T. Hibi, Componentwise linear ideals, *Nagoya Math. J.*, **153**, (1999), 141-153.
6. J. Herzog, T. Hibi, Distributive lattices, bipartite graphs, and Alexander duality, *J. Algebraic Comb.*, **22**, (2005), 289-302.
7. J. Herzog, T. Hibi and X. Zheng, Cohen-Macaulay chordal graphs, *J. Combin. Theory Ser. A*, **113**(5), (2006), 911-916.
8. H. R. Maimani, Median and center of zero-devisor graph of commutative semigroups, *Iranian Journal of Mathematical Sciences and Informatics*, **3**(2), (2008), 69-76.
9. E. Miller, B. Strumfels, *Combinatorial Commutative Algebra*, Springer, 2004.
10. F. Mohammadi, D. Kiani, Sequentially Cohen-Macaulay graphs of form $\theta_{n_1,\ldots,n_k}$, *Bull. Iran. Math. Soc.*, **36**(2), (2010), 109-118.
11. C. Promsakon, C. Uiyyasathian, Edge-chromatic numbers of glued graphs, *Thai J. Math.*, **4**(2), (2010), 395-401.
12. R. P. Stanleyl, *Combinatorics and Commutative Algebra. Second edition*, Progress in Mathematics. 41.Birkhäuser Boston, Inc., Boston, MA, 1996.
13. A. Tehranian, H. R. Maimani, A study of the total graph, *Iranian Journal of Mathematical Sciences and Informatics*, **6**(2), (2011), 75-80.
14. A. Van Tuyl, Sequentially Cohen-Macaulay bipartite graphs, vertex decomposability and regularity, *Archiv der Mathematik*, **93**(5), (2009), 451-459.
15. R. H. Villarreal, *Combinatorial Optimization Methods in Commutative Algebra*, Preliminary version, 2012.
16. D. R. Wood, On the number of maximal independent sets in a graph, *Discrete Math. and Theor. Computer Science*, **13**(3), (2011), 17-20.
17. R. Woodroofe, Vertex decomposable graphs and obstructions to shellability, *Proc. Amer. Math. Soc.*, **137**(8), (2009), 3235-3246.