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## Balanced Degree-Magic Labelings of Complete Bipartite Graphs under Binary Operations

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ABSTRACT. A graph is called supermagic if there is a labeling of edges where the edges are labeled with consecutive distinct positive integers such that the sum of the labels of all edges incident with any vertex is constant. A graph G is called degree-magic if there is a labeling of the edges by integers 1, 2, ..., |E(G)| such that the sum of the labels of the edges incident with any vertex v is equal to  $(1 + |E(G)|) \deg(v)/2$ . Degree-magic graphs extend supermagic regular graphs. In this paper we find the necessary and sufficient conditions for the existence of balanced degree-magic labelings of graphs obtained by taking the join, composition, Cartesian product, tensor product and strong product of complete bipartite graphs.

**Keywords:** Complete bipartite graphs, Supermagic graphs, Degree-magic graphs, Balanced degree-magic graphs.

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#### 1. INTRODUCTION

We consider simple graphs without isolated vertices. If G is a graph, then V(G) and E(G) stand for the vertex set and the edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G.

Let a graph G and a mapping f from E(G) into positive integers be given. The *index mapping* of f is the mapping  $f^*$  from V(G) into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v, e) f(e) \quad \text{for every} \quad v \in V(G),$$

where  $\eta(v, e)$  is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from E(G) into positive integers is called a magic labeling of G for an index  $\lambda$  if its index mapping  $f^*$  satisfies

$$f^*(v) = \lambda$$
 for all  $v \in V(G)$ .

A magic labeling f of a graph G is called a *supermagic labeling* if the set  $\{f(e) : e \in E(G)\}$  consists of consecutive positive integers. We say that a graph G is *supermagic (magic)* whenever a supermagic (magic) labeling of G exists.

A bijective mapping f from E(G) into  $\{1, 2, ..., |E(G)|\}$  is called a *degree-magic labeling* (or only *d-magic labeling*) of a graph G if its index mapping  $f^*$  satisfies

$$f^*(v) = \frac{1+|E(G)|}{2} \operatorname{deg}(v) \quad \text{for all} \quad v \in V(G).$$

A d-magic labeling f of a graph G is called *balanced* if for all  $v \in V(G)$ , the following equation is satisfied

$$|\{e \in E(G) : \eta(v, e) = 1, f(e) \le \lfloor |E(G)|/2 \rfloor \}|$$
  
=  $|\{e \in E(G) : \eta(v, e) = 1, f(e) > ||E(G)|/2|\}|.$ 

We say that a graph G is *degree-magic* (*balanced degree-magic*) or only *d-magic* when a d-magic (balanced d-magic) labeling of G exists.

The concept of magic graphs was introduced by Sedláček [8]. Later, supermagic graphs were introduced by Stewart [9]. There are now many papers published on magic and supermagic graphs; see [6, 7, 10] for more comprehensive references. The concept of degree-magic graphs was then introduced by Bezegová and Ivančo [2] as an extension of supermagic regular graphs. They established the basic properties of degree-magic graphs and characterized degreemagic and balanced degree-magic complete bipartite graphs in [2]. They also characterized degree-magic complete tripartite graphs in [4]. Some of these concepts are investigated in [1, 3, 5]. We will hereinafter use the auxiliary results from these studies.

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**Theorem 1.1.** [2] Let G be a regular graph. Then G is supermagic if and only if it is d-magic.

**Theorem 1.2.** [2] Let G be a d-magic graph of even size. Then every vertex of G has an even degree and every component of G has an even size.

**Theorem 1.3.** [2] Let G be a balanced d-magic graph. Then G has an even number of edges and every vertex has an even degree.

**Theorem 1.4.** [2] Let G be a d-magic graph having a half-factor. Then 2G is a balanced d-magic graph.

**Theorem 1.5.** [2] Let  $H_1$  and  $H_2$  be edge-disjoint subgraphs of a graph G which form its decomposition. If  $H_1$  is d-magic and  $H_2$  is balanced d-magic, then G is a d-magic graph. Moreover, if  $H_1$  and  $H_2$  are both balanced d-magic, then G is a balanced d-magic graph.

**Proposition 1.6.** [2] For p, q > 1, the complete bipartite graph  $K_{p,q}$  is d-magic if and only if  $p \equiv q \pmod{2}$  and  $(p,q) \neq (2,2)$ .

**Theorem 1.7.** [2] The complete bipartite graph  $K_{p,q}$  is balanced d-magic if and only if the following statements hold:

(i) 
$$p \equiv q \equiv 0 \pmod{2};$$

(ii) if  $p \equiv q \equiv 2 \pmod{4}$ , then  $\min\{p,q\} \ge 6$ .

**Lemma 1.8.** [4] Let m, n and o be even positive integers. Then the complete tripartite graph  $K_{m,n,o}$  is balanced d-magic.

2. BALANCED DEGREE-MAGIC LABELINGS IN THE JOIN OF COMPLETE BIPARTITE GRAPHS

For two vertex-disjoint graphs G and H, the *join* of graphs G and H, denoted by G+H, consists of  $G \cup H$  and all edges joining a vertex of G and a vertex of H. For any positive integers p and q, we consider the join  $K_{p,q} + K_{p,q}$  of complete bipartite graphs. Let  $K_{p,q} + K_{p,q}$  be a d-magic graph. Since  $\deg(v)$  is p + 2qor 2p + q and  $f^*(v) = (2pq + (p+q)^2 + 1) \deg(v)/2$  for any  $v \in V(K_{p,q} + K_{p,q})$ , we have

**Proposition 2.1.** Let  $K_{p,q} + K_{p,q}$  be a d-magic graph. Then p or q is even.

**Proposition 2.2.** Let  $K_{p,q} + K_{p,q}$  be a balanced d-magic graph. Then both p and q are even.

**Proposition 2.3.** Let p and q be even positive integers. Then  $K_{p+q,p+q}$  is a balanced d-magic graph.

*Proof.* Applying Theorem 1.7,  $K_{p+q,p+q}$  is a balanced d-magic graph.

In the next result we show the sufficient condition for the existence of balanced d-magic labelings of the join of complete bipartite graphs  $K_{p,q} + K_{p,q}$ . P. Inpoonjai, T. Jiarasuksakun



FIGURE 1. A balanced d-magic graph  $K_{2,6} + K_{2,6}$  with 16 vertices and 88 edges.

**Theorem 2.4.** Let p and q be even positive integers. Then  $K_{p,q} + K_{p,q}$  is a balanced d-magic graph.

*Proof.* Let p and q be even positive integers. We consider the following two cases:

**Case I.** If (p,q) = (2,2), the graph  $K_{2,2} + K_{2,2}$  is decomposable into three balanced d-magic subgraphs isomorphic to  $K_{2,4}$ . According to Theorem 1.5,  $K_{2,2} + K_{2,2}$  is a balanced d-magic graph.

**Case II.** If  $(p,q) \neq (2,2)$ , then  $K_{p+q,p+q}$  is balanced d-magic by Proposition 2.3, and  $2K_{p,q}$  is balanced d-magic by Theorem 1.4. Since  $K_{p,q} + K_{p,q}$  is the graph such that  $K_{p+q,p+q}$  and  $2K_{p,q}$  form its decomposition, by Theorem 1.5,  $K_{p,q} + K_{p,q}$  is a balanced d-magic graph.

We know that  $K_{2,6}$  is d-magic, but it is not balanced d-magic. Applying Theorem 2.4, we can construct a balanced d-magic graph  $K_{2,6} + K_{2,6}$  (see Figure 1) with the labels on edges of  $K_{2,6} + K_{2,6}$  in Table 2.

We will now generalize to find the necessary and sufficient conditions for the existence of balanced d-magic labelings of the join of complete bipartite graphs in a general form. For any positive integers p, q, r and s, we consider the join  $K_{p,q} + K_{r,s}$  of complete bipartite graphs. Let  $K_{p,q} + K_{r,s}$  be a dmagic graph. Since  $\deg(v)$  is p + r + s, q + r + s, p + q + r or p + q + s and  $f^*(v) = (pq + (p+q)(r+s) + rs + 1) \deg(v)/2$  for any  $v \in V(K_{p,q} + K_{r,s})$ , we have

**Proposition 2.5.** Let  $K_{p,q} + K_{r,s}$  be a d-magic graph. Then the following conditions hold:

(i) only one of p, q, r and s is even or

(ii) only two of p, q, r and s are even or

(iii) all of p, q, r and s are even.

Vertices	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$c_1$	$c_2$	$d_1$	$d_2$
$b_1$	15	70	75	26	23	62	18	67	1	88
$b_2$	74	16	17	63	66	24	71	25	11	78
$b_3$	69	19	14	68	61	27	76	22	3	86
$b_4$	36	57	56	37	44	49	29	48	85	4
$b_5$	31	54	59	42	39	46	34	51	84	5
$b_6$	58	32	33	47	50	40	55	41	83	6
$d_1$	20	73	72	21	28	65	13	64	-	-
$d_2$	53	35	30	52	45	43	60	38	-	-
$c_1$	77	87	79	9	8	7	-	-	-	-
$c_2$	12	2	10	80	81	82	-	-	-	-

TABLE 1. The labels on edges of balanced d-magic graph  $K_{2,6} + K_{2,6}$ .

**Proposition 2.6.** Let  $K_{p,q} + K_{r,s}$  be a balanced d-magic graph. Then p, q, r and s are even.

Now we are able to show the sufficient condition for the existence of balanced d-magic labelings of the join of complete bipartite graphs  $K_{p,q} + K_{r,s}$ .

**Theorem 2.7.** Let p, q, r and s be even positive integers. Then  $K_{p,q} + K_{r,s}$  is a balanced d-magic graph.

*Proof.* Let p, q, r and s be even positive integers. We consider the following two cases:

**Case I.** If at least one of p, q, r and s is not congruent to 2 modulo 4. Suppose that p is not congruent to 2 modulo 4. Thus,  $K_{p,q}$  is balanced d-magic by Theorem 1.7. Since r, s and p + q are even,  $K_{r,s,p+q}$  is balanced d-magic by Lemma 1.8. The graph  $K_{p,q} + K_{r,s}$  is decomposable into two balanced d-magic subgraphs isomorphic to  $K_{p,q}$  and  $K_{r,s,p+q}$ . According to Theorem 1.5,  $K_{p,q} + K_{r,s}$  is a balanced d-magic graph.

**Case II.** If p,q,r and s are congruent to 2 modulo 4. Thus q + r, q + s and p + q are not congruent to 2 modulo 4. By Theorem 1.7,  $K_{p,q+r}, K_{r,q+s}$  and  $K_{s,p+q}$  are balanced d-magic. The graph  $K_{p,q} + K_{r,s}$  is decomposable into three balanced d-magic subgraphs isomorphic to  $K_{p,q+r}, K_{r,q+s}$  and  $K_{s,p+q}$ . According to Theorem 1.5,  $K_{p,q} + K_{r,s}$  is a balanced d-magic graph.

**Corollary 2.8.** Let p, q, r and s be even positive integers. If p = q = r = s, then  $K_{p,q} + K_{r,s}$  is a supermagic graph.

Proof. Applying Theorems 1.1 and 2.7.



FIGURE 2. A balanced d-magic graph  $K_{2,4} + K_{2,10}$  with 18 vertices and 100 edges.

Vertices	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$	$a_9$	$a_{10}$	$c_1$	$c_2$	$d_1$	$d_2$
$b_1$	31	70	79	22	57	61	42	41	58	44	85	16	100	1
$b_2$	23	78	71	30	45	52	54	53	49	50	84	17	2	99
$b_3$	77	24	29	72	56	46	48	47	55	51	18	83	3	98
$b_4$	76	25	28	73	39	43	59	60	40	62	19	82	97	4
$d_1$	75	26	27	74	38	64	65	36	67	33	81	20	-	-
$d_2$	21	80	69	32	68	37	35	66	34	63	15	86	-	-
$c_1$	96	6	7	93	92	10	11	89	88	14	-	-	-	-
$c_2$	5	95	94	8	9	91	90	12	13	87	-	-	-	-

TABLE 2. The labels on edges of balanced d-magic graph  $K_{2,4} + K_{2,10}$ .

Since 4 is not congruent to 2 modulo 4, applying Theorem 2.7, a balanced d-magic graph  $K_{2,4} + K_{2,10}$  is constructed (see Figure 2), and the labels on edges of  $K_{2,4} + K_{2,10}$  are shown in Table 2.

### 3. BALANCED DEGREE-MAGIC LABELINGS IN THE COMPOSITION OF COMPLETE BIPARTITE GRAPHS

For two vertex-disjoint graphs G and H, the composition of graphs G and H, denoted by  $G \cdot H$ , is a graph such that the vertex set of  $G \cdot H$  is the Cartesian product  $V(G) \times V(H)$  and any two vertices (u, v) and (x, y) are adjacent in  $G \cdot H$  if and only if either u is adjacent with x in G or u = x and v is adjacent with y in H. For any positive integers p, q, r and s, we consider the composition  $K_{p,q} \cdot K_{r,s}$  of complete bipartite graphs. Let  $K_{p,q} \cdot K_{r,s}$  be a d-magic graph. Since  $\deg(v)$  is (r+s)p+r, (r+s)p+s, (r+s)q+r or (r+s)q+s and

 $f^*(v) = (pq(r+s)^2 + rs(p+q) + 1) \deg(v)/2$  for any  $v \in V(K_{p,q} \cdot K_{r,s})$ , we have

**Proposition 3.1.** Let  $K_{p,q} \cdot K_{r,s}$  be a d-magic graph. Then the following conditions hold:

- (i) only one of p, q, r and s is even or
- (ii) at least both r and s are even.

**Proposition 3.2.** Let  $K_{p,q} \cdot K_{r,s}$  be a balanced d-magic graph. Then at least both r and s are even.

In the next result we find the sufficient condition for the existence of balanced d-magic labelings of the composition of complete bipartite graphs  $K_{p,q}$ .  $K_{r,s}$ .

**Theorem 3.3.** Let p and q be positive integers, and let r and s be even positive integers. Then  $K_{p,q} \cdot K_{r,s}$  is a balanced d-magic graph.

Proof. Let p and q be positive integers, and let  $k = \min\{p,q\}$  and  $h = \max\{p,q\}$ . It is clear that  $K_{r+s,r+s}$ ,  $K_{r,s} + K_{r,s}$  and  $K_{r,s,r+s}$  are balanced d-magic by Proposition 2.3, Theorem 2.4 and Lemma1.8, respectively. The graph  $K_{p,q} \cdot K_{r,s}$  is decomposable into k balanced d-magic subgraphs isomorphic to  $K_{r,s}+K_{r,s}$ , h(k-1) balanced d-magic subgraphs isomorphic to  $K_{r,s,r+s}$ . According to Theorem 1.5,  $K_{p,q} \cdot K_{r,s}$  is a balanced d-magic graph.

Notice that the graph composition  $K_{p,q} \cdot K_{r,s}$  is naturally nonisomorphic to  $K_{r,s} \cdot K_{p,q}$  except for the case (p,q) = (r,s).

**Corollary 3.4.** Let p and q be positive integers, and let r and s be even positive integers. If p = q and r = s, then  $K_{p,q} \cdot K_{r,s}$  is a supermagic graph.

Proof. Applying Theorems 1.1 and 3.3.

The following example is a balanced d-magic graph  $K_{1,2} \cdot K_{2,2}$  (see Figure 3) with the labels on edges of  $K_{1,2} \cdot K_{2,2}$  in Table 3.

4. BALANCED DEGREE-MAGIC LABELINGS IN THE CARTESIAN PRODUCT OF COMPLETE BIPARTITE GRAPHS

For two vertex-disjoint graphs G and H, the *Cartesian product* of graphs G and H, denoted by  $G \times H$ , is a graph such that the vertex set of  $G \times H$  is the Cartesian product  $V(G) \times V(H)$  and any two vertices (u, v) and (x, y) are adjacent in  $G \times H$  if and only if either u = x and v is adjacent with y in H or v = y and u is adjacent with x in G. For any positive integers p, q, r and s, we consider the Cartesian product  $K_{p,q} \times K_{r,s}$  of complete bipartite graphs. Let  $K_{p,q} \times K_{r,s}$  be a d-magic graph. Since  $\deg(v)$  is p+r, p+s, q+r or q+s and  $f^*(v) = (pq(r+s) + rs(p+q) + 1) \deg(v)/2$  for any  $v \in V(K_{p,q} \times K_{r,s})$ , we have



FIGURE 3. A balanced d-magic graph  $K_{1,2} \cdot K_{2,2}$  with 12 vertices and 44 edges.

Vertices	$c_1$	$c_2$	$d_1$	$d_2$	$e_1$	$e_2$	$f_1$	$f_2$	$b_1$	$b_2$	
$a_1$	12	34	43	2	27	26	20	17	35	9	
$a_2$	33	11	1	44	19	18	25	28	10	36	
$b_1$	8	38	39	5	32	14	15	29	-	-	
$b_2$	37	7	6	40	13	31	30	16	-	- (	
$d_1$	4	42	-	-	-	-	-	-		-	
$d_2$	41	3	-	-	-	-	-	-	-	-	
$f_1$	-	-	-	-	23	22	-	-	-	-	
$f_2$	-	-	-	-	21	24	-	) -	-	-	

TABLE 3. The labels on edges of balanced d-magic graph  $K_{1,2} \cdot K_{2,2}$ .

**Proposition 4.1.** Let  $K_{p,q} \times K_{r,s}$  be a d-magic graph. Then the following conditions hold:

(i) only one of p,q,r and s is even or
(ii) all of p,q,r and s are either odd or even.

**Proposition 4.2.** Let  $K_{p,q} \times K_{r,s}$  be a balanced d-magic graph. Then p,q,r and s are either odd or even.

In the next result we are able to find the sufficient condition for the existence of balanced d-magic labelings of the Cartesian product of complete bipartite graphs  $K_{p,q} \times K_{r,s}$ .

**Theorem 4.3.** Let p, q, r and s be even positive integers with  $(p,q) \neq (2,2)$ and  $(r,s) \neq (2,2)$ . Then  $K_{p,q} \times K_{r,s}$  is a balanced d-magic graph.

Proof. Let p, q, r and s be even positive integers with  $(p, q) \neq (2, 2)$  and  $(r, s) \neq (2, 2)$ . Since  $K_{p,q}$  and  $K_{r,s}$  are d-magic by Proposition 1.6,  $2K_{p,q}$  and  $2K_{r,s}$  are balanced d-magic by Theorem 1.4. The graph  $K_{p,q} \times K_{r,s}$  is decomposable into (r + s)/2 balanced d-magic subgraphs isomorphic to  $2K_{p,q}$  and (p + q)/2



FIGURE 4. A balanced d-magic graph  $K_{2,4} \times K_{2,4}$  with 36 vertices and 96 edges.

balanced d-magic subgraphs isomorphic to  $2K_{r,s}$ . According to Theorem 1.5,  $K_{p,q} \times K_{r,s}$  is a balanced d-magic graph.

Observe that the Cartesian product graph  $K_{p,q} \times K_{r,s}$  is naturally isomorphic to  $K_{r,s} \times K_{p,q}$ .

**Corollary 4.4.** Let p, q, r and s be even positive integers with  $(p, q) \neq (2, 2)$ and  $(r, s) \neq (2, 2)$ . If p = q and r = s, then  $K_{p,q} \times K_{r,s}$  is a supermagic graph. *Proof.* Applying Theorems 1.1 and 4.3.

The following example is a balanced d-magic graph  $K_{2,4} \times K_{2,4}$  (see Figure 4), and the labels on edges of  $K_{2,4} \times K_{2,4}$  are shown in Table 4.

Vertices	$b_1$	$b_2$	$b_3$	$b_4$	$e_1$	$e_2$	$e_3$	$e_4$	$a_3$	$a_4$	$a_9$	$a_{10}$
$a_1$	96	2	3	93	-	-	-	-	72	-	25	-
$a_2$	1	95	94	4	-	-	-	-	-	64	-	33
$a_7$		-	-	-	8	90	91	5	27	-	70	-
$a_8$	-	-	-	-	89	7	6	92	-	35	-	62
$c_1$	48	-	-	-	51	-	-	-	88	9	-	-
$c_2$	-	32	-	-	-	67	-	-	10	87	-	-
$c_3$	-	-	40	-	-	-	59	-	11	86	-	-
$c_4$	-	-	-	56	-	-	-	43	85	12	-	-
$f_1$	49	-	-	-	46	-	-	-	-	-	16	81
$f_2$	-	65	-	-	-	30	-	-	-	-	82	15
$f_3$	-	-	57	-	-	-	$\overline{38}$	-	-	-	83	14
$f_4$	-	-	-	41	-	-	-	54	-	-	13	84

Vertices	$d_1$	$d_2$	$d_3$	$d_4$	$g_1$	$g_2$	$g_3$	$g_4$	$a_3$	$a_4$	$a_9$	$a_{10}$
$a_5$	24	74	75	21	-	-	-	-	26	-	71	-
$a_6$	73	23	22	76	-	-	-	-	-	34	-	63
$a_{11}$	-	-	-	-	80	18	19	77	69	-	28	-
$a_{12}$	-	-	-	-	17	79	78	20	-	61	-	36
$c_1$	50	-	-	-	45	-	-	-	-	-	-	-
$c_2$	-	66	-	-	-	29	-	-	-	-	-	-
$c_3$	-	-	58	-	-	-	37	-	-	-	-	-
$c_4$	-	-	-	42	-	-	-	53	-	-	-	-
$f_1$	47	-	-	-	52	-	-	-	-	-	-	-
$f_2$	-	31	-	-	-	68	-	-	-	-	-	
$f_3$	-	-	39	-	-	-	60	-	-	-		-
$f_4$	-	-	-	55	-	-	-	44	-	-	-	-
TABLE	4 T	'he l	abels	s on	edo	es of	' hal	ance	d d-	maoi	c or	anh

TABLE	4.	The	labels	on	edges	ot	balanced	d-magic	graph
$K_{2,4} \times$	$K_{2,2}$	4·							

### 5. Balanced Degree-Magic Labelings in the Tensor Product of Complete Bipartite Graphs

For two vertex-disjoint graphs G and H, the *tensor product* of graphs Gand H, denoted by  $G \oplus H$ , is a graph such that the vertex set of  $G \oplus H$  is the Cartesian product  $V(G) \times V(H)$  and any two vertices (u, v) and (x, y) are adjacent in  $G \oplus H$  if and only if u is adjacent with x in G and v is adjacent with y in H. For any positive integers p, q, r and s, we consider the tensor product  $K_{p,q} \oplus K_{r,s}$  of complete bipartite graphs. Let  $K_{p,q} \oplus K_{r,s}$  be a d-magic graph. Since deg(v) is pr, ps, qr or qs and  $f^*(v) = (2pqrs + 1) \deg(v)/2$  for any  $v \in V(K_{p,q} \oplus K_{r,s})$ , we have

**Proposition 5.1.** Let  $K_{p,q} \oplus K_{r,s}$  be a balanced d-magic graph. Then at least p and q are even or r and s are even.

Now we can prove the sufficient condition for the existence of balanced dmagic labelings of the tensor product of complete bipartite graphs  $K_{p,q} \oplus K_{r,s}$ .

**Theorem 5.2.** Let p and q be positive integers with  $(p,q) \neq (1,1)$ . Then  $K_{p,q} \oplus K_{2,2}$  is a balanced d-magic graph.

Proof. Let p and q be positive integers with  $(p,q) \neq (1,1)$ . Let  $k = \min\{p,q\}$ and  $h = \max\{p,q\}$ . Since  $K_{2,2h}$  is d-magic by Proposition 1.6,  $2K_{2,2h}$  is balanced d-magic by Theorem 1.4. The graph  $K_{p,q} \oplus K_{2,2}$  is decomposable into k balanced d-magic subgraphs isomorphic to  $2K_{2,2h}$ . According to Theorem 1.5,  $K_{p,q} \oplus K_{2,2}$  is a balanced d-magic graph.  $\Box$ 



FIGURE 5. A balanced d-magic graph  $K_{1,3} \oplus K_{2,2}$  with 16 vertices and 24 edges.

$b_1 \mid$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$	$b_7$	$b_8$	$b_9$	$b_{10}$	$b_{11}$	$b_{12}$
-	-	1	11	-	-	3	21	-	-	20	19
-	-	24	14	-	-	22	4	-	-	5	6
13	23	-	-	15	9	-	-	8	7	-	
12	2	-	-	10	16	-	-	17	18	-	-
	$b_1$ - 13 12	$ \begin{array}{cccc} b_1 & b_2 \\ \hline - & - \\ \hline - & - \\ \hline 13 & 23 \\ \hline 12 & 2 \end{array} $	$\begin{array}{cccc} b_1 & b_2 & b_3 \\ \hline - & - & 1 \\ \hline - & - & 24 \\ 13 & 23 & - \\ 12 & 2 & - \end{array}$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$				

TABLE 5. The labels on edges of balanced d-magic graph  $K_{1,3} \oplus K_{2,2}$ .

**Theorem 5.3.** Let p and q be positive integers, and let r and s be even positive integers with  $(r, s) \neq (2, 2)$ . Then  $K_{p,q} \oplus K_{r,s}$  is a balanced d-magic graph.

Proof. Let p and q be positive integers, and let r and s be even positive integers with  $(r, s) \neq (2, 2)$ . Since  $K_{r,s}$  is d-magic by Proposition 1.6,  $2K_{r,s}$  is balanced d-magic by Theorem 1.4. The graph  $K_{p,q} \oplus K_{r,s}$  is decomposable into pqbalanced d-magic subgraphs isomorphic to  $2K_{r,s}$ . According to Theorem 1.5,  $K_{p,q} \oplus K_{r,s}$  is a balanced d-magic graph.

It is clear that the tensor product graph  $K_{p,q} \oplus K_{r,s}$  is isomorphic to  $K_{r,s} \oplus K_{p,q}$ .

**Corollary 5.4.** Let p, q be positive integers with  $(p, q) \neq (1, 1)$ , and let r, s be even positive integers. If p = q and r = s, then  $K_{p,q} \oplus K_{r,s}$  is a supermagic graph.

*Proof.* Applying Theorems 1.1, 5.2 and 5.3.

Below is an example of balanced d-magic graph  $K_{1,3} \oplus K_{2,2}$  (see Figure 5), and the labels on edges of  $K_{1,3} \oplus K_{2,2}$  are shown in Table 5.

### 6. Balanced Degree-Magic Labelings in the Strong Product of Complete Bipartite Graphs

For two vertex-disjoint graphs G and H, the strong product of graphs G and H, denoted by  $G \otimes H$ , is a graph such that the vertex set of  $G \otimes H$  is

the Cartesian product  $V(G) \times V(H)$  and any two vertices (u, v) and (x, y) are adjacent in  $G \otimes H$  if and only if u = x and v is adjacent with y in H, or v = yand u is adjacent with x in G, or u is adjacent with x in G and v is adjacent with y in H. For any positive integers p, q, r and s, we consider the strong product  $K_{p,q} \otimes K_{r,s}$  of complete bipartite graphs. Let  $K_{p,q} \otimes K_{r,s}$  be a d-magic graph. Since deg(v) is p + r + pr, p + s + ps, q + r + qr or q + s + qs and  $f^*(v) = (pq(r+s) + rs(p+q) + 2pqrs + 1) \deg(v)/2$  for any  $v \in V(K_{p,q} \otimes K_{r,s})$ , we have

**Proposition 6.1.** Let  $K_{p,q} \otimes K_{r,s}$  be a d-magic graph. Then the following conditions hold:

(i) only one of p, q, r and s is even or

(ii) all of p, q, r and s are even.

**Proposition 6.2.** Let  $K_{p,q} \otimes K_{r,s}$  be a balanced d-magic graph. Then p, q, r and s are even.

We conclude this paper with an identification of the sufficient condition for the existence of balanced d-magic labelings of the strong product of complete bipartite graphs  $K_{p,q} \otimes K_{r,s}$ .

**Theorem 6.3.** Let p, q, r and s be even positive integers with  $(p, q) \neq (2, 2)$ and  $(r, s) \neq (2, 2)$ . Then  $K_{p,q} \otimes K_{r,s}$  is a balanced d-magic graph.

Proof. Let p, q, r and s be even positive integers with  $(p, q) \neq (2, 2)$  and  $(r, s) \neq (2, 2)$ . Thus,  $K_{p,q} \times K_{r,s}$  is balanced d-magic by Theorem 4.3, and  $K_{p,q} \oplus K_{r,s}$  is balanced d-magic by Theorem 5.3. Since  $K_{p,q} \otimes K_{r,s}$  is the graph such that  $K_{p,q} \times K_{r,s}$  and  $K_{p,q} \oplus K_{r,s}$  form its decomposition, by Theorem 1.5,  $K_{p,q} \otimes K_{r,s}$  is a balanced d-magic graph.

It is clear that the strong product graph  $K_{p,q} \otimes K_{r,s}$  is isomorphic to  $K_{r,s} \otimes K_{p,q}$ .

**Corollary 6.4.** Let p, q, r and s be even positive integers with  $(p, q) \neq (2, 2)$ and  $(r, s) \neq (2, 2)$ . If p = q and r = s, then  $K_{p,q} \otimes K_{r,s}$  is a supermagic graph.

Proof. Applying Theorems 1.1 and 6.3.

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