Iranian Journal of Mathematical Sciences and Informatics
Vol. 4, No. 1 (2009), pp. 27-35

The Basic Theorem and its Consequences

Noha Eftekhari

Department of Mathematics, Faculty of Sciences, ShahreKord University,
P.O.Box 115, ShahreKord, Iran

E-mail: Eftekharinoha@Yahoo.com

Abstract. Let $\mathcal{T}$ be a compact Hausdorff topological space and let $\mathcal{M}$ denote an n–dimensional subspace of the space $\mathbf{C}(\mathcal{T})$, the space of real–valued continuous functions on $\mathcal{T}$ and let the space be equipped with the uniform norm. Zukhovitskii [7] attributes the Basic Theorem to E.Ya.Remez and gives a proof by duality. He also gives a proof due to Shnirel'man, which uses Helly's Theorem, now the paper obtains a new proof of the Basic Theorem. The significance of the Basic Theorem for us is that it reduces the characterization of a best approximation to $f \in \mathbf{C}(\mathcal{T})$ from $\mathcal{M}$ to the case of finite $\mathcal{T}$, that is to the case of approximation in $l^\infty(r)$. If one solves the problem for the finite case of $\mathcal{T}$ then one can deduce the solution to the general case. An immediate consequence of the Basic Theorem is that for a finite dimensional subspace $\mathcal{M}$ of $C_0(\mathcal{T})$ there exists a separating measure for $\mathcal{M}$ and $f \in C_0(\mathcal{T}) \setminus \mathcal{M}$, the cardinality of whose support is not greater than dim $\mathcal{M}+1$. This result is a special case of a more general abstract result due to Singer [5]. Then the Basic Theorem is used to obtain a general characterization theorem of a best approximation from $\mathcal{M}$ to $f \in \mathbf{C}(\mathcal{T})$. We also use the Basic Theorem to establish the sufficiency of Haar's condition for a subspace $\mathcal{M}$ of $\mathbf{C}(\mathcal{T})$ to be Chebyshev.

Keywords: Best Approximation, Separating Measure, Chebyshev set.

2000 Mathematics subject classification: 41A52 and 41A65.

Received 10 March 2009; Accepted 22 May 2009

1. Introduction

Throughout this paper the setting will be as follows: $\mathcal{T}$ will be a compact Hausdorff topological space and $\mathbf{C}(\mathcal{T})$ will denote the space of real continuous functions on $\mathcal{T}$ and $\mathcal{M}$ is an n–dimensional subspace of $\mathbf{C}(\mathcal{T})$. The spaces $\mathbf{C}(\mathcal{T})$ are equipped with the uniform norm. The uniform norm is defined by

$$\|f\| = \max_{t\in\mathcal{T}} \ |f(t)| \text{ for all } f \in \mathbf{C}(\mathcal{T}) \ ,$$

and

$$d(f\ ,\mathcal{M}) = \inf_{g\in\mathcal{M}} \|f-g\| \ \text{ for } f \in \mathbf{C}(\mathcal{T}) \ ,$$

is called the distance from f to $\mathcal{M}$. We denote by

$$P_{\mathcal{M}}(f) = \{g \in \mathcal{M} \ : \ \|f-g\| = d(f\ ,\mathcal{M}) \ \} \ ,$$

the set of best approximations to f from $\mathcal{M}$.
The space $\mathrm{M}(\mathcal{T})$ of bounded real valued Borel measures on $\mathcal{T}$ is isometrically isomorphic to the space of bounded linear functionals on $\mathbf{C}(\mathcal{T})$. There is a nonzero $\lambda \in \mathrm{M}(\mathcal{T})$ such that $\lambda g < \lambda h$, for all $g \in \mathcal{M}$ and $h \in B(f, d(f,\mathcal{M}))$, the open ball with center f and radius $d(f,\mathcal{M})$. We will call such a measure a *separating measure* for f and $\mathcal{M}$. A set W is said to be a *Chebyshev subset* of $\mathbf{C}(\mathcal{T})$ if for each $f \in \mathbf{C}(\mathcal{T})$, the set $P_W(f)$ is a single point.

The paper obtains a characterization of Chebyshev hyperplanes in $l^\infty(n)$ (Theorem 2.1) and then by using it we state a characterization theorem for the best approximation from a Chebyshev hyperplane $\mathcal{M}$ of $l^\infty(n)$ (Theorem 2.3). Cheney [2] gives a necessary and sufficient condition for $g \in \mathcal{M}$ not to be a best approximation to $f \in \mathbf{C}(\mathcal{T})$. We give an alternative proof of Cheney's characterization (Theorem 3.1), and then use it to give a new proof of the Basic Theorem 3.2: there exists a finite subset A of $\mathcal{T}$ such that $d(f|_A, \mathcal{M}|_A) = d(f,\mathcal{M})$ and $\operatorname{card} A \leq \dim \mathcal{M} + 1$. Zukhovitskii [7] attributes the Basic Theorem to E.Ya.Remez and gives a proof by duality. He also gives a proof due to Shnirel'man, which uses Helly's Theorem. The significance of the Basic Theorem for us is that it reduces the characterization of a best approximation to f from $\mathcal{M}$ to the case of finite $\mathcal{T}$, that is to the case of approximation in $l^\infty(r)$. If one solves the problem for the finite case of $\mathcal{T}$ then one can deduce the solution to the general case. A set, the existence of which is asserted by Theorem 3.2, will be called a "basic set". An immediate consequence of the Basic Theorem is that for a finite dimensional subspace $\mathcal{M}$ of $C_0(\mathcal{T})$ there exists a separating measure for $\mathcal{M}$ and $f \in C_0(\mathcal{T}) \setminus \mathcal{M}$, the cardinality of whose support is not greater than $\dim \mathcal{M} + 1$. This result is a special case of a more general abstract result due to Singer [5]. Theorem 3.9 and Theorem 2.3 characterize Chebyshev hyperplanes in $l^\infty(n)$ and best approximations from them. Then the Basic Theorem is used to obtain a general characterization theorem

(Theorem 3.11) of a best approximation from $\mathcal{M}$ to $f \in \mathbf{C}(\mathcal{T})$. The latter theorem is equivalent to Theorem 3.12 which is the extension to a general $\mathcal{M}$, not necessary Chebyshev, of the Chebyshev Alternation Theorem. We also use the Basic Theorem to establish the sufficiency of Haar's condition for a subspace $\mathcal{M}$ of $\mathbf{C}(\mathcal{T})$ to be Chebyshev (Theorem 3.10).

Let $0 \not\equiv f \in \mathbf{C}(\mathcal{T})$. The critical set is

$$\text{crit}\, f \;= \{t \in \mathcal{T} \;:\; |f(t)| = \|f\|\ \}\ .$$

Define

$$e\;:\mathcal{T} \longrightarrow \mathcal{M}^\star \text{ by } e(t)(g) = g(t)\ ,\ \text{for all } g \in \mathcal{M}\ .$$

Let $A \subseteq \mathcal{T}$. The following conditions are equivalent.

(1) For $g \in \mathcal{M} \setminus \{0\}, \text{card}\, g^{-1}(0) \leq n-1$ (the Haar condition).
(2) For each set $A = \{t_1, \dots, t_n\}$ of n distinct points the mapping $S : \mathcal{M} \to \mathbb{R}^n$, defined by $S(g) = (g(t_1), \dots, g(t_n))$, is injective.
(3) For each set $A = \{t_1, \dots, t_n\}$ of n distinct points the mapping $S : \mathcal{M} \to \mathbb{R}^n$ is surjective.
(4) For each set $A = \{t_1, \dots, t_n\}$ of n distinct points $\dim \mathcal{M}|_A = n$.
(5) If $r \leq n$ and $t_1, \dots, t_r$ are distinct points of $\mathcal{T}$ then $e(t_1), \dots, e(t_r)$ are linearly independent points of $\mathcal{M}^\star$.
(6) If $A \subseteq \mathcal{T}$ and $\text{card}\, A \leq n$ then $\mathcal{M}|_A = C(A)$.

The following theorems are required.

Theorem 1.1. *([4, Lemma 2.2.1]) Let $f \in \mathbf{C}(\mathcal{T})$ and let f be not identically zero. Let $\mathcal{M}$ be a subspace of $\mathbf{C}(\mathcal{T})$. A necessary and sufficient condition that $0 \in P_{\mathcal{M}}(f)$ is that, there is no $g \in \mathcal{M} \setminus \{0\}$ such that*

$$f(t)g(t) > 0\ \ , \textit{for all}\ \ t \in \text{crit}\, f.$$

Theorem 1.2. *(The basic separation theorem)([3,Theorem 3.4]) Suppose A and B are disjoint, nonempty, convex sets in a topological vector space $\mathcal{X}$.*

(a) *If A is open there exist $\varphi \in \mathcal{X}^\star$ and $\gamma \in \mathbb{R}$ such that*

$$\varphi(x) < \gamma \leq \varphi(y) \text{ for every } x \in A \text{ and for every } y \in B.$$

(b) *If A is compact, B is closed, and $\mathcal{X}$ is locally convex, then there exist $\varphi \in \mathcal{X}^\star, \gamma_1 \in \mathbb{R}, \gamma_2 \in \mathbb{R}$, such that*

$$\varphi(x) < \gamma_1 < \gamma_2 < \varphi(y) \text{ for every } x \in A \text{ and for every } y \in B.$$

Theorem 1.3. *(Caratheodory's theorem) Let A be a subset of an n–dimensional linear space. Every point of the convex hull of A is expressible as a convex combination of $n+1$ (or fewer) elements of A.*

Theorem 1.4. *(The Chebyshev Alternation Theorem) Let $f \in C([a,b])$. A polynomial $p \in P_{n-1}$ is a best approximation to f if and only if there exist $n+1$ points $a \le t_0 < t_1 < \cdots < t_n \le b$ and $\varepsilon \in \{-1,1\}$ such that*

$$(f-p)(t_j) = \varepsilon(-1)^j \|f-p\| \text{ for } j = 0,\dots,n.$$

($f-p$ has n alternations on $[a,b]$).

2. Chebyshev Hyperplanes in $l^\infty(n)$

In this section we establish those restricted finite dimensional results for approximation by hyperplanes in $l^\infty(n)$ from which, using the Basic Theorem, the classical results for best approximation from a subspace $\mathcal{M}$ of $\mathbf{C}(\mathcal{T})$ will be deduced.

It is a geometrically obvious fact that if $\mathcal{M} = \varphi^{-1}(0)$ is a hyperplane in a normed linear space $\mathcal{X}$ then $\mathcal{M}$ is Chebyshev if and only if $\{x \in \mathcal{X} : \|x\| = 1 \text{ and } \varphi(x) = \|\varphi\|\}$ is a single point. Interpreting this in the case $\mathcal{X} = l^\infty(n)$ we obtain a characterization of Chebyshev hyperplanes in $l^\infty(n)$.

Theorem 2.1. *Let $\varphi = (\varphi_1,\dots,\varphi_n) \in l^1(n) \setminus \{0\}$. A hyperplane $\mathcal{M} = \varphi^{-1}(0)$ of $l^\infty(n)$ is Chebyshev if and only if $\varphi_k \neq 0$ for all $k = 1,\dots,n$.*

Proof. Let $f = (f(1),\dots,f(n)) \in l^\infty(n) \setminus \mathcal{M}$. So $\varphi(f) = \sum_{k=1}^n \varphi_k f(k)$ and $\|\varphi\|\ \|f\|$
$= \sum_{k=1}^n |\varphi_k|.\|f\|$. Thus $\varphi(f) = \|\varphi\|\ \|f\|$ if and only if $f(k) = \operatorname{sgn}\varphi_k.\|f\|$ when $\varphi_k \neq 0$. Therefore $\{f\ : \|f\| = 1, \varphi(f) = \|\varphi\|\}$ is a single point if and only if $\varphi_k \neq 0$ for all $k = 1,\dots,n$. □

Corollary 2.2. *If $\mathcal{M}$ is a Chebyshev hyperplane in $l^\infty(n)$ and $A \subset \{1,2,\dots,n\}$ then $\mathcal{M}|_A = l^\infty(A)$.*

Proof. Let $\mathcal{M} = \varphi^{-1}(0)$ where $\varphi \in l^1(n) \setminus \{0\}$. Then by Theorem 2.1, $\varphi = \sum_{k=1}^n \varphi_k\ e_{\mathcal{T}}(k)$ and $\varphi_k \neq 0$ for $k = 1,\dots,n$. Now suppose, on the contrary, that $\mathcal{M}|_A \subset l^\infty(A)$. Then there exists $\psi \in l^1(A)\setminus\{0\}$ such that $\mathcal{M}|_A \subseteq \psi^{-1}(0)$. So $\psi(g|_A) = 0$ for all $g \in \mathcal{M}$. Let $\psi = \sum_{i\in A} c_i e_A(i)$. Thus $\sum_{i\in A} c_i\ g(i) = 0$ for all $g \in \mathcal{M}$. That is, $(\sum_{i\in A} c_i\ e_{\mathcal{T}}(i))(g) = 0$ for all $g \in \mathcal{M} = \varphi^{-1}(0)$. So $\sum_{i\in A} c_i\ e_{\mathcal{T}}(i) = \alpha\varphi = \alpha\sum_{i=1}^n \varphi_i\ e_{\mathcal{T}}(i)$ for some α. Therefore, $\alpha = 0$ and c_i are all zero and so $\psi \equiv 0$ which is a contradiction. So $\mathcal{M}|_A = l^\infty(A)$. □

The next theorem characterize the best approximation from a Chebyshev hyperplane $\mathcal{M}$ of $l^\infty(n)$.

Theorem 2.3. *Let $\mathcal{M}$ be a Chebyshev hyperplane subspace of $l^\infty(n)$. Let $f \in l^\infty(n)\setminus\mathcal{M}$ and $g \in \mathcal{M}$. Then $g \in P_{\mathcal{M}}(f)$ if and only if $(f-g)(i) = \operatorname{sgn} c(i)\|f-g\|$, for $i = 1,\dots,n$, where $\varphi = (c(1),\dots,c(n)) \in l^1(n)$ and $\|\varphi\|_1 = 1, \varphi \in \mathcal{M}^\perp$ and all $c(i)$ are nonzero.*

Proof. By General Characterization Theorem [1,Theorem 1]

$$\begin{aligned} g \in P_{\mathcal{M}}(f) \quad &\Leftrightarrow \varphi(f-g) = \|\varphi\|\,\|f-g\|,\ \|\varphi\| = 1\ , \varphi \in \mathcal{M}^{\perp}, \\ &\Leftrightarrow (f-g)(i) = \operatorname{sgn} c(i)\|f-g\|,\ \text{for } i = 1,\ldots,n\ , \varphi \in \mathcal{M}^{\perp}. \end{aligned}$$

Also by Theorem 2.1, $c(i) \neq 0$ for $i = 1, \ldots, n$. □

3. The Basic Theorem and its Results

In this section, we will prove "Basic Theorem". It is important for us because investigation of best approximation to $f \in \mathbf{C}(\mathcal{T}) \setminus \mathcal{M}$ from $\mathcal{M}$ reduces to the case of finite $\mathcal{T}$, that is, in finite space $l^{\infty}(r)$. A set, the existence of which is asserted by the theorem, will be called a "basic set". The Basic Theorem has worthwhile results. We develop of the Chebyshev theory of best uniform approximation using the Basic Theorem. The extension to a general (not necessarily Chebyshev) $\mathcal{M}$ of the Chebyshev Alternation Theorem (3.11 and 3.12) will be obtained by exploiting the Basic Theorem.

The following theorem has been proved in Chapter 3 of [2] and we give an alternative proof of it.

Theorem 3.1. (Characterization Theorem) *In order that $g \in \mathcal{M}$ is not a best approximation to $f \in \mathbf{C}(\mathcal{T})$, it is necessary and sufficient that $0 \in \mathcal{M}^{\star}$ is not in the convex hull of the set $\{\ h(t)e(t)\ :\ |h(t)| = \|h\|\ \}$, where $h = g - f$.*

Proof. (Use basic separation theorem) Let $\mathcal{T}_0 = \operatorname{crit} h$. Since h is a continuous function, then $\mathcal{T}_0$ is a closed subset of the compact set $\mathcal{T}$ and so $\mathcal{T}_0$ is a compact set.
Let $A = \{\ h(t)e(t)\ :\ t \in \mathcal{T}_0\ \}$. The function he is continuous on the compact set $\mathcal{T}_0$ and so A is a compact subset of $\mathcal{M}^{\star}$. Since $\mathcal{M}^{\star}$ is finite dimensional then $K = \operatorname{co} A$ is compact in $\mathcal{M}^{\star}$ and K is closed convex subset of $\mathcal{M}^{\star}$. By Theorem 1.2, it follows that, $0 \notin K$ if and only if there exists $\varphi \in (\mathcal{M}^{\star})^{\star} \setminus \{0\}$ such that $\varphi(k) > 0$ for all, $k \in K$. Since $(\mathcal{M}^{\star})^{\star} \cong \mathcal{M}$ then it is equivalent to there exists $g' \in \mathcal{M} \setminus \{0\}$ such that $k(g') > 0$ for all, $k \in K$. So it is equivalent to there exists $g' \in \mathcal{M} \setminus \{0\}$ such that $(\,h(t)\,e(t)\,)(g') > 0$ for all $t \in \mathcal{T}_0$. Since $e(t)\,(g') = g'(t)$, it means that, there exists $g' \in \mathcal{M} \setminus \{0\}$ such that $h(t)\,g'(t) > 0$, for all $t \in \mathcal{T}_0$ and so by Theorem 1.1, $g \in \mathcal{M}$ is not a best approximation to f. □

In the following, we give a new proof of the Basic Theorem.

Theorem 3.2. (Basic Theorem) *Let $f \in \mathbf{C}(\mathcal{T}) \setminus \mathcal{M}$. Then there exist r points $t_1, \ldots, t_r \in \mathcal{T}$ such that*

$$d(f, \mathcal{M}) = d(f|_A, \mathcal{M}|_A)\ ,$$

where $A = \{t_1, \ldots, t_r\}$ and $\operatorname{card} A \leq \dim \mathcal{M} + 1$.

Proof. Let $g \in P_{\mathcal{M}}(f)$ and $h = g - f$. That is, $d(f, \mathcal{M}) = \|f - g\| = \|h\|$. Let $\mathcal{T}_0 = \operatorname{crit} h$. Theorem 3.1 (Characterization Theorem) implies that $0 \in K = \operatorname{co}\{\, h(t)\; e(t) : t \in \mathcal{T}_0 \,\}$. The set $\{\, h(t)\; e(t) : t \in \mathcal{T}_0 \,\}$ is a subset of n-dimensional space $\mathcal{M}^\star$ and so it follows from Caratheodory's theorem (Theorem 1.3) that $\sum_{i=1}^{r} \alpha_i\; h(t_i)\; e(t_i) = 0$, for some $t_1, \ldots, t_r$ in $\mathcal{T}_0$ and some positive numbers $\alpha_1, \ldots, \alpha_r$ with $\sum_{i=1}^{r} \alpha_i = 1$, where $r \leq \dim \mathcal{M} + 1$.
Let $A = \{t_1, \ldots, t_r\}$ and $e_{\mathcal{M}|_A} : \mathcal{T} \to (\mathcal{M}|_A)^\star$. Then $0 \in \operatorname{co}\left\{h(t) e_{\mathcal{M}|_A}(t) : t \in A\right\}$. One can apply Theorem 3.1 (Characterization Theorem) to A, $\mathcal{M}|_A$ and $g|_A \in P_{\mathcal{M}|_A}(f|_A)$ and get $d(f|_A, \mathcal{M}|_A) = \|(f - g)|_A\| = \|h\| = \|f - g\| = d(f, \mathcal{M})$. ☐

Remark 3.3. In the Basic Theorem $1 \leq r \leq n + 1$ in the real case and $1 \leq r \leq 2n + 1$ in the complex case, because of, $\mathbb{C}^n = \mathbb{R}^{2n}$. Also,we call $A \subseteq \mathcal{T}$ a "basic set" for $\mathcal{M}$ and f if it is finite and such that $d(f, \mathcal{M}) = d(f|_A, \mathcal{M}|_A)$.

The significance of the Basic Theorem is that it reduces the characterization of best approximation to f from $\mathcal{M}$ to the case of finite $\mathcal{T}$, that is to the case of approximation in $l^\infty(r)$. If one solves the problem for the finite case of $\mathcal{T}$ then one can deduce the solution to the general case.
The Basic Theorem implies the following corollaries.

Corollary 3.4. *Let $f \in \mathbf{C}(\mathcal{T})$ and $g \in \mathcal{M}$. Let $A \subseteq \mathcal{T}$ be a basic set for $\mathcal{M}$ and f. Then $g \in P_{\mathcal{M}}(f)$ if and only if $\|f - g\| = \|(f - g)|_A\|$ and $g|_A \in P_{\mathcal{M}|_A}(f|_A)$.*

Proof. Let $g \in P_{\mathcal{M}}(f)$. By the Basic Theorem,

$$d(f|_A, \mathcal{M}|_A) = d(f, \mathcal{M}) = \|f - g\| \geq \|(f - g)|_A\| \geq d(f|_A, \mathcal{M}|_A)\ ,$$

which implies that $\|f - g\| = \|(f - g)|_A\|$ and $g|_A \in P_{\mathcal{M}|_A}(f|_A)$. Now assume that, $g|_A \in P_{\mathcal{M}|_A}(f|_A)$ and $\|f - g\| = \|(f - g)|_A\| = d(f|_A, \mathcal{M}|_A) = d(f, \mathcal{M})$ (the Basic Theorem implies the last equality). Therefore $g \in P_{\mathcal{M}}(f)$. ☐

Corollary 3.5. *Let $f \in \mathbf{C}(\mathcal{T}) \setminus \mathcal{M}$. Let $A \subseteq \mathcal{T}$ be a minimal basic set for $\mathcal{M}$ and f. Then $A \subseteq \operatorname{crit}(f - P_{\mathcal{M}}(f)) = \cap_{g \in P_{\mathcal{M}}(f)} \operatorname{crit}(f - g)$.*

Proof. Let $g \in \operatorname{relint} P_{\mathcal{M}}(f)$. From Corollary 3.4 it follows that

$$\|(f - g)|_A\| = \|f - g\| = \|(f - g)|_{\operatorname{crit}(f-g)}\|.$$

So $\emptyset \neq A \cap \operatorname{crit}(f - g) \subseteq A$ and it will be shown that $B = A \cap \operatorname{crit}(f - g)$ is a basic set for $\mathcal{M}$ and f. If $B = A$ then there is nothing to prove. Now if $a \in A \setminus B$ then $|(f - g)(a)| < \|f - g\|$. Suppose, on the contrary, that $d(f|_B, \mathcal{M}|_B) < d(f|_A, \mathcal{M}|_A)$. Choose $g' \in \mathcal{M}$ such that $g'|_B \in P_{\mathcal{M}|_B}(f|_B)$. So

$$\|(f - g')|_B\| = d(f|_B, \mathcal{M}|_B) < d(f|_A, \mathcal{M}|_A) = \|(f - g)|_A\|.$$

Now for $\theta \in (0, 1)$, consider

$$\|(f - ((1 - \theta)g' + \theta g))|_A\| = \max\{\max_{a \in A \setminus B} |(f - ((1 - \theta)g' + \theta g))(a)|,$$
$$\|(f - ((1 - \theta)g' + \theta g))|_B\|\}.$$

Since $|(f-g)(a)| < \|f-g\|$ and the set $\{|(f-g')(a)| \ : \ a \in A \setminus B\}$ is bounded so for θ close to 1, $\max_{a\in A\setminus B} |(f-((1-\theta)g'+\theta g))(a)| < \|f-g\|$, also $\|(f-g')|_B\| < \|f-g\|$ and $\|(f-g)|_B\| \le \|(f-g)|_A\| = \|f-g\|$. Thus

$$\|(f-((1-\theta)g'+\theta g))|_A\| < \|f-g\|,$$

which is a contradiction. Therefore, $B = A \subseteq \operatorname{crit}(f-g)$ is a basic set for $\mathcal{M}$ and f and so by next remark $A \subseteq \operatorname{crit}(f - P_{\mathcal{M}}(f))$. □

Remark 3.6. If $g \in \operatorname{relint} P_{\mathcal{M}}(f)$ then

$$\operatorname{crit}(f-g) = \operatorname{crit}(f - P_{\mathcal{M}}(f)).$$

Because, let $t \in \operatorname{crit}(f-g)$ and let $g' \in P_{\mathcal{M}}(f)\setminus\{g\}$. Then $g \in (g', g'')$ for some $g'' \in P_{\mathcal{M}}(f)$. That is, for some $\theta \in (0,1)$, $g = (1-\theta)g' + \theta g''$ and

$$\begin{aligned} d(f,\mathcal{M}) = \|f-g\| = |(f-g)(t)| &\le (1-\theta)|(f-g')(t)| + \theta|(f-g'')(t)| \\ &\le (1-\theta)\|f-g'\| + \theta\|f-g''\| = d(f,\mathcal{M}). \end{aligned}$$

So $t \in \operatorname{crit}(f-g')$. That is, $\operatorname{crit}(f-g) = \operatorname{crit}(f-P_{\mathcal{M}}(f))$.

The following theorem is an immediate consequence of the Basic Theorem.

Theorem 3.7. *Let $f \in C_0(\mathcal{T}) \setminus \mathcal{M}$. Then there exists a separating measure φ, for f and $\mathcal{M}$, such that $|\operatorname{supp}\varphi| \le \dim \mathcal{M} + 1$.*

Proof. Let A be a minimal basic set for $\mathcal{M}$ and f. If $\varphi \in (\mathcal{M}|_A)^\star$ is a separating measure for $f|_A$ and $\mathcal{M}|_A$. Then φ is of the form $\varphi = \sum_{i\in A} c(i)\ e(i), (e(i) \in C(A)^\star)$. The functional φ has the natural extension $\bar\varphi = \sum_{i\in A} c(i)\ e(i), (e(i) \in C_0(\mathcal{T})^\star)$ and $\bar\varphi$ is a separating measure for f and $\mathcal{M}$. Therefore, $|\operatorname{supp}\bar\varphi| = \operatorname{card} A \le \dim\mathcal{M} + 1$. (By the Basic Theorem.) □

Theorem 3.8. *Let $f \in \mathbf{C}(\mathcal{T}) \setminus \mathcal{M}$. Let A be a minimal basic set for $\mathcal{M}$ and f. Then $\mathcal{M}|_A$ is a Chebyshev hyperplane in $C(A)$.*

Proof. Apply the Basic Theorem to $C(A), \mathcal{M}|_A$ and $f|_A$ then there exists a minimal basic set $A_1 \subseteq A$ such that $\operatorname{card} A_1 \le \dim \mathcal{M}|_A + 1$ and $d(f,\mathcal{M}) = d(f|_A, \mathcal{M}|_A) = d(f|_{A|_{A_1}}, \mathcal{M}|_{A|_{A_1}}) = d(f|_{A_1}, \mathcal{M}|_{A_1})$. By minimality of A, it follows that $A_1 = A$.So

$$\dim C(A) \le \operatorname{card} A \le \dim \mathcal{M}|_A + 1.$$

But $f|_A \notin \mathcal{M}|_A$ so $\dim\mathcal{M}|_A = \dim C(A) - 1$. That is, $\mathcal{M}|_A$ is a hyperplane in $C(A)$.

By Corollary 3.5, $A = A_1 \subseteq \operatorname{crit}(f|_A - P_{\mathcal{M}|_A}(f|_A))$ and so all functions of $P_{\mathcal{M}|_A}(f|_A)$ coincide on A, that is $P_{\mathcal{M}|_A}(f|_A)$ is a single point. Thus $\mathcal{M}|_A$ is Chebyshev in $C(A)$. □

Theorem 3.9. *Let $n > 1$. A hyperplane $\mathcal{M}$ of $l^\infty(n)$ is Chebyshev if and only if $A = \{1, 2, \ldots, n\}$ is the only basic set.*

Proof. (⇒) By Corollary 2.2.
(⇐) By Theorem 3.8. □

Theorem 3.10. *(Haar's Theorem) Let $\mathcal{M}$ be a finite dimensional subspace of $\mathbf{C}(\mathcal{T})$. Then $\mathcal{M}$ satisfies the Haar Condition if and only if $\mathcal{M}$ is a Chebyshev subspace of $\mathbf{C}(\mathcal{T})$.*

Proof. (⇒) Let $f \in \mathbf{C}(\mathcal{T}) \setminus \mathcal{M}$ and $\dim \mathcal{M} = n$. Let $A = \{t_1, \ldots, t_r\}$ be a minimal basic set for $\mathcal{M}$ and f. Suppose that $r \leq n$. Then $e_{\mathcal{M}|_A}(t_1), \ldots, e_{\mathcal{M}|_A}(t_r)$ are linearly independent(equivalent to Haar Condition). So $\dim \mathcal{M}|_A = r$ and $\mathcal{M}|_A = C(A)$ which contradicts $d(f|_A, \mathcal{M}|_A) = d(f, \mathcal{M}) \neq 0$. Thus $r = n + 1$. So the restriction mapping $r_A : \mathcal{M} \longrightarrow \mathcal{M}|_A$ is injective and $\mathcal{M}|_A$ is Chebyshev in $C(A)$ (Theorem 3.8) and $r_A(P_{\mathcal{M}}(f)) \subseteq P_{\mathcal{M}|_A}(f|_A)$. Thus $P_{\mathcal{M}}(f)$ is a single point. That is, $\mathcal{M}$ is a Chebyshev subspace of $\mathbf{C}(\mathcal{T})$.
Now(⇐), by any known proof of Haar's Theorem. □

Now by Corollary 3.4, Theorem 3.8 and Theorem 2.3, one can obtain the following general characterization theorem. Singer [6] obtained a more general abstract characterization theorem of which, this is a special case.

Theorem 3.11. *Let $f \in \mathbf{C}(\mathcal{T}) \setminus \mathcal{M}$ and $g \in \mathcal{M}$. Then $g \in P_{\mathcal{M}}(f)$ if and only if there exists a nonempty finite subset $A = \{t_1, \ldots, t_r\}$ of $\mathcal{T}$, $1 \leq r \leq n + 1$, and nonzero $c(t)$ for $t \in A$ with $\sum_{t \in A} |c(t)| = 1$ such that*

(1) $\sum_{t \in A} c(t)e(t) \in \mathcal{M}^{\perp}$, *and;*
(2) $f(t) - g(t) = \operatorname{sgn} c(t)\ \|f - g\|$, *for $t \in A$.*

Proof. Let A be a minimal basic set for $\mathcal{M}$ and f ($\operatorname{card} A \leq \dim \mathcal{M} + 1 = n + 1$). By Corollary 3.4, $g \in P_{\mathcal{M}}(f)$ if and only if $g|_A \in P_{\mathcal{M}|_A}(f|_A)$ and $\|f - g\| = \|(f - g)|_A\|$. By Theorem 3.8 and Theorem 2.3, it is equivalent to there exists a non-zero $c(t)$ for $t \in A$ with $\sum_{t \in A} |c(t)| = 1$ such that

(1) $\sum_{t \in A} c(t)e(t) \in \mathcal{M}^{\perp}$, and;
(2) $f(t) - g(t) = \operatorname{sgn} c(t)\ \|f - g\|$, for $t \in A$.

□

If r is the smallest integer such that Theorem 3.11 is satisfied then we obtain the following characterization theorem.

Theorem 3.12. *Let $f \in \mathbf{C}(\mathcal{T}) \setminus \mathcal{M}$ and $g \in \mathcal{M}$. Then $g \in P_{\mathcal{M}}(f)$ if and only if there exists a nonempty finite subset $A = \{t_1, \ldots, t_r\}$ of $\mathcal{T}$, where $1 \leq r \leq n+1$ with the following properties,*

(i) *The rank of the matrix:*

$$G = \begin{bmatrix} g_1(t_1) & \ldots & g_1(t_r) \\ \vdots & \vdots & \vdots \\ g_n(t_1) & \ldots & g_n(t_r) \end{bmatrix}$$

is less than r, where $\{g_1, \ldots, g_n\}$ *is a basis of* $\mathcal{M}$.

(ii) *The matrix*

$$\begin{bmatrix} g_1(t_1) & \ldots & g_1(t_r) \\ \vdots & \vdots & \vdots \\ g_n(t_1) & \ldots & g_n(t_r) \\ f(t_1) & \ldots & f(t_r) \end{bmatrix}$$

is of rank r.

(iii) *Among the minors of order* r *of the matrix in part (ii), there exists at least a minor* $\triangle \neq 0$ *in which all cofactors* $\triangle_j$ *of the elements* $f(t_j)$, $j = 1, \ldots, r$ *are nonzero.*

(iv) *The following equalities are satisfied,*

$$f(t_j) - g(t_j) = (\operatorname{sgn} \frac{\triangle_j}{\triangle}) \left\| f - g \right\|, \textit{ for } j = 1, \ldots, r.$$

Proof. The Theorem 3.12 is a translation of the Theorem 3.11 (modified if necessary).
(i) , (iii) $\Leftrightarrow$ (1) and the fact that all $c(t)$ are nonzero for $t \in A$. Also, r is minimal and $\dim \mathcal{M}|_A = r - 1$.
(ii) $\Leftrightarrow$ $f|_A \notin \mathcal{M}|_A$ which relates to A is minimal.
(iv) $\Leftrightarrow$ (2) □

This result, attributed by Zukhovitskii to Remez is the generalization of the Chebyshev Alternation Theorem for Chebyshev $\mathcal{M} \subseteq C([0,1])$ to a general (not necessary Chebyshev) $\mathcal{M} \subseteq \mathbf{C}(\mathcal{T})$. If the theorem is specialized to $\mathcal{T} = [0,1]$ and $\mathcal{M}$ Chebyshev, then it yields the alternation theorem.

Acknowledgement. The author acknowledges with gratitude the role played by Dr. A.L.Brown in editing original thesis; without his encouragement and guidance neither the original thesis nor this adaptation of it could have been produced. I am thankful to Prof. Dr. H.L.Vasudeva.

References

[1] A.L.Brown, Best Approximation by Smooth Function and Related Problems, *International Series of Numerical Mathematics, Birkhauser Verlag Basel*, **72** (1984), 70-82.

[2] E.W.Cheney, *Introduction to Approximation Theory*, Mc Graw–Hill, 1966.

[3] W.Rudin, *Functional Analysis, second Edition*, Mc Graw-Hill, 1991.

[4] H.S.Shapiro, *Topics in Approximation Theory*, Springer–Verlag, 1971 .

[5] I.Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer–Verlag, 1970 .

[6] I.Singer, Caracterisation des elements des meilleure approximation dans un espace de Banach guelconque, *Acta Sci. Mat.*, **17** (1956), 181-189.

[7] S.I.Zukhovitskii, On the Approximation of Real Functions in the Sense of P.L.Chebyshev, *Translations. Amer. Math. Soc.*, **19** (1962), 221-252.