Iranian Journal of Mathematical Sciences and Informatics
Vol. 4, No. 1 (2009), pp. 43-54

Some remarks on weakly invertible functions in the unit ball and polydisk

Romi F. Shamoyan[a,*] and Olivera R. Mihić[b]

[a]Bryansk University, Bryansk, Russia
[b]Fakultet organizacionih nauka, Jove Ilića 154, Belgrade, Serbia

E-mail: rsham@mail.ru
E-mail: oliveradj@fon.rs

Abstract. We will present an approach to deal with a problem of existence of (not) weakly invertible functions in various spaces of analytic functions in the unit ball and polydisk based on estimates for integral operators acting between functional classes of different dimensions.

Keywords: polydisk, unit ball, integral operator, weakly invertible function.

2000 Mathematics subject classification: Primary 32A18.

1. Introduction

Let $\mathbf{B}$ be as usual the unit ball and let $\mathbf{S}$ be a unit sphere. Let also $U^n = \{z = (z_1, \ldots, z_n) : |z_j| < 1,\ j = 1, \ldots, n\}$ be the unit polydisk in n- dimensional complex space $\mathbb{C}^n$ and $\mathbf{T}^n$ be a boundary of U^n (see [3], [15]). Let further X be some topological space of analytic functions in U^n or $\mathbf{B}^n$ in which the set of polynomials in $z_1, \ldots, z_n$ is dense. We will assume that the operators $S(f)(z) = z_1 \cdots z_n f(z_1, \ldots, z_n)$, $\Phi_z(f) = f(z)$, $z = (z_1, \ldots, z_n) \in U^n$ or $z \in \mathbf{B}^n$ are continuous in X.

*Corresponding Author

Accepted 23 June 2009

Definition 1. Let $f \in X$ and assume there exists a sequence of polynomials $\{P_k\}_1^\infty$ such that

$$\lim_{k\to\infty} P_k(z_1,\ldots,z_n)f(z_1,\ldots,z_n) = 1,$$

on the topology of the space X, then we say f is weakly invertible in X.

Weakly invertibility in one dimension has been studied by many authors (see [8], [5] and references there). In this research area in spaces of function of one variable the most intensive investigation was done in fundamental work of Nikolai Nikolski (see [8]).

The problem of extension of one variable results to higher dimension appears naturally. Recently some research was done in this direction (see for example [13] and references there). Our intention is to provide an approach that will work in the unit ball and polydisk and will solve some natural questions concerning the problem of generalization of one variable results to the case of unit ball and polydisk.

We use m_{2n} to denote the volume measure on U^n and m_n to denote the normalized Lebesgue measure on T^n. When $n = 1$, we simply denote U^1 by U, T^1 by T, m_{2n} by m_2, m_n by m. Let dv denote the volume measure on $\mathbf{B}$, normalized so that $v(\mathbf{B}) = 1$, and let $d\sigma$ denote the surface measure on $\mathbf{S}$ normalized so that $\sigma(\mathbf{S}) = 1$. For $\alpha > -1$ the weighted Lebesgue measure dv_α is defined by $dv_\alpha = c_\alpha(1-|z|^2)^\alpha dv(z)$ where $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$ is a normalizing constant so that $v_\alpha(\mathbf{B}) = 1$ (see [15]).

We denote by $H(\mathbf{B})$ the class of all holomorphic functions on $\mathbf{B}$. Let also $H(U)$ be a space of all holomorphic functions in the unit disk U, and similarly $H(U^n)$ be a space of all holomorphic functions in U^n.

For $\alpha > -1$ and $p > 0$ the weighted Bergman space $A^p_\alpha(\mathbf{B})$ consists of holomorphic functions f in $L^p(\mathbf{B}, dv_\alpha)$, that is, $A^p_\alpha(\mathbf{B}) = L^p(\mathbf{B}, dv_\alpha) \cap H(\mathbf{B})$.

As usual, we denote by $\overrightarrow{\alpha}$ the vector $(\alpha_1,\ldots,\alpha_n)$.

For $\alpha_j > -1$, $j = 1,\ldots,n$, $0 < p < \infty$, recall that the weighted Bergman space $A^p_{\overrightarrow{\alpha}}(U^n)$ consists of all holomorphic functions on the polydisk satisfying the condition $\|f\|^p_{A^p_{\overrightarrow{\alpha}}} = \int_{U^n} |f(z)|^p \prod_{i=1}^n (1-|z_i|^2)^{\alpha_i} dm_{2n}(z) < \infty$. When $\alpha_1 = \ldots = \alpha_n = \alpha$ then we use notation $A^p_\alpha(U^n)$.

For any f function from $L^1(\mathbf{B})$ we denote by $P[f]$ the Bergman projection of f function (see [15, Chapter 2]).

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

The notation $A \asymp B$ means that there is a positive constant C such that $\frac{B}{C} \le A \le CB$. We will write for two expressions $A \lesssim B$ if there is a positive constant C such that $A < CB$.

Remark 1. Note that weakly invertible functions in higher dimensions can be defined also otherwise (we consider for simplicity only case of bidisk) via following ,mixed" conditions

(1) $$\sup_{|z_j|<1} \|P_m(z_i)f(z_1,z_2)-1\|_{X_{z_i}(U)} \to 0,\ m\to\infty,\ i,j=1,2,\ i\neq j \ \text{ or}$$

(2) $$\|\|P_m(z_i)f(z_1,z_2)-1\|_{X_{z_i}(U)}\|_{X_{z_j}(U)} \to 0,\ m\to\infty,\ i,j=1,2,\ i\neq j.$$

2. Weakly invertible functions in higher dimensions

The goal of this section is to provide some generalizations of known onedimensional assertions on weakly invertible functions in higher dimensions. First we present general arguments then we consider concrete situations. For the proof of our main results we will need several lemmas.

Lemma A. [15] *There exists a positive integer N such that for any $0<r\leq 1$ we can find a sequence $\{a_k\}$ in $\mathbf{B}$ with the following properties:*
(1) $\mathbf{B}=\bigcup_k \mathcal{D}(a_k,r)$;
(2) The sets $\mathcal{D}(a_k,\frac{r}{4})$ are mutually disjoint;
(3) Each point $z\in\mathbf{B}$ belongs to at most N of the sets $\mathcal{D}(a_k,4r)$.

We are going to call as usual $\{a_k\}$ an r-lattice in the Bergman metric or sampling sequence.

Lemma B. [15] *For each $r>0$ there exists a positive constant C_r such that*

$$C_r^{-1}\leq \frac{1-|a|^2}{1-|z|^2}\leq C_r, \qquad C_r^{-1}\leq \frac{1-|a|^2}{|1-\langle z,a\rangle|}\leq C_r,$$

for all a and z such that $\beta(a,z)<r$. Moreover, if r is bounded above, then we may choose C_r independent of r.

Lemma C. [7, page 126] *Let $Q_m(\rho)=\{x=(x_1,\dots,x_m)\in\mathbb{R}^m : (\sum_{i=1}^m |x_i|^p)^{\frac{1}{p}}\leq \rho\},\ p\in[1,\infty)$, h be positive measurable function and $\rho\leq 1$. Then*

$$\int_{Q_m(\rho)} h(\|x\|)dx = C(m)\int_0^\rho t^{m-1}h(t)dt,$$

where $\|x\|=(\sum_{i=1}^m |x_i|^p)^{\frac{1}{p}}$ and $C(m)$ is a constant depending on m.

Lemma 1. *Let $f\in H(\mathbf{B})$ and $F(z_1,\dots,z_m)=C(\alpha)\int_{\mathbf{B}}\frac{f(z)(1-|z|)^\alpha dm_2(z)}{\prod_{k=1}^m(1-\langle z,z_k\rangle)^{\frac{\alpha+1+n}{m}}}$, $\alpha>-1$, $C(\alpha)$ is a constant of Bergman representation formula.*
1° *Let $p\leq 1$. Then $F\in H(\mathbf{B}\times\cdots\times\mathbf{B})$ and*

(3) $$|F(z_1,\dots,z_m)|^p\leq C\int_{\mathbf{B}}\frac{|f(\widetilde{w})|^p(1-|\widetilde{w}|)^t dv(\widetilde{w})}{\prod_{k=1}^m|1-\langle\overline{z_k},\widetilde{w}\rangle|^{\frac{\alpha+1+n}{m}p}},$$

where $t=(n+1+\alpha)p-(n+1),\ z_j\in\mathbf{B}, \jmath=1,\dots,m,\ t>-1.$

2° *Let* $p>1,\ \tau=p\left(\frac{\alpha+n+1}{mp'}-\tau_2\right),\ \tau<0,\ \tau_1+\tau_2=\frac{\alpha+n+1}{m},\ \tau_1,\tau_2>0,\ \frac{1}{p}+\frac{1}{p'}=1.$ *Then* $F\in H(\mathbf{B}\times\cdots\times\mathbf{B})$ *and*

(4)

$$|F(z_1,\dots,z_m)|^p\lesssim\int_{\mathbf{B}}\frac{|f(w)|^p(1-|w|)^\alpha(1-|z_1|^2)^\tau\cdots(1-|z_m|^2)^\tau}{\prod_{k=1}^m|1-\langle z_k,\overline{w}\rangle|^{p\tau_1}}dv(w),$$

where $z_j\in\mathbf{B},\ j=1,\dots,m.$

Proof. Using known properties of sampling sequence $\{a_k\}$ we get the following chain estimates ($p\le1$) from Lemma A and Lemma B and the fact that

$$\left(\sum_{k=1}^\infty a_k\right)^p\le\sum_{k=1}^\infty a_k^p,\quad a_k\ge0,\quad p\le1,$$

$$|F(z_1,\dots,z_m)|^p\lesssim\sum_{k\ge0}\max_{\mathcal{D}(a_k,r)}|f(w)|^p\left(\int_{\mathcal{D}(a_k,r)}\frac{(1-|w|)^\alpha}{\prod_{j=1}^m|1-\langle\overline{w},z_j\rangle|^{\frac{\alpha+1+n}{m}}}dv(w)\right)^p$$

$$\lesssim\sum_{k\ge0}\max_{\mathcal{D}(a_k,r)}|f(w)|^p\frac{(1-|a_k|)^{p\alpha}\,(v(\mathcal{D}(a_k,r)))^p}{\prod_{j=1}^m|1-\langle\overline{a_k},z_j\rangle|^{\frac{\alpha+1+n}{m}p}}.$$

Then using the relation

$|1-\langle w,z\rangle|\asymp|1-\langle a_k,z\rangle|,\quad w\in\mathcal{D}(a_k,r),\quad z\in\mathbf{B},$ (see [15, page 63])

and Lemma 2.24 from [15] and Lemma A we finally get

$$|F(z_1,\dots,z_m)|^p\le C\int_{\mathbf{B}}\frac{|f(\widetilde{w})|^p(1-|\widetilde{w}|)^t dv(\widetilde{w})}{\prod_{k=1}^m|1-\langle\overline{z_k},\widetilde{w}\rangle|^{\frac{\alpha+1+n}{m}p}}.$$

For $p>1$ the proof is based on Hölder's inequality applied twice and the estimate (see [15, Theorem 1.12]): $\int_{\mathbf{B}}\frac{(1-|z|)^\nu}{|1-\langle w,z\rangle|^{s_1}}dv(z)\le\frac{C}{(1-|w|)^{s_1-n-1-\nu}}, w\in\mathbf{B}, \nu>-1,\ s_1>\nu+n+1,$ applied m times for $s_1=\tau_2p'm$.

$$|F(z_1,\dots,z_m)|^p\lesssim\int_{\mathbf{B}}\frac{|f(w)|^p(1-|w|)^\alpha(1-|z_1|^2)^\tau\cdots(1-|z_m|^2)^\tau}{\prod_{k=1}^m|1-\langle z_k,\overline{w}\rangle|^{p\tau_1}}dv(w),$$

$z_j\in\mathbf{B},\ j=1,\dots,m,\ \tau=p\left(\frac{\alpha+n+1}{mp'}-\tau_2\right),\ \tau<0.$ □

The estimate in the following lemma can be obtained without difficulties.

Lemma 2. *Let* $t>-1,\ \beta_j>\frac{t+1}{n},\ j=1,\dots,n.$ *Then*

$$\int_0^1\frac{(1-|w|)^t d|w|}{\prod_{j=1}^n|1-|w|e^{i\varphi}z_j|^{\beta_j}}\le\frac{C}{\prod_{j=1}^n|1-z_je^{\overline{i\varphi}}|^{\beta_j-\frac{t+1}{n}}},\ z_j\in U,\ e^{i\varphi}\in T.$$

Let us turn now to some general observations concerning weakly invertible functions. Our arguments can be applied in the unit ball and unit polydisk. Our first observation is the following. In the mentioned paper [8] a series of results of the following type were proved.

Let $X\subset H(U),\ X_1\subset X$ be topological subspaces of $H(U)$. Then there exist a function $f,\ f \in X_1$ which is not weakly invertible in X. Our intention is to find an expansion of this f (and a class X_1) from diagonal $(z, \dots, z)$ using an integral operator T_n acting between two functional classes of different dimensions such that if $T_n f = \Phi$ then $\Phi(z, \dots, z) = f(z),\ T_1 f = f,\ n \geq 1,\ n \in \mathbb{N}$, and to show that $\Phi \in X_n(\mathbf{B})$, $n \in \mathbb{N}$. Then we obtain the estimate of the type

$$\|\mathcal{D}f_1\|_X \leq C\|f_1\|_{\widetilde{X}_n(\mathbf{B})}$$

where $\widetilde{X}_1 = X,\ X_n \subset \widetilde{X}_n,\ (\mathcal{D}f_1)(z) = f_1(z, \dots, z)$. Putting into last estimate $f_1 = P_m\Phi - 1$ we will immediately get Φ is not weakly invertible in $\widetilde{X}_n$. Moreover, it is easy to see for $n = 1$ our assertion will coincide with above mentioned onedimensional result.

The integral operators acting between dimensions unit disk and unit ball and unit polydisk that we will consider are of the following type

$$(T_n^\alpha f)(z_1, \dots, z_n) = C(\alpha)\int_U \frac{f(w)(1-|w|)^\alpha dm_2(w)}{\prod_{k=1}^n (1 - \langle w, z_k\rangle)^{\frac{\alpha+2}{n}}},\ \alpha > -1,$$

where $C(\alpha)$ is a constant of well-known Bergman representation formula, $z = (z_1, \dots, z_n)$ is in unit ball $\mathbf{B}$ or unit polydisk. Note in the case of polydisk this integral operators are not new (see for example [3]). For the case of unit ball $\mathbf{B}$ these are new integral operators.

We turn now to our second general observation concerning the concept of weak invertibility in spaces of analytic functions in higher dimensions, unit ball and polydisk. Various assertions for weakly invertible functions have the following form (see [8]): Every function from a class $X_1,\ f \neq 0,\ f \in X_1$ is weakly invertible in $X,\ X_1 \subset X$, X and X_1 are subspaces of $H(U)$. We would like to propose a general approach in higher dimension to this problem by following scheme. From our definition of weakly invertible functions via $\|P_k f - 1\|_X$ in higher dimension it follows that to show that f is weakly invertible in $X \subset H(\mathbf{B})$ or $X \subset H(U^n)$ it is enough to show for any linear bounded functional Φ on X such that $\Phi(g) = 0$ for any $g \in E(f)$ where $E(f)$ is a closure of a set $\{Pf\}$, P is a set of polynomials in n variables the following equality holds $\Phi(1) = 0$. For that reason we apply the following general procedure. We find a sequence $\widetilde{G}_k(z)$, $\widetilde{G}_k \in X$ such that $\|\widetilde{P}_m f - \widetilde{G}_k\|_X \leq C\|f\|_{X_1}\|\widetilde{P}_m - \widetilde{G}_k\|_X$. Then using the fact that polynomials are dense in X we find $\widetilde{P}_m : \|\widetilde{P}_m - \widetilde{G}_k\|_X \to 0$ as $|m| \to \infty$ ($\widetilde{P}_m$ is a polynomial of degree m). So that $\widetilde{G}_k \in E(f)$. Hence for all Φ linear bounded functionals such that $\Phi(\widetilde{f}) = 0$ for all $\widetilde{f} \in E(f)$ we have $\Phi(\widetilde{G}_k) = 0$, for all $\overrightarrow{k} = (k_1, \dots, k_n)$.

Since we can represent all linear bounded functionals on X for many known analytic X spaces in unit ball $\mathbf{B}$ and unit polydisk U^n by Causchy duality we will get

$$\Phi(G_{\overrightarrow{k}}) = \lim_{\rho \to 1}\int_{\mathbf{S}} G_{\overrightarrow{k}}(\rho\xi)g(\rho\overline{\xi})d\sigma(\xi) = 0,$$

$$\Phi(G_{\overrightarrow{k}}) = \lim_{\rho\to 1}\int_{T^n} G_{\overrightarrow{k}}(\rho\xi)g(\rho\overline{\xi})dm_n(\xi) = 0, \tag{5}$$

where $g(z) = \Phi(l_z),\ l_z(\xi) = \frac{1}{\prod_{k=1}^{n}(1-\langle\xi_k, z_k\rangle)},\ \xi_k, z_k \in U^n$.

Finally from (5) using properties of f function and X space we will show that $\Phi(1) = 0$ which will complete the proof of the fact that f is weakly invertible in X. In dimension one in the unit disk the modification of this scheme is known (see [8, p.132-134]).

We provide now concrete applications of approaches we mentioned above connected with weakly invertibility in higher dimension.

The following proposition use ideas we indicate above for the study of weak invertibility of generalization of $\exp\left(\frac{z+1}{z-1}\right)$ function in polydisk. In unit disk the weak invertibility of $\exp\left(\frac{z+1}{z-1}\right)$ was studied in [8] systematically.

Proposition 1. *Let λ be an increasing positive function on $[0,1)$, $\log(\lambda(r)) = \varphi\left(\log\frac{1}{1-r}\right)$, $\psi = \log\varphi$ and ψ is convex near $+\infty$ and let*

$$\int_0^1 \left(\frac{\log\lambda(r)}{1-r}\right)^{\frac{1}{2}} dr < \infty,$$

then the Φ function, $\Phi = T_n^\alpha(f_0),\ \alpha > -1,\ n \geq 1,\ f_0(z) = e^{\frac{z+1}{z-1}},\ z \in U$, is not weakly invertible in

$$A_n(\lambda) = \{f \in H(U^n) : \sup_{z\in U^n}\frac{|f(z)|}{\widetilde{\lambda}(|z|)} < \infty\},\ (\widetilde{\lambda}(|z|) = \left(\prod_{k=1}^{n}\lambda(|z_k|)\right)^{\frac{1}{n}}.$$

Proof. Since $\Phi = T_n^\alpha(f_0)$ using Hölder's inequality for n functions we get

$$\|\Phi\|_{A_n(\lambda)} \leq const\|f_0\|_{A_1(\lambda)}. \tag{6}$$

We assumed in addition that

$$\int_U \frac{\lambda(|z|)(1-|z|)^\alpha dm_2(z)}{|1-\langle w,\overline{z}\rangle|^{\alpha+2}} \leq C\lambda(|w|),\ w \in U. \tag{7}$$

It is easy to note that

$$\|\widetilde{P}_m f_0 - 1\|_{A_1(\lambda)} \leq C_1\|P_m\Phi - 1\|_{A_n(\lambda)},\ \widetilde{P}_m = P_m(z,\dots,z),\ P_m = P_m(z_1,\dots,z_n).$$

It remains to use the fact that f_0 is not weakly invertible in $A_1(\lambda)$ (see [8]). □

Remark 2. For $n = 1$ Proposition 1 were obtained in [8, page 163].

Remark 3. Various other assertions can be obtained similarly for $\Phi = T_n^\alpha(f_0)$ function concerning it is weakly invertibility based on results of [8] for $n = 1$.

Remark 4. Note that if $\exp\left(\frac{z+1}{z-1}\right)$ is not weakly invertible in $X(U)$ then $f_0 = \exp\left(\frac{\sum_{k=0}^{n}\frac{z_k+1}{z_k-1}}{n}\right)$ and $f_0^1 = \exp\left(\frac{\frac{z_1+\cdots z_n}{n}+1}{\frac{z_1+\cdots+z_n}{n}-1}\right)$ are not weakly invertible in

$Y(U^n)$ or $Y(\mathbf{B})$ as soon as $\|g(z,\dots,z)\|_X \le const\|g(z_1,\dots,z_n)\|_Y,\ g\in H(\mathbf{B})$ or $g\in H(U^n)$. The Proposition 1 we formulated gives such an example.

If f is weakly invertible in X and if $(T_n^\alpha f)(z_1,\dots,z_n) = C\int_U \frac{f(z)(1-|z|)^\alpha dm_2(z)}{\prod_{k=1}^n (1-\langle \overline{z}, z_k\rangle)^{\frac{\alpha+2}{n}}}$ is acting from $X(U)$ to $X_1(U^n)$ for $\alpha>\alpha_0$. Then any Φ function

$$\Phi(z_1,\dots,z_n) = \frac{1}{\widetilde{P}_m(z_1,\dots,z_n)}\int_U \frac{(f(z)P_m(z)-1)dm_2(z)}{\prod_{k=1}^n (1-\langle z,\overline{z}_k\rangle)^{\frac{\alpha+2}{n}}} + \frac{1}{\widetilde{P}_m(z_1,\dots,z_n)}$$

is weakly invertible in X_1 as soon as f is weakly invertible in $X(U)$, where P_m is a sequence of polynomials such that $\|P_m f-1\|_X\to 0,\ m\to\infty$ and $\widetilde{P}_m$ is any sequence of polynomials of n variables.

Using arguments we provided above we formulate the following proposition.

Proposition 2. *Let f be an inner function and $\widetilde{\Phi} = T_n^{\widetilde{\alpha}}(f^{-1})$ for some fixed $\widetilde{\alpha}>-1$. Then let $\widetilde{\Phi}\in A^p_\alpha(U^n),\ \alpha>-1$. Then there exist a sequence of polynomials $\{P_m(z)\}$ such that for any $\gamma>\frac{(\alpha+2)n}{p}-2$, the Φ function*

$$\Phi(z_1,\dots,z_n) = \frac{1}{\widetilde{P}_m(z_1,\dots,z_n)}\int_U \frac{(f(z)P_m(z)-1)(1-|z|)^\gamma dm_2(z)}{\prod_{k=1}^n (1-\langle \overline{z},z_k\rangle)^{\frac{\gamma+2}{n}}} + \frac{1}{\widetilde{P}_m(z_1,\dots,z_n)}$$

is weakly invertible function in $A^p_\alpha(U^n)$, for any sequence of polynomials $\widetilde{P}_m(z_1,\dots,z_n)$ so that $\Phi\in A^p_\alpha(U^n)$.

Remark 5. For $n=1$ it is was proved in [8, page 92].

Proof. The proof follows from theorem on diagonal map in $A^p_\alpha(U)$ classes (see [3]) and the following onedimensional assertion from [8]:
Let f be an inner function $f\in A^p_\alpha(U),\ \frac{1}{f}\in A^p_\alpha(U)$, then f is weakly invertible in $A^p_\alpha(U)$.

Indeed since $\widetilde{\Phi}\in A^p_\alpha(U^n),\ f^{-1}\in A^p_{\alpha n+2n-2}(U)$, also $f\in A^p_{\alpha n+2n-2}(U)$, $\alpha>-1$, $n\in\mathbb{N}$, hence there exists a sequence of polynomials such that

$$\|P_m f-1\|_{A^p_{\alpha n+2n-2}}\to 0 \ \text{ as } \ m\to\infty.$$

It remains again apply the theorem on diagonal map in A^p_α classes (see [3]). $\square$

Note any theorem on traces and diagonal map in $A^p_\alpha,\ H^p,\ Q_p$, Bloch type classes (see [3], [4], [14] and references there) can be applied in this situation or more generally as we see estimates connecting functional spaces in various dimensions can be applied in problems connected with weak invertibility.

In recent note [9] a result of the similar nature was proved. Namely it was proved by authors in [9] that if $f\in H^2(U^n),\ n\in\mathbb{N}$, then $f\in H^{2n}(\mathbf{B})$ and there holds

$$\sup_{0<r<1}\int_{\mathbf{S}} |f(r\xi)|^{2n} d\sigma_n(\xi) \le C\sup_{0<r<1}\left(\int_{T^n}|f(r\xi)|^2 dm_n(\xi)\right)^n.$$

Note for $n=1$ the estimate we provided above is obvious. We add now new results in this direction.

Theorem 1. *1°* *Let* $p\in(0,\infty),\ t>-1,\ \alpha=-1+\frac{t+2}{n},\ k=1,\dots,n,$ $f\in H(U^n)$. *Then* $\int_U|f(z,\dots,z)|^p(1-|z|)^t dm_2(z)$

$$\le C\int_0^1\int_0^1\int_T(1-\min r_k)^{-n}\prod_{k=1}^n(1-r_k)^\alpha|f(r_1\xi,\dots,r_n\xi)|^p dm(\xi)dr_1\cdots dr_n;$$

2° *Let* $\beta=\alpha-n,\ \beta>-1,\ p\in(\frac{\alpha+1}{\alpha+n+1},1],\ \rho\le1,\ f\in H(\mathbf{B})$. *Then*

$$\int_{Q_m(\rho)}(\sup_{\xi\in T^n}(P[f])(R_1,\dots,R_n))^p(1-(\sum_{k=1}^n R_k^2)^{\frac12})^\alpha dR_1\cdots dR_n\lesssim C\int_{\mathbf{B}}|f(w)|^p(1-|w|)^\beta dv(w);$$

3° *Let* $p\le1,\ \alpha>n-1,\ f\in H(U^n)$. *Then*

$$\int_0^1(1-r)^\alpha\sup_{\xi\in\mathbf{S}}|f(r\xi)|^p dr\le C\int_{U^n}|f(w)|^p\prod_{k=1}^n(1-|w_k|)^{\frac{\alpha+1}{n}-2}dm_{2n}(w);$$

4° *Let* $p\le1,\ s>0,\ (\alpha+2)p>s+1,\ f\in H(U)$. *Then*

$$\int_0^1(1-r)^s\sup_{\xi\in\mathbf{S}}|(T_n^\alpha f)(r\xi)|^p dr\le C\int_U|f(z)|^p(1-|z|)^{s-1}dm_2(z).$$

Proof. **1°** The following dyadic decomposition of subframe and polydisk were introduced in [3] and will be also used by us.

$$U_{k_1,\dots,k_n,l_1,\dots,l_n}=U_{k_1,l_1}\times\cdots\times U_{k_n,l_n}=\{(\tau_1\xi_1,\dots,\tau_n\xi_n):\tau_j\in(1-\frac{1}{2^{k_j}},1-\frac{1}{2^{k_j+1}}],$$

$$k_j=0,1,2,\dots;\ \frac{\pi l_j}{2^{k_j}}<\xi_j\le\frac{\pi(l_j+1)}{2^{k_j}},\ l_j=-2^{k_j},\dots,2^{k_j}-1,\ j=1,\dots,n\}.$$

We have
$\int_U|f(z,\dots,z)|^p(1-|z|)^t dm_2(z)\lesssim\sum_{k\ge0}\sum_{j=-2^k}^{2^k-1}\left(\int_{U_{j,k}}|f(z,\dots,z)|^p(1-|z|)^t dm_2(z)\right)$

$$\lesssim C\sum_{k_1,\dots,k_n\ge0}\sum_{j=-2^{\min k_j}}^{2^{\min k_j}-1}\left(\sup_{z\in U_{j,k_1,\dots,k_n}}|f(z)|^p\right)2^{-\frac{(k_1+\cdots k_n)t}{n}}2^{-2\frac{k_1+\cdots k_n}{n}}$$

$$\lesssim C\sum_{k_1,\dots,k_n\ge0}\sum_{j=-2^{\min k_j}}^{2^{\min k_j}-1}2^{\min k_j n}2^{k_1+\cdots k_n}2^{-(\sum_{k=1}^n k_j)(\frac{t}{n}+\frac{2}{n})}\int_{U_{j,k_1,\dots,k_n}}|f(z)|^p dm_2(z)$$

$$\lesssim C\int_{U^n}|f(z)|^p\prod_{k=1}^n(1-|z_k|)^\alpha(1-\min_k|z_k|)^{-n}dm_{2n}(z),\ \alpha=-1+\frac{t+2}{n},$$

where we used the fact that
$\sup_{z\in U_{j,k_1,\dots,k_n}}|f(z)|^p\le C2^{\min k_j n}2^{k_1+\cdots k_n}\int_{U^*_{j,k_1,\dots,k_n}}|f(z)|^p dm_{2n}(z)$ which follows from subharmonicity of $|f(z)|^p,\ 0<p<\infty$ and where $U^*_{j,k_1,\dots,k_n}$ are enlarged dyadic cubes (see [3]). We used at the last step also the fact that $U^*_{j,k_1,\dots,k_n}$ is a finite covering of polydisk U^n (see [3]).

2° According to Bergman representation formula (see [15, Chapter 2]) in ball we have

$$f(z_1,\dots,z_n)=C_\alpha\int_{\mathbf{B}}\frac{f(w)(1-|w|)^\alpha dv(w)}{(1-\langle\overline{w},z\rangle)^{\alpha+n+1}},\ \alpha>-1,\ z\in\mathbf{B},\ z=(z_1,\dots,z_n).$$

For $p\le 1$ consider the same integral but with $(z_1,\dots,z_n)\in U^n$ we will have

$$(\sup_{\xi\in T^n}(P[f])(R_1,\dots,R_n))^p\lesssim\int_0^1\int_{\mathbf{S}}\frac{|f(r\xi)|^p(1-r)^{\alpha p+(p-1)(n+1)}d\sigma(\xi)dr}{(1-r(\sum_{k=1}^n R_k^2)^{\frac{1}{2}})^{(\alpha+n+1)p}},$$

then using Lemma C we can easily get what we need.

Part **3°** and **4°** can be obtained similarly as **2°**, but instead of integral representation in the unit ball we have to use known integral representation in the polydisk and disk respectively. The proof of **3°** follows from Lemma 1 for $p\le 1$ and Lemma 2. Theorem 1 is proved. □

Estimates connecting quasinorms like

$$\sup_{r<1}\sup_{\xi\in\mathbf{S}}|f(r\xi)|(1-r)^\alpha,\ \alpha>0,\ f\in H(\mathbf{B}),$$

$$\sup_{r_k<1}\sup_{\xi\in\mathbf{T^n}}|f(r\xi)|\prod_{k=1}^n(1-r_k)^\alpha,\ \alpha>0,\ f\in H(U^n),$$

$$\text{with }\sup_{z\in U}|\mathcal{D}f(z)|(1-|z|)^t\text{ or }\sup_{z\in U}|f(z)|(1-|z|)^t,\ f\in H(U)$$

with

$$\sup_{z\in\mathbf{B}}|T_n^\alpha f(z)|(1-|z|^2)^\alpha,$$

can be obtained similarly and easier.

Remark 6. The complete analogues of assertions in Theorem 1 are true also for $p>1$. In this case we should use Lemma 1 for $p>1$ and repeat arguments we provided for $p\le 1$ above.

Remark 7. All estimates of Theorem 1 are obvious for $n=1$ or coincide with classical assertions from the theory of analytic functions of one variable (see [15]).

Remark 8. Various other estimates connecting functional classes in the unit disk, with classes on subframe and expanded disk can be found also in [6] and [14]. They also allow to get various assertions on weakly invertible functions in higher dimensions directly from onedimensional results.

The problem of weak invertibility of function were considered in [8] in particular in the following classes in the unit disk (see [8], p.148-150):

$$AI_\lambda^{p,q}=\{f\in H(U):\int_I\left(\sup_{|z|<1}|f_r(z)|(1-|z|)^q\right)^p\lambda(r)dr<\infty\},\ 0<p,q<\infty,$$

$$BI_\lambda^{p,q} = \{f \in H(U) : \int_I \left(\int_U |f_r(z)|^q dm_2(z) \right)^{\frac{p}{q}} \lambda(r) dr < \infty\},\ 0 < p, q < \infty,$$

$I = (0, 1)$,

under some conditions on $\lambda = \lambda(r)$ function, where $f_r(z) = f(rz),\ r \in (0,1),\ z \in U$.

We showed that the concept of weak invertibility is closely related in higher dimension with integral T_n^α operators.

We will obtain sharp estimates for these integral operators acting in $AI_\lambda^{p,q}$ and $BI_\lambda^{p,q}$ classes to get information on weak invertibility in $AI_\lambda^{p,q}$ and $BI_\lambda^{p,q}$ in polydisk.

Let S be a set of positive on $(0,1)$ functions $w,\ w \in L^1(0,1)$ such that $m_w \le \frac{w(\lambda r)}{w(r)} \le M_w$, for all $r \in (0,1),\ \lambda \in [q_w, 1]$ and some fixed $M_w,\ m_w,\ q_w$ such that $m_w, q_w \in (0,1),\ M_w > 0$. S classes were studied in [12]. We note if $w \in S$ then $w(x) \in [x^{\alpha_w}, x^{-\beta_w}],\ x \in (0,1),\ \alpha_w = \frac{\ln m_w}{\ln q_w}, \beta_w = \frac{\ln M_w}{\ln \frac{1}{q_w}}$.,

Theorem 2. (*a*) *Let $0 < p < \infty,\ \alpha_j > -1,\ j = 1, \dots, n$, then if $f \in H(U^n)$ and if $\|f\|_{AI_\lambda^{p,\vec{\alpha}}(U^n)} = \int_I \left(\sup_{|z_j|<1} |f_r(z)|(1-|z_1|)^{\alpha_1} \cdots (1-|z_n|)^{\alpha_n} \right)^p \lambda(r) dr < \infty,\ \lambda \in S$, then $f(z, \dots, z) \in AI_\lambda^{p,|\alpha|}(U),\ |\alpha| = \sum_{j=1}^n \alpha_j$. And the reverse is also true:*
If $f \in AI_\lambda^{p,|\alpha|}(U)$, then there exists a function $g \in AI_\lambda^{p,\vec{\alpha}}(U^n)$ such that $g(z, \dots, z) = f(z),\ z \in U$.
(*b*) *Let $p \le q \le 1,\ \lambda \in S$, then if $f \in H(U^n)$ and if $\|f\|^p_{BI_\lambda^{p,q}(U^n)} = \int_I (\int_{U^n} |f_r(z_1, \dots, z_n)|^q\ dm_{2n}(z))^{\frac{p}{q}} \lambda(r) dr < \infty$,*
then $\|f\|_{BI_\lambda^{p,q,n}(U)} = \int_I (\int_U |f_r(z, \dots, z)|^q (1-|z|)^{2n-2} dm_2(z))^{\frac{p}{q}}\ \lambda(r) dr < \infty$. And the reverse is also true: If $f \in BI_\lambda^{p,q,n}(U)$, then there exists a function $g \in BI_\lambda^{p,q}(U^n)$ such that $g(z, \dots, z) = f(z),\ z \in U$.

One part of Theorem 2 follows immediately from known results on diagonal map in Bergman classes (see [3]). For the proof of the second part of Theorem 2 (the reverse statements) we use standard methods applied in proofs of Diagonal map theorems (see [14], [4], [3] and [10]).

Different corollaries of these results for weak invertibility in $AI_\lambda^{p,q}$ and $BI_\lambda^{p,q}$ classes in higher dimension can be obtained immediately. Namely, we have to repeat arguments applied above for other classes of functions in polydisk and to use known onedimensional results from [8].

Below we present another technique to get results on weakly invertible functions in higher dimension from various onedimensional results. It is based on arguments connected with slice functions (see [11] and [15, Chapter 1]). We give only one example in this direction.

First we consider Toeplitz operators defined with the help of Bergman projection $T_\Phi(f) = P[\Phi f]$, $\Phi \in L^1(\mathbf{B})$, $f \in A^p(\mathbf{B})$. We consider the particular case of analytic symbol Φ. The general case can be considered similarly.

We are going to prove that if $(f_\xi)(z) \in A^p_\alpha(U),\ f_\xi(z) = f(\xi z),\ \xi \in \mathbf{S},\ z \in U$ and $(\Phi_\xi(z)) \in X(U) \subset H(U)$ for all $\xi \in \mathbf{S}^n$ and $(\Phi_\xi)(z)(f_\xi)(z) = (T_\Phi f)(z)$ is acting and bounded from $A^p_\alpha(U)$ to $A^p_\alpha(U)$, $\alpha > -1$, $\xi \in \mathbf{S}$, $0 < p < \infty$, then $(T_\Phi f)(z) = \Phi(z) f(z),\ z \in \mathbf{B}^n$ is also bounded operator which is acting from $A^p_\alpha(\mathbf{B})$ to $A^p_\alpha(\mathbf{B}^n)$, $0 < p < \infty$, $\alpha > -1$.

We have by formula from survey of A.B.Aleksandrov (see [1], [2]):

$$\begin{aligned}
&\int_{\mathbf{B}} |T_\Phi f(w)|^p (1-|w|)^\alpha dv(w) \\
&\quad = \frac{1}{2\pi} \int_{\mathbf{S}} d\sigma(\xi) \int_{\mathbf{D}} |T_\Phi f(z\xi)|^p |z|^{2n-2} (1-|z|)^\alpha dm_2(z) \\
&\quad \le \frac{C}{2\pi} \int_{\mathbf{S}} d\sigma(\xi) \int_{\mathbf{D}} |f(z\xi)|^p |\Phi(z\xi)|^p |z|^{2n-2} (1-|z|)^\alpha dm_2(z) \\
&\quad \le \frac{C}{2\pi} \int_{\mathbf{S}} d\sigma(\xi) \int_{\mathbf{D}} |f(z\xi)|^p |z|^{2n-2} (1-|z|)^\alpha dm_2(z) \\
&\quad \le C \int_{\mathbf{B}} |f(w)|^p (1-|w|)^\alpha dv(w).
\end{aligned}$$

These estimates will hold if $\Phi_\xi(z) := \Phi(z\xi) \in X(U)$ for every fixed $\xi \in \mathbf{S}^n$.
We now apply the arguments we presented above for $\Phi = 1$ and $f = P_n \widetilde{f} - 1$.

Theorem 3. *Let $f \in H(\mathbf{B}),\ p \in (0,\infty),\ \alpha > -1,\ f_\xi(z) = f(\xi z),\ \xi \in \mathbf{S}$, $z \in U$. $(P_n^\xi)(z) = P_n(\xi z)$ be a sequence of polynomials such that $\|(f_\xi)(z)(P_n^\xi)(z) - 1\|_{A^p_\alpha(U)} \to 0,\ n \to \infty$ uniformly by $\xi \in \mathbf{S}$. Then f is weakly invertible in $A^p_\alpha(\mathbf{B})$.*

References

[1] A. V. Aleksandrov, Function Theory in the Ball, in Several Complex Variables II ,Springer-Verlag, New York, 1994.

[2] A. E. Djrbashian and A. O. Karapetyan, Integral inequalities between conjugate pluriharmonic functions in multydimensional domains, Izvestia Acad Nauk Armenii, (1988), 216-236

[3] A. E. Djrbashian, F. A. Shamoyan, Topics in the Theory of A^p_α Spaces , Leipzig, Teubner, 1988.

[4] M. Jevtic, M. Pavlovic, R. Shamoyan, A note on diagonal mapping theorem in spaces of analytic functions in the unit polydisk, Publ. Math. Debrecen, **74** no. 1-2 (2009), 1-14.

[5] B. Korenblum, H. Hedenmalm and K. Zhu, Theory of Bergman spaces, Springer-Verlag, New York, 2000.

[6] S. Li and R. Shamoyan, On some properties of a differential operator on the polydisk , Banach Journal of Math. Analysis, no. 1, 2009.

[7] B. M. Makarov, M. G. Goluzina, A. A. Lodkin and A. N. Podkoritov, Selected problems of real Analysis, Moscow , Nauka (In Russian), 432 pages, 1992.

[8] N. K. Nikolski, Selected problems of weighted approximation and spectral analysis , Proceedings Steklov Mat. Inst., volume 120 (In Russian), 270 pages, 1974.

[9] M. Pavlović and M. Dostanić, On the Inclusion $H^2(U^n) \subset H^{2n}(B^n)$ and the Isoperimetric Inequality, Journal of Mathematical Analysis and Applications, **226** (1998), 143-149.

[10] G. Ren and J. Shi, The diagonal mapping in mixed norm spaces, Studia Math., **163** no. 2 (2004), 103-117.

[11] W. Rudin, Function Theory in the Unit Ball of $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

[12] E. Seneta, Regularly varying functions, Springer-Verlag, New York, 1976.

[13] F. A. Shamoian, On weak invertibility in weighted anisotropic spaces of holomorphic functions in the polydisk, Russian Math. Surveys, **55** no. 2 (2000), 152-154.

[14] R. Shamoyan, O. Mihić, On some inequalities in holomorphic function theory in polydisk related to diagonal mapping, Czech. Math. Journal, accepted, 2010.

[15] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Graduate Texts in Mathematics, 226. Springer-Verlag, New York, 2005.