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# $BCK$-Algebras and Hyper $BCK$-Algebras Induced by a Deterministic Finite Automaton

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Abstract. In this note first we define a $BCK$-algebra on the states of a deterministic finite automaton. Then we show that it is a $BCK$-algebra with condition (S) and also it is a positive implicative $BCK$-algebra. Then we find some quotient $BCK$-algebras of it. After that we introduce a hyper $BCK$-algebra on the set of all equivalence classes of an equivalence relation on the states of a deterministic finite automaton and we prove that this hyper $BCK$-algebra is simple, strong normal and implicative. Finally we define a semi continuous deterministic finite automaton. Then we introduce a hyper $BCK$-algebra $S$ on the states of this automaton and we show that $S$ is a weak normal hyper $BCK$-algebra.

**Keywords:** Deterministic finite automaton, $BCK$-algebra , hyper $BCK$-algebra, quotient $BCK$-algebra.

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## 1. Introduction

The hyper algebraic structure theory was introduced by F. Marty [9] in 1934. Imai and Iseki [6] in 1966 introduced the notion of $BCK$-algebra. Meng and

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Jun [10] defined the quotient hyper $BCK$-algebras in 1994. Torkzadeh, Roodbari and Zahedi [12] introduced the hyper stabilizers and normal hyper $BCK$-algebras. Corsini and Leoreanu [4] found some connections between a deterministic finite automaton and the hyper algebraic structure theory. Now in this note first we introduce a $BCK$-algebra on the states of a deterministic finite automaton and we prove some theorems and obtain some related results. Also we define a hyper $BCK$-algebra on the set of all equivalence classes of an equivalence relation on the states of a deterministic finite automaton. Finally we introduce a hyper $BCK$-algebra on the states of a semi continuous deterministic finite automaton.

## 2. Preliminaries

**Definition 2.1**. [10] Let $X$ be a set with a binary operation " $*$ " and a constant "0". Then $(X, *, 0)$ is called a $BCK$-algebra if it satisfies the following condition:
(i) $((x * y) * (x * z)) \ * \ (z * y) = 0$,
(ii) $(x * (x * y)) * y = 0$,
(iii) $x * x = 0$,
(iv) $0 * x = 0$,
(v) $x * y = 0$ and $y * x = 0$ imply $x = y$.
For all $x, y, z \in X$.
For brevity we also call $X$ a $BCK$-algebra. If in $X$ we define a binary relation"$\leq$ " by $x \leq y$ if and only if $x * y = 0$, then $(X, *, 0)$ is a $BCK$-algebra if and only if it satisfies the following axioms for all $x, y, z \in X$; ;
(I) $(x * y) * (x * z) \leq \ z * y$,
(II) $x * (x * y) \leq y$,
(III) $x \leq x$,
(IV) $0 \leq x$,
(V) $x \leq y$ and $y \leq x$ imply $x = y$.
**Definition 2.2**. [10] Given a $BCK$-algebra $(X, *, 0)$ and given elements $a, b$ of $X$, we define

$$A(a, b) = \{x \in X| \ x * a \leq b\}.$$

If for all $x, y$ in $X$, $A(x, y)$ has a greatest element then the $BCK$-algebra is called to be with condition $(S)$.
**Definition 2.3**. [10] Let $(X, *, 0)$ be a $BCK$-algebra and let $I$ be a nonempty subset of $X$. Then $I$ is called to be an ideal of $X$ if, for all $x, y$ in $X$,
(i) $0 \in I$,
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.
**Theorem 2.4.** [10] Let $I$ be an ideal of $BCK$-algebra $X$. if we define the relation $\sim_I$ on $X$ as follows:

$$x \sim_I y \text{ if and only if } x\ o\ y \in I \text{ and } y\ o\ x \in I.$$

Then $\sim_I$ is a congruence relation on $H$.

**Definition 2.5.** [10] Let $(X, *, 0)$ be a $BCK$-algebra, $I$ be an ideal of $X$ and $\sim_I$ be an equivalence relation on $X$. we denote the equivalence class containing $x$ by $C_x$ and we denote $X/I$ by $\{C_x : x \in H\}$. Also we define the operation $* \quad : X/I \times X/I \to X/I$ as follows:

$$C_x * \ C_y \longrightarrow C_{x*y}.$$

**Theorem 2.6.** [10] Let $I$ be an ideal of $BCK$-algebra $X$. Then $I$=$C_0$.

**Theorem 2.7.** [10] Let $(X, *, 0)$ be a $BCK$-algebra and $I$ be an ideal of $X$ . Then $(X/I, *,\ C_0)$ is a $BCK$-algebra.

**Definition 2.8.** [10] A $BCK$-algebra $(X, *, 0)$ is called positive implicative if it satisfies the following axiom:

$$(x * z) * (y * z) = (x * y) * z$$

for all $x, y, z \in X$.

**Definition 2.9.** [10] A nonempty subset $I$ of a $BCK$-algebra $X$ is called a varlet ideal of $X$ if

(VI1) $x \in I$ and $y \leq x$ imply $y \in I$,

(VI2) $x \in I$ and $y \in I$ imply that there exists $z \in I$ such that $x \leq z$ and $y \leq z$.

**Definition 2.10.** [8] Let $H$ be a nonempty set and "o" be a hyper operation on $H$, that is "o" is a function from $H \times H \ \ to \ \ \mathcal{P}^*(H) = \mathcal{P}(H) - \{\emptyset\}$. Then $H$ is called a hyper $BCK$-algebra if it contains a constant "0" and satisfies the following axioms:

(HK1) $(x\ o\ z)\ \ o\ \ (y\ o\ z) \ll \ x\ o\ y$,

(HK2) $(x \text{ o } y) \text{ o } z =(x \text{ o } z) \text{ o } y$,

(HK3) $x\ o\ H \ll \{x\}$,

(HK4) $x \ll y$, $y \ll x \Longrightarrow x = y$.

For all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x\ o\ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a\ \in A$, $\exists b\ \in B$ Such that $a \ll b$.

**Theorem 2.11.** [2] In a hyper $BCK$-algebra $(H, o, 0)$ , the condition (HK3) is equivalent to the condition:

$x\ o\ y \ll \{x\}$ for all $x, y \in H$.

**Definition 2.12.** [7] Let $I$ be a non-empty subset of a hyper $BCK$-algebra $H$ and $0 \in I$. Then,

(1) $I$ is called a weak hyper $BCK$-ideal of $H$ if $x\ o\ y\ \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

(2)$I$ is called a hyper $BCK$-ideal of $H$ if $x\ o\ y\ \ll I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

(3) $I$ is called a strong hyper $BCK$-ideal of $H$ if $(x\ o\ y) \cap\ I \neq \emptyset$ and $y \in I$ imply that $x \in I,$ for all $x, y \in H.$

**Theorem 2.13.** [7] Any strong hyper $BCK$-ideal of a hyper $BCK$-algebra $H$ is a hyper $BCK$-ideal and a weak hyper $BCK$-ideal. Also any hyper $BCK$-ideal of a hyper $BCK$-algebra $H$ is a weak hyper $BCK$-ideal.

**Definition 2.14.** [12] Let $H$ be a hyper $BCK$-algebra and $A$ be a nonempty subset of $H$. Then the sets$_l A = \{x \in H|\ \ a \in a\ o\ x\ \ \ \forall a \in A\}$ and $A_r = \{x \in H|\ \ x \in x\ o\ a\ \ \forall a \in A\}$ are called left hyper $BCK$-stabilizer of $A$ and right hyper $BCK$-stabilizer of $A$, respectively.

**Definition 2.15.** [12] A hyper $BCK$-algebra $H$ is called:

(i) Weak normal, if $a_r$ is a weak hyper $BCK$-ideal of $H$ for any element $a \in H$.

(ii) Normal, if $a_r$ is a hyper $BCK$-ideal of $H$ for any element $a \in H$.

(iii) Strong normal, if $a_r$ is a strong hyper $BCK$-ideal of $H$ for any element $a \in H$.

**Definition 2.16.** [11] A hyper $BCK$-algebra $(H$, o, $0)$ is called simple if for all distinct elements $a, b \in H - \{0\}\,,\ \ \ a \nleq b$ and $b \nleq a$.

**Definition 2.17.** [2] A hyper $BCK$-algebra $(H, o, 0)$ is called:

(i) Weak positive implicative (resp. positive implicative), if it satisfies the axiom

$$(x\ o\ z)\ \ o\ \ (y\ o\ z) \ll ((x\ o\ y)\ \ o\ z)\ \ \text{(resp.}\ \ (x\ o\ z)\, o\ \ (y\ o\ z) = (x\ o\ y)\, o\ z)$$

for all $x, y, z \in H$.

(ii) Implicative. if $x \ll x\ o\ \ (y\ o\ x)$ , for all $x, y, z \in H$.

**Definition 2.18.** [5] A deterministic finite automaton consists of:

(i) A finite set of states, often denoted by $S$.

(ii) A finite set of input symbols, often denoted by $M$.

(iii) A transition function that takes as arguments a state and an input symbol and returns a state. The transition function will commonly be denoted by $t$, and in fact $t : S \times M \to S$ is a function.

(iv) A start state, one of the states in $S$ such as $s_0$.

(v) A set of final or accepting states $F$. The set $F$ is a subset of $S$.

For simplicity of notation we write $(S,\ M, s_0,\ F,\ t)$ for a deterministic finite automaton.

**Remark 2.19.** [5] Let $(S, M, s_0, F, t)$ be a deterministic finite automaton. A word of $M$ is the product of a finite sequence of elements in $M$, $\lambda$ is empty word and $M^*$ is the set of all words on $M$. We define recursively the extended transition function, $t^* : S \times M^* \longrightarrow S$, as follows:

$$\forall\ s\ \in S,\ \forall\ a\ \in M,\ t^*\,(s, a) = t\,(s, a)\,,$$

$$\forall\ s\ \in S,\ t^*\,(s, \lambda) = s,$$

$$\forall\ s\ \in S,\ \forall\ x \in M^*,\ \forall\ a\ \in M,\ t^*\left(s, ax\right) = t^*\left(t\left(s, a\right), x\right).$$

Note that the length $\ell\left(x\right)$ of a word $x \in M^*$ is the number of its letters. so $\ell\left(\lambda\right) = 0$ and $\ell\left(a_1 a_2\right) = 2$, where $a_1,\ a_2 \in M$.

**Definition 2.20.** [4] The state $s$ of $S - \{s_0\}$ will be called connected to the state $s_0$ of $S$ if there exists $x \in M^*$, such that $s = t^*\left(s_0, x\right)$.

## 3. *BCK*-algebras induced by a deterministic finite automaton

In this section we present some relationships between *BCK*-algebras and deterministic finite automata.

**Definition 3.1.** Let $(S, M, s_0, F, t)$ be a deterministic finite automaton. If $s \in S - \{s_0\}$ is connected to $s_0$ , then the order of a state $s$ is the natural number $l + 1$, where $l = \min\ \{\ell(x) \mid t^*\left(s_0, x\right) = s,\quad x \in M^*\}$ , and if $s \in S - \{s_0\}$ is not connected to $s_0$ we suppose that the order of $s$ is 1. Also we suppose that the order of $s_0$ is 0.

We denote the order of a state $s$ by $ord\ s$.

Now, we define the relation $\sim$ on the set of states $S$, as follows:

$$s_1 \sim s_2 \Leftrightarrow ord\ s_1 = ord\ s_2$$

It is obvious that this relation is an equivalence relation on $S$.

Note that we denote the equivalence class of $s$ by $\overline{s}$. Also we denote the set of all these classes by $\overline{S}$.

**Theorem 3.2.** Let $(S, M, s_0, F, t)$ be a deterministic finite automaton. We define the following operation on $S$:

$$\forall\left(s_1, s_2\right) \in S^2,\ \ s_1 o s_2 = \begin{cases} s_0, & \text{if } ord\ s_1 < ord\ s_2, \quad s_1, s_2 \neq s_0, \quad s_1 \neq s_2 \\ s_1, & \text{if } ord\ s_1 \geq ord\ s_2, \quad s_1, s_2 \neq s_0, \quad s_1 \neq s_2 \\ s_0, & \text{if } s_1 = s_2 \\ s_0, & \text{if } s_1 = s_0\ , \quad s_2 \neq s_0 \\ s_1, & \text{if } s_2 = s_0\ , \quad s_1 \neq s_0 \end{cases}$$

Then $(S,\ o,\ s_0)$ is a *BCK*-algebra and $s_0$ is the zero element of $S$.

Proof. By definition of the operation 'o', we know that $t\ o\ t = s_0$ and $s_0\ o\ t = s_0$ for all $t \in S$. So $(S,\ o,\ s_0)$ satisfies (III) and (IV).

Now we consider the following situations to show that $(S,\ o,\ s_0)$ satisfies (I) and (II).

(i) Let $s_1, s_2, s_3 \neq s_0$ and $ord\ s_1 < ord\ s_2\ < ord\ s_3$. Then $\left(s_1\ o\ s_2\right) o \left(s_1\ o\ s_3\right) = s_0\ o\ s_0 =\ s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we obtain that in this case (I) holds.

On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Hence, in this case (II) holds.
(ii) Let $s_1, s_2, s_3 \neq s_0$ and $ord\ s_2 < ord\ s_1 < ord\ s_3$. Then $(s_1\ o\ s_2)\ o\ (s_1\ o\ s_3) = s_1\ o\ s_0 = s_1$ and $s_3\ o\ s_2 = s_3$. Since $s_1\ o\ s_3 = s_0$ we get that $s_1 \leq s_3$. Thus in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Therefore in this case (II) holds.
(iii) Let $s_1, s_2, s_3 \neq s_0$ and $ord\ s_2 < ord\ s_3 < ord\ s_1$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. So in this case (II) holds.
(iv) Let $s_1, s_2, s_3 \neq s_0$ and $ord\ s_1 < ord\ s_3 < ord\ s_2$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Hence, in this case (II) holds.
(v) Let $s_1, s_2, s_3 \neq s_0$ and $ord\ s_3 < ord\ s_1 < ord\ s_2$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Thus in this case (II) holds.
(vi) Let $s_1, s_2, s_3 \neq s_0$ and $ord\ s_3 < ord\ s_2 < ord\ s_1$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. So in this case (II) holds.
(vii) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_2 < ord\ s_3$ and $s_1 \neq s_2$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_0 = s_1$ and $s_3\ o\ s_2 = s_3$. Since $s_1\ o\ s_3 = s_0$ we get that $s_1 \leq s_3$. So in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Therefore in this case (II) holds.
(viii) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_2 > ord\ s_3$ and $s_1 \neq s_2$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Hence, in this case (II) holds.
(ix) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_3 < ord\ s_2$ and $s_1 \neq s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Thus in this case (II) holds.
(x) $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_3 > ord\ s_2$ and $s_1 \neq s_3$.
Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we get that in this case (I) holds.

Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. So in this case (II) holds.
(xi) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_2 = ord\ s_3 > ord\ s_1$ and $s_2 \neq s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Therefore in this case (II) holds.
(xii) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_2 = ord\ s_3 < ord\ s_1$ and $s_2 \neq s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Hence, in this case (II) holds.
(xiii) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_2 = ord\ s_3$ and $s_1 \neq s_2 \neq s_3 \neq s_1$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Thus in this case (II) holds.
(xiv) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_3$, $s_1 \neq s_3$ and $s_1 = s_2$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. So in this case (II) holds.
(xv) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_2$, $s_1 \neq s_2$ and $s_1 = s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_0 = s_1$ and $s_3\ o\ s_2 = s_3$. Since $s_1\ o\ s_3 = s_0$ we get that $s_1 \leq s_3$ . So in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Therefore in this case (II) holds.
(xvi) Let $s_1, s_2, s_3 \neq s_0$, $ord\ s_1 = ord\ s_2$, $s_1 \neq s_2$ and $s_2 = s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Hence, in this case (II) holds.
(xvii) Let $s_1, s_2, s_3 \neq s_0$, $ord\, s_1 < ord\, s_3$ and $s_1 = s_2$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Thus in this case (II) holds.
(xviii) Let $s_1, s_2, s_3 \neq s_0$, $ord\, s_1 > ord\, s_3$ and $s_1 = s_2$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. So in this case (II) holds.
(xix) Let $s_1, s_2, s_3 \neq s_0$, $ord\, s_1 < ord\, s_2$ and $s_1 = s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.

On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Therefore in this case (II) holds.
(xx) Let $s_1, s_2, s_3 \neq s_0$, $ord\, s_1 > ord\, s_2$ and $s_1 = s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_0 = s_1$ and $s_3\ o\ s_2 = s_3 = s_1$. Since $s_1 \leq s_1$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Hence, in this case (II) holds.
(xxi) Let $s_1, s_2, s_3 \neq s_0$, $ord\, s_1 < ord\, s_2$ and $s_2 = s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Thus in this case (II) holds.
(xxii) Let $s_1, s_2, s_3 \neq s_0$, $ord\, s_1 > ord\, s_2$ and $s_2 = s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. So in this case (II) holds.
(xxiii) Let $s_1 = s_2 = s_3$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_0$. Therefore in this case (II) holds.
(xxiv) Let $s_1 = s_0$ and $s_2, s_3 \neq s_0$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$. Let $s_3\ o\ s_2 = t$ and $t \in S$. Since $s_0 \leq t$ we get that in this case (I) holds.
Also $s_1\ o\ (s_1\ o\ s_2) = s_0\ o\ s_0 = s_0$ and $s_0\ o\ s_2 = s_0$. Hence, in this case (II) holds.
(xxv) Let $s_2 = s_0$, $s_1, s_3 \neq s_0$. Since $s_1\ o\ s_3 = s_1$ or $s_1\ o\ s_3 = s_0$, we have two cases:
(6) $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$. We know that $s_3\ o\ s_2 = s_3$. Since $s_0 \leq s_3$ we conclude that in this case (I) holds.
(7) $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_0 = s_1$. We know that $s_3\ o\ s_2 = s_3$ and in this case $s_1\ o\ s_3 = s_0$. So $s_1 \leq s_3$ and (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Thus in this case (II) holds.
(xxvi) Let $s_3 = s_0$ and $s_1, s_2 \neq s_0$. Since $s_1\ o\ s_2 = s_1$ or $s_1\ o\ s_2 = s_0$, we obtain that $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ or $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_1 = s_0$. Also $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we conclude that in this case (I) holds.
The proof of (II) is studied in other cases.
(xxvii) Let $s_1 \neq s_0$ and $s_2 = s_3 = s_0$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_1\ o\ s_1 = s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2) = s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_2 = s_0$. Therefore in this case (II) holds.
(xxviii) Let $s_3 \neq s_0$ and $s_1 = s_2 = s_0$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 = s_0$. and $s_1\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we get that in this case (I) holds.

Also $s_1\ o\ (s_1\ o\ s_2)\ = s_0\ o\ s_0 = s_0$ and $s_0\ o\ s_2 = s_0$. Hence, in this case (II) holds.
(xxix) Let $s_2 \neq s_0$ and $s_1 = s_3 = s_0$. Then $(s_1\ o\ s_2)\, o\, (s_1\ o\ s_3) = s_0\ o\ s_0 =\ s_0$ and $s_3\ o\ s_2 = s_0$. Since $s_0 \leq s_0$ we obtain that in this case (I) holds.
On the other hand, $s_1\ o\ (s_1\ o\ s_2)\ = s_0\ o\ s_0 = s_0$ and $s_0\ o\ s_2 = s_0$. Thus in this case (II) holds.
So we conclude that $(S,\ o,\ s_0)$ satisfies (I) and (II).
To prove (V), Let $s_1 \leq s_2$ and $s_2 \leq s_1$. If $s_1 = s_2$, then we are done. Otherwise, since $s_1 \leq s_2$, there exist two cases:
(i) $ord\ s_1 < ord\ s_2,\quad s_1, s_2 \neq s_0,\quad s_1 \neq s_2$ . Then $s_2 o\ s_1 = s_2$ . Therefore $s_2 \nleq s_1$, which is a contradiction.
(ii) $s_1 = s_0\ ,\ s_2 \neq s_0$. Then $s_2 o\ s_1 = s_2 o\ s_0 = s_2$. Thus $s_2 \nleq s_1$, which is a contradiction.
So we show that $(S,\ o,\ s_0)$ is a $BCK$-algebra.
**Example 3.3.** Let $A = (S, M, s_0, F, t)$ be a deterministic finite automaton such that $S = \{q_0, q_1, q_2, q_3\}\,,\ \ M = \{a, b\}\,,\ \ s_0 = q_0,\ \ F = \{q_1, q_3\}$ and $t$ is defiend by

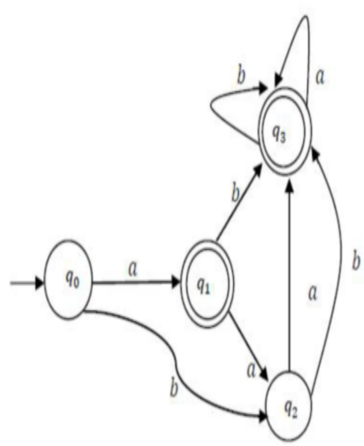


FIGURE 1

$$t\,(q_0, a) = q_1,\ t\,(q_0, b) = q_2,\ t\,(q_1, a) = q_2, t\,(q_1, b) = q_3,$$

$$t\,(q_2, a) = q_3,\ t\,(q_2, b) = q_3,\ t\,(q_3, a) = q_3,\ t\,(q_3, b) = q_3.$$

It is easy to see that $ord\ q_1 = ord\ q_2 = 2, ord\ q_3 = 3$ and $ord\ q_0 = 0$. According to the definition of operation "o" which is defined in Theorem 3.2, we have the following table:

Table 1.

| O | $q_0$ | $q_1$ | $q_2$ | $q_3$ |
|---|---|---|---|---|
| $q_0$ | $q_0$ | $q_0$ | $q_0$ | $q_0$ |
| $q_1$ | $q_1$ | $q_0$ | $q_1$ | $q_0$ |
| $q_2$ | $q_2$ | $q_2$ | $q_0$ | $q_0$ |
| $q_3$ | $q_3$ | $q_3$ | $q_3$ | $q_0$ |

In this section we suppose that $(S,\ o,\ s_0)$ is the $BCK$-algebra, which is defined in Theorem 3.2.

**Notation.** We denote the class of all states which their order is n by $\overline{s_n}$.

**Theorem 3.4.** $(S,\ o, s_0)$ is a $BCK$-algebra with condition (S).

Proof: Let $s_1, s_2 \in S, ord\ s_1 = n$ and $ord\ s_2 = m$. Then we should consider following situations:

(1) Let $ord\ s_1 < ord\ s_2, \quad s_1, s_2 \neq s_0, \quad s_1 \neq s_2$. Then $A(s_1, s_2) = \bigcup_{i=0}^{m-1} \overline{s_i} \cup \{s_2\}$ and the greatest element of $A(s_1, s_2)$ is $s_2$.

(2) Let $ord\ s_1 \geq ord\ s_2, \quad s_1, s_2 \neq s_0, \quad s_1 \neq s_2$. Then $A(s_1, s_2) = \bigcup_{i=0}^{n-1} \overline{s_i} \cup \{s_1\}$ and the greatest element of $A(s_1, s_2)$ is $s_1$.

(3) $s_1 = s_2$. Then $A(s_1, s_2) = \bigcup_{i=0}^{n-1} \overline{s_i} \cup \{s_1\}$ and the greatest element of $A(s_1, s_2)$ is $s_1$.

(4) Let $s_1 = s_0\ , \quad s_2 \neq s_0$. Then $A(s_1, s_2) = \bigcup_{i=0}^{m-1} \overline{s_i} \cup \{s_2\}$ and the greatest element of $A(s_1, s_2)$ is $s_2$.

(5) Let $s_1 \neq s_0\ , \quad s_2 = s_0$. Then $A(s_1, s_2) = \bigcup_{i=0}^{n-1} \overline{s_i} \cup \{s_1\}$ and the greatest element of $A(s_1, s_2)$ is $s_1$.

**Theorem 3.5.** Let $I_n = \{s \in S \mid s \in \bigcup_{i=0}^{n} \overline{s_i}\}$ for any $n \in N$ . Then $I_n$ is an ideal of $(S,\ o, s_0)$.

Proof. Suppose that $s_1\ o\ s_2 \in I_n$ and $s_2 \in I_n$, then we have the following situations:

(1) $\qquad s_1 \neq s_2,\ s_2 \neq s_0$ and $ord s_2 < ord s_1$.

By definition of the operation "o", we know that $s_1 o\ s_2 = s_1$. So $s_1 \in I_n$.

(2) $\qquad s_1 \neq s_2,\ s_2 \neq s_0$ and $ord s_2 = ord s_1$.

Since $s_2 \in I_n$and $\overline{s_2} \subseteq I_n$, we obtain that $s_1 \in I_n$.

(3) $s_1 \neq s_2,\ s_1 \neq s_0$ and $ords_1 < ords_2$.

By definition of $I_n$, it is easy to see that $s_1 \in I_n$.

(4) $s_1 = s_2$.

It is clear that $s_1 \in I_n$.

(5) $s_2 = s_0$.

By definition of the operation "o", we know that $s_1 o\ s_2 = s_1$. So $s_1 \in I_n$.
(6) $s_1 = s_0$.
*Since* $s_0 \in I_n$, we get that $s_1 \in I_n$.
Also by definition of $I_n$, we know that $s_0 \in I_n$. So $I_n$ is an ideal of $S$.
**Theorem 3.6.** Let $I_n$ be a set, which is defined in Theorem 3.5. Then $C_x = \{x\}$ for all $x \notin I_n$.
Proof. Let $x \notin I_n$. By Theorem 2.6, we know that $I_n = C_{s_0}$. So $s_0 \notin C_x$. Now we suppose that $y \in C_x$ and $y \neq x$ . By definition of the equivalence relation $\sim_{I_n}$, we know that $x\ o\ y \in I_n$ and $y\ o\ x \in I_n$. Since $x \notin I_n$ and $x\ o\ y \in I_n$, we obtain that $ord\ x \ngeq ord\ y$. So $ord\ y > ord\ x$ and $y\ o\ x = y \in I_n = C_{s_0}$, which is a contradiction. Hence, $y = x$.
**Theorem 3.7.** Let $I_n$be the ideal of $S$ which is defined in Theorem 3.5. Then $(S/I_n, *,\ C_{s_0})$ is a *BCK*-algebra.
Proof. By Theorem 2.7, it is obvious that $(S/I_n, *,\ C_{s_0})$ is a *BCK*-algebra.
**Theorem 3.8.** $(S,\ o,\ s_0)$ is a positive implicative *BCK*-algebra.
Proof. By considering 29 situations which have been stated in the proof of Theorem 3.2, we get that in all cases $(s_1\ o\ s_3)\,o\ (s_2\ o\ s_3) = (s_1\ o\ s_2)\,o\ s_3$, for all $s_1, s_2, s_3 \in S$. So $(S,\ o,\ s_0)$ is a positive implicative *BCK*-algebra.
**Theorem 3.9.** Let $n = \max\ \{ord\ s \mid s \in S\}$. Then $I = \bigcup_{i=0}^{m-1} \overline{s_i} \cup \{z\}$ for $1 \leq m \leq n$ and $z \in s_m$, is a varlet ideal of $(S,\ o,\ s_0)$.
Proof. To prove (VI1), we suppose that $x \in I$ and $y \leq x$. Then $s_0 = y\ o\ x$ and we have three cases:
(6) Let $ord\ y < ord\ x$, $\ x, y \neq s_0$ and $x \neq y$. Then by definition of $I$, it is obvious that $y \in I$.
(7) Let $x = y$. Then it is clear that $y \in I$.
(3) Let $y = s_0$ , $x \neq s_0$. Then by definition of $I$, it is easy to see that $s_0 = y \in I$. Therefore (VI1) holds.
Now we show that $I$ satisfies (VI2). let $x \in I,\ y \in I$ and $x, y \neq z$. Since $ord\ x < ord\ z$ and $ord\ y < ord\ z$, we get that $x\ o\ z = s_0$ and $y\ o\ z = s_0$. So $x \leq z$ and $y \leq z$. Also if $x \in I,\ y \in I$, $x = z$ and $y \neq z$, then $x\ o\ z = z\ o\ z = s_0$ and $y\ o\ z = s_0$. Thus $x \leq z$ and $y \leq z$. Similarly we can prove that $x \leq z$ and $y \leq z$ for the following cases:

(6) $$x \in I,\ y \in I, x \neq z \text{ and } y = z,$$

(7) $$x \in I,\ y \in I, x = z \text{ and } y = z.$$

So (VI2) holds.

## 4. Hyper $BCK$-algebras induced by a deterministic finite automaton

**Theorem 4.1.** Let $(S, M, s_0, F, t)$ be a deterministic finite automata. We define the following hyper operation on $\overline{S}$ :

$$\forall\ (\overline{s_1}, \overline{s_2}) \in \overline{S}^2,\ \ \overline{s_1} o\, \overline{s_2} = \begin{cases} \overline{s_1}\ , & \text{if} \quad \overline{s_1} \neq\ \overline{s_2},\ \overline{s_2} \neq \overline{s_0} \neq \overline{s_1} \\ \{\overline{s_0},\ \overline{s_1}\}\,, & \text{if} \quad \overline{s_1} =\ \overline{s_2} \\ \overline{s_0}\ , & \text{if} \quad \overline{s_1} = \overline{s_0},\ \overline{s_2} \neq \overline{s_0} \\ \overline{s_1}\ , & \text{if} \quad \overline{s_1} \neq \overline{s_0},\ \overline{s_2} = \overline{s_0}. \end{cases}$$

Then $(\overline{S},\ o,\ \overline{s_0})$ is a hyper $BCK$-algebra and $\overline{s_0}$ is the zero element of $\overline{S}$.

Proof. First we have to consider the following situations to show that $(\overline{S},\ o,\ \overline{s_0})$satisfies (HK1) and (HK2).

(i) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_3} \neq\ \overline{s_2} \neq \overline{s_1} \neq \overline{s_3}$. Then $(\overline{s_1}\ o\, \overline{s_3})\ o\ (\overline{s_2}\ o\, \overline{s_3}) = \overline{s_1}\ o\ \overline{s_2}$. Since $\overline{s}\ o\ \overline{s} = \{\overline{s_0},\ \overline{s}\}$ we obtain that $\overline{s} \ll \overline{s}$ for any $\overline{s} \in \overline{S}$. So $(\overline{s_1}\ o\, \overline{s_3})\ o\ (\overline{s_2}\ o\, \overline{s_3}) \ll \overline{s_1}\ o\, \overline{s_2}$ and in this case (HK1) holds.

Also $(\overline{s_1}\ o\, \overline{s_2})\, o\, \overline{s_3} = \overline{s_1}\ o\, \overline{s_3} = \overline{s_1}$ and $(\overline{s_1}\ o\, \overline{s_3})\, o\, \overline{s_2} = \overline{s_1}\ o\, \overline{s_2} = \overline{s_1}$. Thus in this case (HK2) holds.

(ii) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_1}{=}\overline{s_2} \neq \overline{s_3}$. Then$(\overline{s_1}\ o\, \overline{s_3})\ o\ (\overline{s_2}\ o\, \overline{s_3}) = \overline{s_1}\ o\, \overline{s_2}$. So $(\overline{s_1}\ o\, \overline{s_3})\ o\ (\overline{s_2}\ o\, \overline{s_3}) \ll \overline{s_1}\ o\, \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\, \overline{s_2})\, o\, \overline{s_3} = \{\overline{s_0},\ \overline{s_1}\}\ o\, \overline{s_3} = \{\overline{s_0},\ \overline{s_1}\}$ and $(\overline{s_1}\ o\, \overline{s_3})\, o\, \overline{s_2} = \overline{s_1}\ o\, \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$. Therefore in this case (HK2) holds.

(iii) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_1}{=}\overline{s_3} \neq \overline{s_2}$.

Then $(\overline{s_1}\ o\, \overline{s_3})\ o\ (\overline{s_2}\ o\, \overline{s_3}) = \{\overline{s_0},\ \overline{s_1}\}\ o\, \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$ and $\overline{s_1}\ o\, \overline{s_2} = \overline{s_1}$. Since $\overline{s_0}\ o\, \overline{s_1} = \overline{s_0}$ we obtain that $\overline{s_0} \ll\ \overline{s_1}$ and also we know that $\overline{s_1} \ll \overline{s_1}$. Hence, $(\overline{s_1}\ o\, \overline{s_3})\ o\ (\overline{s_2}\ o\, \overline{s_3}) \ll \overline{s_1}\ o\, \overline{s_2}$ and in this case (HK1) holds.

Also $(\overline{s_1}\ o\, \overline{s_2})\, o\, \overline{s_3} = \overline{s_1}\ o\, \overline{s_3} = \{\overline{s_0},\ \overline{s_1}\}$ and $(\overline{s_1}\ o\, \overline{s_3})\, o\, \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}\ o\, \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$. So in this case (HK2) holds.

(iv) Let $\overline{s_1}, \overline{s_2}, \overline{s_3} \neq \overline{s_0}$ and $\overline{s_2}{=}\overline{s_3} \neq \overline{s_1}$.

Then$(\overline{s_1}\ o\, \overline{s_3})\ o\ (\overline{s_2}\ o\, \overline{s_3}) = \overline{s_1}\ o\ \{\overline{s_0},\ \overline{s_2}\} =\ \overline{s_1}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_1}$. Thus $(\overline{s_1}\ o\, \overline{s_3})\ o\ (\overline{s_2}\ o\, \overline{s_3}) \ll \overline{s_1}\ o\, \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\, o\, \overline{s_3} = \overline{s_1}\ o\ \overline{s_3} = \overline{s_1}$ and $(\overline{s_1}\ o\ \overline{s_3})\, o\, \overline{s_2} = \overline{s_1}\ o\ \overline{s_2} = \overline{s_1}$. Therefore in this case (HK2) holds.

(v) Let $\overline{s_1} = \overline{s_2} = \overline{s_3}$. Then $(\overline{s_1}\, o\, \overline{s_3})\ \ o\ \ (\overline{s_2}\, o\, \overline{s_3}) = \{\overline{s_0},\ \overline{s_1}\}\ \ o\ \ \{\overline{s_0},\ \overline{s_1}\} = \{\overline{s_0},\ \overline{s_1}\}$ and $\overline{s_1}\ o\ \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$. So $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case$(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1).

Also $(\overline{s_1}\ o\ \overline{s_2})\, o\, \overline{s_3} = (\overline{s_1}\ o\ \overline{s_1})\, o\, \overline{s_1} = (\overline{s_1}\ o\ \overline{s_3})\, o\, \overline{s_2}$. Hence, in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK2).

(vi) Let $\overline{s_2}, \overline{s_3} \neq \overline{s_0}$, $\ \overline{s_1} = \overline{s_0}$ and $\overline{s_2} \neq \overline{s_3}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$. Thus $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\, o\, \overline{s_3} = \overline{s_0}\ o\ \overline{s_3} = \overline{s_0}$ and $(\overline{s_1}\ o\ \overline{s_3})\, o\, \overline{s_2} = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$. So in this case (HK2) holds.

(vii) Let $\overline{s_2}, \overline{s_3} \neq \overline{s_0}$, $\ \overline{s_1} = \overline{s_0}$ and $\overline{s_2} = \overline{s_3}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_0}\ o\ \{\overline{s_0},\ \overline{s_2}\} = \overline{s_0}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$. Therefore $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case$(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1).

Also$(\overline{s_1}\ o\ \overline{s_2})\, o\, \overline{s_3} = \overline{s_0}\ o\ \overline{s_3} = \overline{s_0}$ and $(\overline{s_1}\ o\ \overline{s_3})\, o\, \overline{s_2} = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$. So in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK2).

(viii) Let $\overline{s_1}, \overline{s_3} \neq \overline{s_0}$, $\ \overline{s_2} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_3}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$. Hence, $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\, o\, \overline{s_3} = \overline{s_1}\ o\ \overline{s_3} = \overline{s_1}$ and $(\overline{s_1}\ o\ \overline{s_3})\, o\, \overline{s_2} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$. Thus in this case (HK2) holds.

(ix) Let $\overline{s_1}, \overline{s_3} \neq \overline{s_0}$, $\ \overline{s_2} = \overline{s_0}$ and $\overline{s_1} = \overline{s_3}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) = \{\overline{s_0},\ \overline{s_1}\}\ \ o\ \overline{s_0} = \ \{\overline{s_0},\ \overline{s_1}\}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$. Since $\overline{s_0} \ll\ \overline{s_1}$ and $\overline{s_1} \ll\ \overline{s_1}$ we obtain that $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1).

Also $(\overline{s_1}\ o\ \overline{s_2})\, o\, \overline{s_3} = \overline{s_1}\ o\ \overline{s_3} = \{\overline{s_0},\ \overline{s_1}\}$ and $(\overline{s_1}\ o\ \overline{s_3})\, o\, \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}\, o\, \overline{s_0} = \{\overline{s_0},\ \overline{s_1}\}$. Hence, in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK2).

(x) Let $\overline{s_1}, \overline{s_2} \neq \overline{s_0}$, $\ \overline{s_3} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_2}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_1}\ o\ \overline{s_2} = \overline{s_1}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_1}$. Therefore $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\, o\, \overline{s_3} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$ and $(\overline{s_1}\ o\ \overline{s_3})\, o\, \overline{s_2} = \overline{s_1}\ o\ \overline{s_2} = \overline{s_1}$. So in this case (HK2) holds.

(xi) Let $\overline{s_1}, \overline{s_2} \neq \overline{s_0}$, $\ \overline{s_3} = \overline{s_0}$ and $\overline{s_1} = \overline{s_2}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_1}\ o\ \overline{s_2} = \ \{\overline{s_0},\ \overline{s_1}\}$ and $\overline{s_1}\ o\ \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$. Since $\overline{s_0} \ll\ \overline{s_0}$ and $\overline{s_1} \ll\ \overline{s_1}$ we get that $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1).

Also $(\overline{s_1}\ o\ \overline{s_2})\, o\, \overline{s_3} = \{\overline{s_0},\ \overline{s_1}\}\ \ o\ \overline{s_0} = \{\overline{s_0},\ \overline{s_1}\}$ and $(\overline{s_1}\ o\ \overline{s_3})\, o\, \overline{s_2} = \overline{s_1} o\ \overline{s_2} = \{\overline{s_0},\ \overline{s_1}\}$. Thus in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK2).

(xii) Let $\overline{s_1} = \overline{s_2} = \overline{s_0}$ and $\ \overline{s_3} \neq \overline{s_0}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_0}\ o\ \overline{s_0} = \overline{s_0}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_0}$. Therefore $(\overline{s_1}\ o\ \overline{s_3})\ \ o\ \ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case (HK1) holds.

On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_0}\ o\ \overline{s_3} = \overline{s_0}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_0}\ o\ \overline{s_0} = \overline{s_0}$. Hence, in this case (HK2) holds.
(xiii) Let $\overline{s_1} = \overline{s_3} = \overline{s_0}$ and $\overline{s_2} \neq \overline{s_0}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_0}$ . So $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1).
On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_0}\ o\ \overline{s_0} = \overline{s_0}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_0}\ o\ \overline{s_2} = \overline{s_0}$. Thus this case $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK2).
(xiv) Let $\overline{s_2} = \overline{s_3} = \overline{s_0}$ and $\overline{s_1} \neq \overline{s_0}$. Then $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$ and $\overline{s_1}\ o\ \overline{s_2} = \overline{s_1}$ . Therefore $(\overline{s_1}\ o\ \overline{s_3})\ o\ (\overline{s_2}\ o\ \overline{s_3}) \ll \overline{s_1}\ o\ \overline{s_2}$ and in this case (HK1) holds.
On the other hand, $(\overline{s_1}\ o\ \overline{s_2})\ o\ \overline{s_3} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$ and $(\overline{s_1}\ o\ \overline{s_3})\ o\ \overline{s_2} = \overline{s_1}\ o\ \overline{s_0} = \overline{s_1}$. Hence, in this case (HK2) holds.
So we show that $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK1) and (HK2).
Now we should prove that $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK3). By Theorem 2.11, it is enough to show that $\overline{s_1}\ o\ \overline{s_2} \ll \overline{s_1}$ for all $\overline{s_1}$ , $\overline{s_2} \in \overline{S}$. By definition of the hyper operation "o" we know that $\overline{s_1}\ o\ \overline{s_2}$ is equal to $\overline{s_1}$ or $\{\overline{s_0},\ \overline{s_1}\}$ or $\overline{s_0}$ for any $\overline{s_1}$ , $\overline{s_2} \in \overline{S}$. Also we know that $\overline{s_1} \ll \overline{s_1}$ and $\overline{s_0} \ll \overline{s_1}$.
Hence $(\overline{S},\ o,\ \overline{s_0})$ satisfies (HK3).
To prove (HK4), Let $\overline{s_1} \ll \overline{s_2}$ and $\overline{s_2} \ll \overline{s_1}$. If $\overline{s_1} = \overline{s_2}$, then we are done. Otherwise, since $\overline{s_1} \ll \overline{s_2}$ , we obtain that $\overline{s_1} = \overline{s_0}$, $\overline{s_2} \neq \overline{s_0}$ . So $\overline{s_2}\ o\ \overline{s_1} = \overline{s_2}\ o\ \overline{s_0} = \overline{s_2}$. Therefore $\overline{s_2} \not\ll \overline{s_1}$, which is a contradiction.
**Example 4.2.** Consider the deterministic finite automaton $A = (S, M, s_0, F, t)$ in Example 3.3. Then the structure of the hyper $BCK$-algebra $(\overline{S},\ o,\ \overline{s_0})$ induced on $\overline{S}$ according to Theorem 4.1 is as follows:

Table 2.

| O | $\overline{q_0}$ | $\overline{q_1}$ | $\overline{q_3}$ |
|---|---|---|---|
| $\overline{q_0}$ | $\overline{q_0}$ | $\overline{q_0}$ | $\overline{q_0}$ |
| $\overline{q_1}$ | $\overline{q_1}$ | $\{\overline{q_0}, \overline{q_1}\}$ | $\overline{q_1}$ |
| $\overline{q_3}$ | $\overline{q_3}$ | $\overline{q_3}$ | $\{\overline{q_0}, \overline{q_3}\}$ |

**Theorem 4.3.** Let $(\overline{S},\ o,\ \overline{s_0})$ be the hyper $BCK$-algebra, which is defined in Theorem 4.1. Then $(\overline{S},\ o,\ \overline{s_0})$ is a strong normal hyper $BCK$-algebra.
Proof. By definition of the hyper operation "o", we obtain that $\overline{a} \in \overline{a}\ o\ \overline{t}$, for any $\overline{a}$ and $\overline{t}$ in $\overline{S}$. So we have:

$${}_l\overline{a} = \{\overline{t} \in \overline{S} \mid \overline{a} \in \overline{a}\ o\ \overline{t}\} = \overline{S}, \quad \overline{a}_r = \{\overline{t} \in \overline{S} \mid \overline{t} \in \overline{t}\ o\ \overline{a}\} = \overline{S}, \quad \forall\ \overline{a} \in \overline{S}.$$

It is clear that $\overline{S}$ is a strong hyper $BCK$-ideal. So $(\overline{S},\ o,\ \overline{s_0})$ is a strong normal hyper $BCK$-algebra.

**Theorem 4.4.** Let $(\overline{S},\ o,\ \overline{s_0})$ be the hyper $BCK$-algebra, which is defined in Theorem 4.1. Then $(\overline{S},\ o,\ \overline{s_0})$ is a simple hyper $BCK$-algebra.
Proof. Let $\overline{s_1} \neq \overline{s_2}$ and $\overline{s_1}, \overline{s_2} \neq \overline{s_0}$. Then $\overline{s_1} o\ \overline{s_2} = \overline{s_1}$ and $\overline{s_2} o\ \overline{s_1} = \overline{s_2}$. Hence, $\overline{s_1} \nleq\ \overline{s_2}$ and $\overline{s_2} \nleq\ \overline{s_1}$. So $(\overline{S},\ o,\ \overline{s_0})$ is a simple hyper $BCK$-algebra.
**Theorem 4.5.** Let $(\overline{S},\ o,\ \overline{s_0})$ be the hyper $BCK$-algebra, which is defined in Theorem 4.1. Then $(\overline{S},\ o,\ \overline{s_0})$ is an implicative hyper $BCK$-algebra.
Proof. Since $\overline{s_1} \in \overline{s_1}\ o\ \overline{s_2}$ and $\overline{s_1}\ o\ \overline{s_2} \neq \emptyset$ for all $\overline{s_1}\ ,\ \overline{s_2} \in \overline{S}$, we obtain that $\overline{s_1} \in \overline{s_1}\ o(\overline{s_2}\ o\ \overline{s_1}\ )$ . So $\overline{s_1} \ll \overline{s_1}\ o(\overline{s_2}\ o\ \overline{s_1}\ )$ and $(\overline{S},\ o,\ \overline{s_0})$ is an implicative hyper $BCK$-algebra.
**Definition 4.6.** A deterministic finite automaton $(S, M, s_0, F, t)$ is called semi continuous if for all distinct elements $s, s' \in S$, the following implication holds: If $\exists x \in M^*,$ such that $s' = t^*(s,x)\ \Rightarrow \nexists x^{'\in M^*,}$ such that $s = t^*\left(s', x^{'}\right)$.
**Theorem 4.7.** Let $(S, M, s_0, F, t)$ be a semi continuous deterministic finite automata. We define the following hyper operation on $S$:

$$\forall\ (s_1, s_2) \in S^2,\quad s_1 o s_2 = \begin{cases} \{s_1, s_0\}, & \text{if } s_2 \text{ is connected to } s_1\ ,\quad s_1, s_2 \neq s_0 \text{ and } s_1 \neq s_2 \\ s_1, & \text{if } s_2 \text{ is not connected to } s_1\ ,\quad s_1, s_2 \neq s_0 \text{ and } s_1 \neq s_2 \\ s_0, & \text{if } s_1 = s_2 \\ s_0, & \text{if } s_1 = s_0\ ,\quad s_2 \neq s_0 \\ s_1, & \text{if } s_2 = s_0\ ,\quad s_1 \neq s_0. \end{cases}$$

Then $(S,\ o,\ s_0)$ is a hyper $BCK$-algebra and $s_0$ is the zero element of $S$.
Proof. First we consider the following situations to prove (HK1) and (HK2).
(i) Let $s_1, s_2, s_3 \neq s_0,\ \ s_3 \neq s_1 \neq s_2 \neq s_3,\ \ s_2$ is connected to $s_1$ , $s_3$ is connected to $s_1$ and $s_3$ is connected to $s_2$ .
Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ \ o\ \ \{s_2, s_0\} = \ \{s_1, s_0\}$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_1\ o\ s_1 = s_0$ and $s_0\ o\ s_1 = s_0$, we obtain that $s_1 \ll s_1$and $s_0 \ll s_1$. So in this case (HK1) holds.
On the other hand, $(s_1\ o\ s_2)\ o\, s_3 = \{s_1, s_0\}\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\, s_2 = \{s_1, s_0\}\, o\ s_2 = \{s_1, s_0\}$. Thus in this case (HK2) holds.
(ii) Let $s_1, s_2, s_3 \neq s_0,\ \ s_3 \neq s_1 \neq s_2 \neq s_3,\ \ s_2$ is not connected to $s_1$ , $s_3$ is connected to $s_1$ and $s_3$ is connected to $s_2$ .
Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ \ o\ \ \{s_2, s_0\} = \ \{s_1, s_0\}$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$and $s_0 \ll s_1$, we conclude that in this case (HK1) holds.
Also $(s_1\ o\ s_2)\ \ o\ s_3 = \{s_1\}\ \ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ \ o\ s_2 = \{s_1, s_0\}\, o\ s_2 = \{s_1, s_0\}$. Therefore in this case (HK2) holds.
(iii) Let $s_1, s_2, s_3 \neq s_0,\ \ s_3 \neq s_1 \neq s_2 \neq s_3,\ \ s_2$ is connected to $s_1$ , $s_3$ is not connected to $s_1$ and $s_3$ is connected to $s_2$ . Since $s_2$ is connected to $s_1$ and $s_3$ is connected to $s_2$ , we get that $s_3$ is connected to $s_1$ . So this case does not happen.
(iv) Let $s_1, s_2, s_3 \neq s_0,\ \ s_3 \neq s_1 \neq s_2 \neq s_3,\ \ s_2$ is connected to $s_1$ , $s_3$ is connected to $s_1$ and $s_3$ is not connected to $s_2$ .
Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ \ \ o\ s_2 = \ \{s_1, s_0\}$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_1 \ll s_1$and $s_0 \ll s_1$, we obtain that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\, s_3 = \{s_1, s_0\}\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\, s_2 = \{s_1, s_0\}\ o\, s_2 = \{s_1, s_0\}$. Hence, in this case (HK2) holds.
(v) Let $s_1, s_2, s_3 \neq s_0$, $s_3 \neq s_1 \neq s_2 \neq s_3$, $s_2$ is not connected to $s_1$, $s_3$ is not connected to $s_1$ and $s_3$ is connected to $s_2$.
Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ \{s_2, s_0\} = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we conclude that in this case (HK1) holds.
On the other hand, $(s_1\ o\ s_2)\ o\, s_3 = s_1\ o\ s_3 = s_1$ and $(s_1\ o\ s_3)\ o\, s_2 = s_1 o\, s_2 = s_1$. Thus in this case (HK2) holds.
(vi) Let $s_1, s_2, s_3 \neq s_0$, $s_3 \neq s_1 \neq s_2 \neq s_3$, $s_2$ is not connected to $s_1$, $s_3$ is connected to $s_1$ and $s_3$ is not connected to $s_2$.
Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ o\ s_2 = \{s_1, s_0\}$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ and $s_0 \ll s_1$, we get that in this case (HK1) holds.
Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\ s_2 = \{s_1, s_0\}\ o\ s_2 = \{s_1, s_0\}$. So in this case (HK2) holds.
(vii) Let $s_1, s_2, s_3 \neq s_0$, $s_3 \neq s_1 \neq s_2 \neq s_3$, $s_2$ is connected to $s_1$, $s_3$ is not connected to $s_1$ and $s_3$ is not connected to $s_2$.
Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_2 = \{s_1, s_0\}$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_1 \ll s_1$ and $s_0 \ll s_1$, we obtain that in this case (HK1) holds.
On the other hand, $(s_1\ o\ s_2)\ o\, s_3 = \{s_1, s_0\}\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\, s_2 = s_1 o\ s_2 = \{s_1, s_0\}$. Therefore in this case (HK2) holds.
(viii) Let $s_1, s_2, s_3 \neq s_0$, $s_3 \neq s_1 \neq s_2 \neq s_3$, $s_2$ is not connected to $s_1$, $s_3$ is not connected to $s_1$ and $s_3$ is not connected to $s_2$.
Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_2 = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we conclude that in this case (HK1) holds.
Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = s_1$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_2 = \mathrm{s}_1$. Hence, in this case (HK2) holds.
(ix) Let $s_1, s_2, s_3 \neq s_0$, $s_1 = s_2 \neq s_3$ and $s_3$ is connected to $s_1$.
Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ o\ \{s_2, s_0\}$
$= s_0$ and $s_1\ o\ s_2 = s_0$. Since $s_0 \ll s_0$ we get that in this case (HK1) holds.
On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = s_0\ o\, s_3 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = \{s_1, s_0\}\ o\ s_1 = s_0$. Thus in this case (HK2) holds.
(x) Let $s_1, s_2, s_3 \neq s_0$, $s_1 = s_2 \neq s_3$ and $s_3$ is not connected to $s_1$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_2 = s_0$ and $s_1\ o\ s_2 = s_0$. Since $s_0 \ll s_0$ we obtain that in this case (HK1) holds.
Also $(s_1\ o\ s_2)\ o\ s_3 = s_0\ o\ s_3 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_1 = \mathrm{s}_0$. So in this case (HK2) holds.
(xi) Let $s_1, s_2, s_3 \neq s_0$, $s_1 = s_3 \neq s_2$ and $s_3$ is connected to $s_2$. By definition of semi continuous automaton we know that when $s_3$ is connected to $s_2$ then $s_2$ is not connected to $s_3$ or $s_1$.
So $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_0\ o\ \{s_2, s_0\} = s_0$ and $s_1\ o\ s_2 = s_1$. Since $s_0 \ll s_1$ we conclude that in this case (HK1) holds.

On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_1 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_2 = s_0$. Hence, in this case (HK2) holds.

(xii) Let $s_1, s_2, s_3 \neq s_0$, $s_1 = s_3 \neq s_2$, $s_3$ is not connected to $s_2$ and $s_2$ is connected to $s_3$. Then we have $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_0\ o\ s_2 = s_0$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_0 \ll s_1$ we get that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = \{s_1, s_0\}\ o\ s_1 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_2 = s_0$. Therefore in this case (HK2) holds.

(xiii) Let $s_1, s_2, s_3 \neq s_0$, $s_1 = s_3 \neq s_2$, $s_3$ is not connected to $s_2$ and $s_2$ is not connected to $s_3$. Then we have $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_0\ o\ s_2 = s_0$ and $s_1\ o\ s_2 = s_1$. Since $s_0 \ll s_1$ we obtain that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_1 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_2 = s_0$. Thus in this case (HK2) holds.

(xiv) Let $s_1, s_2, s_3 \neq s_0$, $s_1 \neq s_2 = s_3$ and $s_3$ is connected to $s_1$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ o\ s_0 = \{s_1, s_0\}$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_1 \ll s_1$ and $s_0 \ll s_0$ we conclude that in this case (HK1) holds.

On the other hand, $(s_1\ o\ s_2)\ o\, s_3 = \{s_1, s_0\}\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\, s_2 = \{s_1, s_0\}\, o\ s_2 = \{s_1, s_0\}$. So in this case (HK2) holds.

(xv) Let $s_1, s_2, s_3 \neq s_0$, $s_1 \neq s_2 = s_3$ and $s_3$ is not connected to $s_1$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we get that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = s_1$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_2 = s_1$. Hence, in this case (HK2) holds.

(xvi) Let $s_1 = s_2 = s_3$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_1\ o\ s_2 = s_0$. Since $s_0 \ll s_0$ we obtain that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = s_0\ o\ s_3 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_2 = s_0$. Therefore in this case (HK2) holds.

(xvii) Let $s_1 = s_0$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_0\ o\ (s_2\ o\ s_3) = s_0$ and $s_1\ o\ s_2 = s_0$. Since $s_0 \ll s_0$ we conclude that in this case (HK1) holds.

On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = s_0\ o\ s_3 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_2 = s_0$. Thus in this case (HK2) holds.

(xviii) Let $s_2 = s_0$, $s_3 \neq s_1, s_1 \neq s_0 \neq s_3$ and $s_3$ is connected to $s_1$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = \{s_1, s_0\}\ o\ s_0$
$= \{s_1, s_0\}$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ and $s_0 \ll s_1$, we get that in this case (HK1) holds.

Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_3 = \{s_1, s_0\}$. So in this case (HK2) holds.

(xix) Let $s_2 = s_0$, $s_3 \neq s_1, s_1 \neq s_0 \neq s_3$ and $s_3$ is not connected to $s_1$. Then $(s_1\ o\ s_3)\, o\, (s_2\ o\ s_3) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we obtain that in this case (HK1) holds.

On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = s_1$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_2 = s_1$. Hence, in this case (HK2) holds.

(xx) Let $s_2 = s_0$, $s_3 = s_1$ and $s_1 \neq s_0 \neq s_3$. Then $(s_1\ o\ s_3)\,o\,(s_2\ o\ s_3) = s_0\ o\ s_0 = s_0$ and $s_1\ o\ s_2 = s_1$. Since $s_0 \ll s_1$ we conclude that in this case (HK1) holds.
Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_0 o\ s_0 = s_0$. Therefore in this case (HK2) holds.
(xxi) Let $s_3 = s_0$, $s_2 \neq s_1, s_1 \neq s_0 \neq s_2$ and $s_2$ is connected to $s_1$. Then $(s_1\ o\ s_3)\,o\,(s_2\ o\ s_3) = s_1\ o\ s_2 = \{s_1, s_0\}$ and $s_1\ o\ s_2 = \{s_1, s_0\}$. Since $s_1 \ll s_1$ and $s_0 \ll s_0$, we get that in this case (HK1) holds.
On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = \{s_1, s_0\}\ o\ s_3 = \{s_1, s_0\}$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_2 = \{s_1, s_0\}$. So in this case (HK2) holds.
(xxii) Let $s_3 = s_0$, $s_2 \neq s_1, s_1 \neq s_0 \neq s_2$ and $s_2$ is not connected to $s_1$. Then $(s_1\ o\ s_3)\,o\,(s_2\ o\ s_3) = s_1\ o\ s_2 = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we obtain that in this case (HK1) holds.
Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_3 = s_1$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_2 = s_1$. Hence, in this case (HK2) holds.
(xxiii) Let $s_3 = s_0$, $s_2 = s_1$ and $s_1 \neq s_0 \neq s_2$. Then $(s_1\ o\ s_3)\,o\,(s_2\ o\ s_3) = s_1\ o\ s_2 = s_0$ and $s_1\ o\ s_2 = s_0$. Since $s_0 \ll s_0$ we conclude that in this case (HK1) holds.
On the other hand, $(s_1\ o\ s_2)\ o\ s_3 = s_0\ o\ s_0 = s_0$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_2 = s_0$. Therefore in this case (HK2) holds.
(xxiv) Let $s_2 = s_3 = s_0$ and $s_1 \neq s_0$. Then $(s_1\ o\ s_3)\,o\,(s_2\ o\ s_3) = s_1\ o\ s_0 = s_1$ and $s_1\ o\ s_2 = s_1$. Since $s_1 \ll s_1$ we get that in this case (HK1) holds.
Also $(s_1\ o\ s_2)\ o\ s_3 = s_1\ o\ s_0 = s_1$ and $(s_1\ o\ s_3)\ o\ s_2 = s_1 o\ s_0 = s_1$. Thus in this case (HK2) holds.
So we obtain that $(S,\ o,\ s_0)$ satisfies (HK1)and (HK2).
Now we should prove that $(S,\ o,\ s_0)$ satisfies (HK3). By Theorem 2.11, it is enough to show that $s_1 o\ s_2 \ll \{s_1\}$ for all $s_1, s_2 \in S$. By definition of the hyper operation "o" we know that $s_1 o\ s_2$ is equal to $s_1$ or $\{s_1, s_0\}$ or $s_0$ for any $s_1, s_2 \in S$. Also we know that $\mathrm{s}_1 \ll \mathrm{s}_1$ and $\mathrm{s}_0 \ll \mathrm{s}_1$.
Hence $(S,\ o,\ s_0)$ satisfies (HK3).
To prove (HK4), Let $s_1 \ll s_2$ and $s_2 \ll s_1$. If $s_1 = s_2$, then we are done. Otherwise, since $s_1 \ll s_2$, there exist two cases:
(i) $s_2$ is connected to $s_1$, $s_1, s_2 \neq s_0$ and $s_1 \neq s_2$. Then by definition of semi continuous automaton we know that $s_2$ is not connected to $s_1$ and we have $s_2 o\ s_1 = s_2$. Therefore $s_2 \not\ll s_1$, which is a contradiction.
(ii) $s_1 = s_0$, $s_2 \neq s_0$. Then $s_2 o\ s_1 = s_2 o\ s_0 = s_2$. Thus $s_2 \not\ll s_1$, which is a contradiction.
So we show that $(S,\ o,\ s_0)$ is a hyper $BCK$-algebra.
**Theorem 4.8.** Let $(S,\ o, s_0)$ be a hyper $BCK$-algebra which is defined in Theorem 4.7. Then $(S,\ o, s_0)$ is a weak normal hyper $BCK$-algebra.
Proof. By definition of the hyper operation "o", we know that $a_r = \{t \in S \mid t \in t\ o\ a\} = S - \{a\}$ for all $a \neq s_0$ and $a \in S$. Also $a_r = S$ for $a = s_0$.

It is clear that $S$ is a weak hyper $BCK$-ideal. So it is enough to show that $S-\{s\}$ for all $s \neq s_0$ and $s \in S$, is a weak hyper $BCK$-ideal.
It is easy to see that $s_0 \in S-\{s\}$. Let $s_1\ o\ s_2 \subseteq S-\{s\}$ and $s_2 \in S-\{s\}$. Then we have to consider the following situations:

(1) $s_2$ is connected to $s_1$, $s_1, s_2 \neq s_0$ and $s_1 \neq s_2$.
$Since s_1 o\ s_2 = \{s_1, s_0\}$ and $s_1\ o\ s_2 \subseteq S-\{s\}$, we obtain that $s_1 \in S-\{s\}$.
(2) $s_2$ is not connected to $s_1$, $s_1, s_2 \neq s_0$ and $s_1 \neq s_2$.
$Since s_1 o\ s_2 = s_1$ and $s_1\ o\ s_2 \subseteq S-\{s\}$, we get that $s_1 \in S-\{s\}$.
(3) $s_1 = s_2$.
$Since$ $s_2 \in S-\{s\}$, it is clear that $s_1 \in S-\{s\}$.
(4) $s_1 = s_0$, $s_2 \neq s_0$.
$Since s_1 o\ s_2 = s_0$ and $s_0 \in S-\{s\}$, we obtain that $s_1 \in S-\{s\}$.
(5) $s_2 = s_0$, $s_1 \neq s_0$.
$Since s_1 o\ s_2 = s_1$ and $s_1\ o\ s_2 \subseteq S-\{s\}$, we conclude that $s_1 \in S-\{s\}$.
So $(S,\ o, s_0)$ is a weak normal hyper $BCK$-algebra.
**Example 4.9.** Consider the deterministic finite automaton $A=(S,M,s_0,F,t)$ in Example 3.3. Then the structure of the hyper $BCK$-algebra $(S,\ o,\ s_0)$ induced on the states of this automaton according to Theorem 4.7 is as follows:

Table 3.

| O | $q_0$ | $q_1$ | $q_2$ | $q_3$ |
|---|---|---|---|---|
| $q_0$ | $q_0$ | $q_0$ | $q_0$ | $q_0$ |
| $q_1$ | $q_1$ | $q_0$ | $\{q_0, q_1\}$ | $\{q_0, q_1\}$ |
| $q_2$ | $q_2$ | $q_2$ | $q_0$ | $\{q_0, q_2\}$ |
| $q_3$ | $q_3$ | $q_3$ | $q_3$ | $q_0$ |

Thus $(S,\ o,\ s_0)$ is a hyper $BCK$-algebra.
**Remark 4.10.** Let $(S,\ o,\ s_0)$ be the hyper $BCK$-algebra which is defined in Theorem 4.7. In example 4.9, we saw that $q_0 \in q_1 o\ q_3$ and $q_0 \notin q_3 o\ q_1$. So $q_1 \ll q_3$ and $q_3 \not\ll q_1$. Hence, $(S,\ o, s_0)$ may not be simple.

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