Naṣīr al-Dīn al-Ṭūsī's Version of *The Measurement of the Circle of Archimedes from his Revision of the Middle Books*

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Abstract

The mathematical treatises known as *Taḥrīr al-mutawassiṭāt*, "Revision of the *Middle books*", are a collection of Greek and Arabic treatises which were re-edited by Naṣīr al-Dīn al-Ṭūsī. This paper explores the nature of the editorial work Ṭūsī did, showing that it is not a simple rendering of the original text but that it involves major revisions and in fact original additions. For the first time, a collated version of Ṭūsī's revision on the treatise *on the measurement of the circle* (by Archimedes) is provided. A translation of the Arabic text is put side by side with a rendition of the treatise as it circulated before Ṭūsī's time. An in-depth analysis of the text elucidates the nature of Ṭūsī's contributions to the text. This article also provides modest corrections to earlier studies by W. R. Knorr and P. Luckey.

Keywords: Naṣīr al-Dīn al-Ṭūsī, *Middle books*, quadrature of circles, Archimedes, *Measurement of the circle*

Introduction[1]

Next to Naṣir al-Dīn al-Ṭūsī's (d. 1274) impressive contributions to science in the fields of philosophy and astronomy,[2] stand his equally notable contributions to mathematics. The collection called *Revision of the 'Middle books'* (*Taḥrīr al-mutawassiṭāt*) is such a contribution, being a collection of Arabic versions of mostly Greek treatises and to a lesser extent treatises of Arabic origin. Ṭūsī reorders propositions in a more logical way, resolves defective translations, provides comments, and introduces new proofs. The large collection of extant manuscripts shows in itself that it had a significant influence on mathematicians in the Islamic world in the centuries to come (see Sezgin, p. 131; Brockelmann, vol. I, pp. 510-511; Rosenfeld, pp. 212-213. Compare with Kheirandish, pp. 131-144). Influence on Western Europe is known to exist as well, as for example Gerard of Cremona's Latin translation of *The measurement of the Circle* also indicates Ṭūsī's additions (Clagett, pp. 40-55).

This article is meant foremost to show that indeed Ṭūsī did all of the aforementioned things, and next also show the kind of interventions he made on the original text. For this aim, the small treatise on the measurement of the circle by Archimedes will suffice. We will first provide some introductory comments and then provide the text of Ṭūsī's revision of *The measurement of the circle*, together with a translation which is compared with a rendition of Archimedes' treatise as it was circulated before Ṭūsī's time. This comparative presentation will virtually instantly reveal a lot of the nature of Ṭūsī's editorial interventions and his additions. Afterwards we will elucidate the mathematics that are used and discuss some main characteristics of Ṭūsī's revisions.

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1. The research for this paper was partly conducted as a B.Sc. thesis at the Department of Mathematics, Utrecht University. I would like to thank Prof. J. P. Hogendijk and the anonymous reviewer for their valuable comments on previous versions of this paper.
2. For philosophy most notably his super-commentary on Ibn Sīnā's *al-Ishārāt wa'l-Tanbīhāt*, cf. Ibn Sīnā, Naṣīr al-Dīn Ṭūsī, *al-Ishārāt wa'l-Tanbīhāt*, ed. S. Dunya, 4 vols., Cairo: Dār al-Maʿārif, 1957; for astronomy most notably his *al-Tadhkirah fī ʿilm al-hayʾ at*, cf. Ragep, J., *Naṣīr al-Dīn al-Ṭūsī's Memoir on Astronomy (al-Tadhkira fī ʿilm al-hayʾ a)*, 2 vols., New York: Springer-Verlag, 1993.

Preliminary notes

Perhaps it is good to first briefly go over some background information, which will help us contextualize Ṭūsī's revisions of the '*Middle books*' in general, and of *The measurement of the circle* specifically. The term '*Middle books*', *Mutawassiṭāt*, refers to a collection of Arabic translations of Greek mathematical treatises. Little is known about this term, the best study of it dating from 1865 by Moritz Steinschneider (see pp. 54-97). From this study we know that the term was used as early as the 10th century C.E. (Ibid, p. 56) In the centuries after that, the term was used as if the collection were one book in contrast to Euclid's *Elements* and Ptolemy's *Almagest*, which would point to an explanation of the term *mutawassiṭāt* as being the required study material after the *Elements* and before the *Almagest* (Ibid, pp. 56-58). This is confirmed by Nasawī (fl. 11th c.) who states at the beginning of his rendering of (pseudo-) Archimedes' *Lemmas*;

> "… the *Middle books* […] which it is necessary to read between
> the book of Euclid and the *Almagest*." (Ibid, p. 78)

As it became used as a didactical term over the years, so it also included treatises by Islamic authors rather than only Arabic translations of Greek treatises.

The measurement of the circle was written by Archimedes (ca. 250 B.C.E.) and underwent subsequently a remarkable history as a text. Over time, different versions began to circulate, supplanting previous ones. The original we have no more; what we nowadays consider as Archimedes' treatise stems from Late Antiquity (Heiberg, pp. 232-234), at the latest from before the 6th century, since Eutocius' famous commentary relies on a version close to it (Knorr, esp. p. 405 and p. 431). This treatise consists in its extant form of three propositions. The first one explains that a circle is equal to a right triangle of which one of the sides is equal to the radius and the other equal to the perimeter. The second proposition (in Ṭūsī's text the third) relies on the third proposition and gives 22 over 7 as an approximation of π. The third proposition (in Ṭūsī's text the second) gives for the first time in history a very good approximation of π (pi). The treatise as a whole is notoriously short. On several occasions the text merely states the

result and leaves it to the reader to check the reasoning.[1] We do not know much about the reception into the Islamic world; though it seems that the Arabic translation was made from an older version than the Greek text we have now (Knorr, p. 431). We at least know that *The measurement of the circle* was available before 857 C.E., as Kindī supposed to have corresponded on its content with the scientist Ibn Māsawayh (who died in 242/857; see Rashed, p. 15). The details of the original translation remain uncertain, although the text was most probably translated first by Isḥāq ibn Ḥunayn and afterwards retranslated or improved by Thābit ibn Qurrah (Sezgin, pp. 128-129). In his study of Kindī's treatise, Rashed suspects there might have been another translation by Qustā ibn Lūqā (Rashed, pp. 15-16; see Lorch, pp. 94-114, who warns for a too ready acceptance of this). The commentary by Eutocius was also translated into Arabic. As we will see, it was not used by Ṭūsī to write his revision. Perhaps he did not have access to it.[2]

Ṭūsī wrote his revision of *The measurement of the circle* at around 661/1262-63. It appears that he wrote the revision of Menelaus' *Spherics* later (663/1265; see Krause). Nevertheless, *The measurement of the circle* is the last treatise in both the print and facsimile used for this study and thus the actual compilation of the *Taḥrīr al-mutawassiṭāt* was either conducted by Ṭūsī at the very end of his life or, more probable, after he had died. At the beginning of his revision of *On the sphere and cylinder* by Archimedes Ṭūsī states that he read it first from a poor copy of Thābit ibn Qurra and later in an old codex of Isḥāq ibn Ḥunayn. He then states:

> "…In that codex I found what I had been looking for, and I thought of editing the book correctly, setting out its contents precisely, explaining its postulates by means of geometrical principles, adding its necessary premises, and providing a commentary on what is problematic about it, based on what I could take from the commentary of Eutocius or other books of

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1. Heath, for example, writes: "How did Archimedes arrive at these particular approximations? No puzzle has exercised more fascination upon writers interested in the history of mathematics.", p. 51.
2. Knorr assumes that he did consult Eutocius' commentary, but does not present convincing evidence, cf. Knorr, p. 549.

this discipline […] I added at the end of the book the treatise of Archimedes on the measurement of the circle, since this depends on postulates which are brought up in the former treatise." (Aṭ-Ṭūsī, pp. 2-3)

Ṭūsī here not only provides us with a short description of his methodology, but also proposes that *The measurement of the circle* should be seen as an appendix to *On the sphere and cylinder*. Indeed, in the print and facsimile *The measurement of the circle* comes directly after *On the sphere and cylinder*. On the other hand, the lay-out of *The measurement of the circle* gives the impression of being a text on its own. The commentary of *On the sphere and cylinder* is properly concluded, and the new text is properly introduced (and likewise properly concluded).[1]

Formal remarks

For the text and translation of Ṭūsī's revision of *The measurement of the circle* a printed edition was used. In this edition it functions as an appendix of the larger 'Book on the sphere and cylinder with revisions' (*Kitāb al-kura wa-l-usṭuwāna li-Arkhimīdis bi-taḥrīr*), other texts in this book include revisions on Autolycos, Aristarchus, Hypsicles and another text of Archimedes. Unfortunately, this print cannot be trusted on its own, especially when it comes to the numbers written in *Abjad* system. For a better understanding of the text and to correct the Hyderabad printed edition, I collated it with a facsimile published in Tehran (H= Hyderabad print, T= Tehran facsimile).[2]

For the rendition of the Arabic translation of Archimedes' text, as it must have circulated in Late Antiquity, I have made use of the facsimile of MS Fatih 3134, f. 2v-6v, as printed in Knorr's study on the textual tradition of Archimedes' text (Knorr, pp. 455-463). Since its use here is primarily to compare the mathematical content, a translation suffices. It should be pointed out that the Arabic translation is very close to the Greek text we have today, only departing from it in

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1. Though it may be pointed out that whereas *On the sphere and cylinder* does start with a *bismillah* (in God's name)*, The measurement of the circle* does not.
2. al-Tusi, Nasir al-Din, *Taḥrīr-e Mutawassīṭāt*, Introduction by Dr. Jafar Aghayanī-Chavoshī, Insitute for Humanities and Cultural Studies, Tehran, 2005. This is a facsimile of Ms. Tabriz, Melli Library, no. 3484.

a significant way in the first proposition. Knorr supplies an English translation of both the Arabic translation and Ṭūsī's text, but the quality of this is disputable and thus I have here provided my own, though still drawing from Knorr, in particular for the rendition of the Arabic translation. The order of the propositions in the Arabic translation is the same as in the extant Greek text, but I have rearranged it here as to align it with Ṭūsī's text. The following table shows the transliteration of the labels in the geometrical figures.

Usage of labels

Arabic	ا	ب	ج	د	ه	ز	ح	ط	ى
Transliteration	a	b	g	d	e	z	h	T	y
Arabic	ک	ل	م	ن	ص	ع	ق	س	
Transliteration	k	l	m	n	S	o	q	s	

Text and translation of *Taksīr al-dāʾirah*

Rendition of Arabic translation ***	Translation of Ṭūsī's text ***	Arabic text of Naṣīr al-Dīn al-Ṭūsī ***
Book of Archimedes on the measurement of the circle	Essay of Archimedes on the measurement of the circle, consisting of three propositions.	مقالة ارشميدس فى تكسير الدائرة وهى ثلاثة اشكال كل دائرة فهى مساوية لمثلث قائم الزاوية يكون احد ضلعيه المحيطين بالزاوية القائمة مساويا لنصف قطر تلک الدائرة والثانى مساوياً¹ لمحيطها والحاصل انها يساوى² سطح نصف قطرها فى الخط المساوى لنصف محيطها
1) Any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the base is equal to the circumference.	Every circle is equivalent to a right triangle of which one of the two sides which surround the right angle is equal to the half of the diameter of that circle, and the other [side] is equal to its perimeter. The result is that it is equal to the rectangle contained by half of its diameter and the line that is equal to half of its perimeter.	
Let the circle **abgd** have to the triangle **e** the stated relation; I say that is equal. For if it is not, then the circle is bigger or smaller than it.	So let the circle be a circle **abgd** and the said triangle a triangle **e**. If the circle is not equal to it, then it is either bigger or smaller.	فلتكن الدائرة دائرة³ ابجد والمثلث المذكور مثلث ه فان لم تكن الدائرة مساوية له فهى إما اعظم منه وإما اصغر

Let it be bigger first. We make in the circle the square **ag**. Then from the circle **abgd** a part bigger than its half has been separated, which is the square **ag**. We bisect the arc **afb** and arcs similar to it at point **f** and points similar to it, and we connect **af**, **fb**, and similarly for the others. Then from the remainder of the segments of the circle **a bgd** again more than their half is removed, namely **afb** and those similar to it. If we have done like that according to what follows, there will remain sections which are smaller than the measure of the excess of the circle over the triangle **e**.

The rectilinear figure that is contained in the circle is thus bigger than the triangle.

We make **n** be the center and extend the perpendicular **ns**;

The line **ns** is then less than one of the two sides of the triangle containing the right angle.

The perimeter of the rectilineal figure is less than the other side of the two, because it is

Let it be bigger first. We draw in the circle a square **ag**. It separates from it a bigger part than the half of it [i.e. the circle]. We bisect **ab** at **f** and similarly the four arcs and we connect the chords.

So the resulting triangles separate more than half of the sections as has been proved before.

This is repeated until there remains sections of the circle that are smaller than the measure of the excess of the circle over the triangle **e**.

So the equilateral figure that is in the circle is [then] bigger than the triangle.

Let the center be **n** and we draw from there to one of the sides a perpendicular line, let it be **ns**.

Then it is less than the line **nx** which is equal to one of the sides of the triangle **e** [i.e. the radius].

The perimeter of the equilateral figure is less than the perimeter of the circle [which] is

وليكن اولا اعظم ونرسم فى الدائرة مربع اج٤ وهو يفصل منها اعظم من نصفها وننصف اب على فـ وهكذا القسى الاربع و نصل الاوتار فنفصل المثلثات الحادثة اعظم من نصف القطع لما مر بيانه وهكذا مرة بعد اخرى الى ان تبقى من الدائرة قطع هى اصغر من مقدار زيادة الدائرة على مثلث ه

فيكون الشكل المتساوى الاضلاع الذى فى الدائرة اعظم من المثلث

وليكن المركز نـ ونخرج منه على احد الاضلاع عمودا٥ وليكن نس

وهو اصغر من نص المساوى لاحد ضلعى مثلث ه

ومحيط الشكل المتساوى الاضلاع اصغر من محيط الدائرة المساوى٦ للضلع الآخر من مثلث ه فسطح نس

less than the perimeter of the circle. The product of one of the sides of the triangle containing the right angle with the other, which is double the area of the triangle, is bigger than the product of **ns** with the perimeter of the polygon, which is double the area of the polygon. The same holds for half of that. Thus, the triangle is bigger than the polygon, while earlier it was smaller.

This is no doubt a contradiction.

Let the circle be, if possible, less than the triangle **e**, and let the square be circumscribed, which is **oq**. Then more than half of the square **oq** has been separated, because of the circle.

We divide the arc **ba** in two halves at **f**, similarly we divide the other arcs in two halves, two halves. Let points of the sections be passed by lines perpendicular to the circle, so that line **zT** has been divided into two halves at point **f**. Then line **nq** is perpendicular to **zT**, the other lines being similarly.

Because **qz** and **qT** are

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equal to the other side of the triangle **e**. So the rectangle contained by **ns** times the perimeter of the figure, I mean, twice the magnitude of the figure is less than twice the triangle. So the figure is less than the triangle and it is greater than it.

This is a contradiction.

Then let the circle be smaller than the triangle and we draw around it a square **oq**. [The circle] separates from the square more than half of it.

We bisect the arc **ba** at **f** and draw **zfT**, tangent to the circle at **f**. The radius **nf** is perpendicular to it [i.e. the tangent line]. We do this likewise with the other arcs.

Because **qb** and **qa** are

فى محيط الشكل اعنى ضعف مقدار الشكل اصغر من ضعف المثلث فالشكل اصغر من مثلث وكان اعظم منه

هذا خلف

ثم لتكن الدائرة اصغر من المثلث ونرسم عليها مربع عق فهى تفصل من المربع اعظم من نصفه

وننصف⁷ قوس با على ف ونخرج زفط مماساً⁸ للدائرة على ف ويكون نصف قطر نف عمودا عليه وهكذا نعمل فى سائر القسى

ولأن قب قا متساويان وكذلك

bigger than **Tz**, their halves are bigger than its half, so that line **qT** is bigger than **Tf** which is like **Tb**.

Thus, the triangle **qfT** is bigger than half of the triangle **qfb**, so all the more it is bigger than half of the figure **qfyb** which is contained by the line **bq**, **qf**, and the arc **byf**. Similarly, the triangle **qfz** is bigger than **fSaz**.

All of **Tqr** is bigger than half of the figure **aSfybq**, and similarly it is for the other triangles, similarly even more so for the half of the other segments.

If we have done like that according to what follows, there will remain segments outside of the circle that, when combined, constitute less than the excess of the triangle **e** over the circle **abgd**.

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equal to each other and likewise **Tb**, **Tf**, **zf** and **za**, [all] four are equal to each other. But **Tq** and **qz** are equal. The two are together longer than **Tz**. So **qT** is longer than **bT**,

so triangle **qfT** is bigger than triangle **Tfb** which is bigger than segment **Tfyb**, which is outside the circle, and it is the same with the others.

The four triangles at the angular points of the square separate a bigger [part] than half of the remainder of the square after subtraction of the circle. We divide the arcs exactly so, repeatedly, and we draw tangent lines to the circle until the remaining segments outside the circle together are smaller than the excess of the triangle **e** over the circle. So the polygon that lies on the circle is smaller than the triangle **e**.

طب طف زف زا الاربعة متساوية يكون طق قز متساويين وهما معاً⁹ اطول من طز فـ قط١٠ اطول من بط

فمثلث قفط اعظم من مثلث طفب الذى هو اعظم من قطعة طفيب الخارجة من الدائرة وكذلك فى الباقى١١

والمثلثات١٢ الاربعة التى على زوايا المربع تفصل من باقى المربع بعد نقصان الدائرة منه اعظم من النصف ونصف١٣ القسى هكذا مرة بعد اخرى وتخرج الخطوط المماسة للدائرة الى ان تبقى قطع خارجة من الدائرة مجموعها اصغر من زيادة مثلث ه على الدائرة فيكون الشكل الكثير الاضلاع الذى على الدائرة اصغر من مثلث ه

So let there remain the segment **fza** and similar segments, then the rectilinear figure which contains the circle is smaller than the triangle **e**. [But] this is not possible because it was bigger than it. That is because **na** is equal to the perpendicular of the triangle, while the perimeter of the polygon is bigger than the other side of the triangle which contains the right angle,

because it is bigger than the perimeter of the circle, and the product of **fn** into the perimeter of the polygon is bigger than the product of of one of the two sides of the triangle containing the right angle into the other. Thus, the circle is not smaller than the triangle **e**, while it has previously been proved that it is not bigger than it.

Therefore the circle **ab gd** is equal to the triangle **e**.

Also, since the area of the triangle **e** is equal to the product of its perpendicular into half its base, and its perpendicular is equal to half the diameter of the circle **abgd** and its base

But the rectangle contained by **nf**, the radius, and the perimeter of the figure that circumscribes the circle, I mean, twice the magnitude of the figure is greater than twice the triangle, because of the fact that the perimeter of the figure is greater than the perimeter of the circle. The figure is bigger than the triangle but it was smaller than it. That is a contradiction.

So the circle is equal to the triangle **e**, so the rectangle contained by the radius and half of the perimeter [of the circle] is equal to the surface of the circle and that is what we wanted.

ولكن١٤ سطح نف نصف القطر فى محيط الشكل الذى على الدائرة اعنى ضعف مقدار الشكل اعظم من ضعف المثلث لكون محيط الشكل اعظم من محيط الدائرة فالشكل اعظم من المثلث وكان اصغر منه هذا خلف

فاذاً١٥ الدائرة مساوية بمثلث ه فسطح نصف القطر فى نصف المحيط مساو لسطح الدائرة وذلك ما اردناه

is equal to the circumference of the circle **abgd**, then the product of half the diameter into half the circumference of the circle **abg** is equal to the area of the triangle **e**, and that is what we wanted to prove.

For this reason it is that the product of half the diameter into half the section of the circumference is the area of the figure which is contained by that section and the two lines which proceed from the two ends of the section to the center.

In addition it has been demonstrated with this [proof] that the rectangle contained by the radius and half of a part of the perimeter is equal to the circular sector that is contained by that part and the two lines by that part and the two lines issuing from the center to the two endpoints of that part.

وقد بان من ذلك ايضا ان سطح نصف القطر فى نصف قطعة من المحيط يكون مساوياً١٦ للقطاع الذى محيط به تلك القطعة مع الخطين الخارجين من المركز الى طرفى تلك القطعة

2) Every circumference of a circle is greater than three times the diameter by less than a seventh of the diameter but greater than $^{10}/_{71}$ of the diameter.

The perimeter of the circle is longer than three times its diameter by [a magnitude] less than one-seventh of the diameter and more than $^{10}/_{71}$ of the diameter.

محيط الدائرة اطول من ثلاثة اضعاف قطرها باقل من سبع القطر واكثر من عشرة اجزاء من احد وسبعين جزءا من القطر

Let **ag** be the diameter of a circle, and its center **e**, the line **dz** be tangent to the circle, the angle **z eg** be a third of a right angle.

So let **ag** be the diameter of the circle and **e** its center and **dz** a tangent to the circle. The angle **zeg** is a third of a right angle. I mean half of an angle [i.e. of one of the angles] of an equilateral triangle.

فليكن اج قطر الدائرة و ه مركزها و دز مماساً١٧ للدائرة وزاوية زهج ثلث زاوية قائمة اعنى نصف زاوية من زوايا المثلث المتساوى الاضلاع

So the ratio **ez** to **zg** is

So the ratio **ez** to **zg** is

فنسبة ـهز الى زـج هى نسبة الاثنين

Left column:

as the ratio 306:153, while the ratio of **eg** to **zg** is bigger than the ratio 265:153.

Let us divide angle **zeg** in halves by the line **eh**, so that the ratio of **ze** to **eg** is as the ratio of **zh** to **hg**.

The ratio of **ze** and **eg** together to **zg** is as the ratio of **eg** to **gh**. Thus the ratio **ge** to **gh** becomes greater than the ratio 571 to 153.

Then the ratio of **eh** squared to **hg** squared is as the ratio of [34]9450 to [2]3409. The ratio of length being greater than the ratio 591[$^1/_8$] to 153.

Again, let the angle **heg** be bisected by **eT**; then by the same reasoning

Middle column:

the ratio 2:1 and let this be as the ratio 306:153. If we subtract the square of the number that is opposite to **zg** from the square of the number that is opposite to **ez**, and we take the root of what is left, **eg** is in this magnitude more than 265 and the difference is some fraction [i.e. is less than one].

We bisect the angle **zeg** at **h** by a line **eh**, then the ratio **ze** to **eg** is as the ratio **zh** to **hg**.

And *componendo* and *separando* the ratio **ze** and **eg** combined to **zg** is as the ratio **eg** to **gh**. If we add up the numbers that belong to **ze** [and] **eg**, [then] that is more than 571. So we make it opposite to **eg** and what is opposite at **gh** becomes in this measure 153.

If we add their squares and we take the root of that, [then] **eh** in this measure is greater than 591$^1/_8$.

And again we bisect the angle **heg** at **T** by a line **eT**. As has been done before; the ratio

Right column (Arabic):

الى الواحد ولتكن كنسبة ۳۰۶ الى ۱۵۳ واذا القينا[18] مربع العدد الذى بازاء زج من مربع العدد الذى بازاء هز واخذنا جذر الباقى كان هج بذلك المقدار اكثر من ۲۶۵ بكسر ما

وننصف زاوية زهج على ح بخط هح فنسبة زه الى هج كنسبة زح الى حج

واذا ركبنا[19] وابدلنا كانت نسبة زه هج معا الى زج كنسبة هج الى حج فاذا جمعنا العددين اللذين بازاء زه هج كان اكثر من ۵۷۱ فنجعله بازاء هج ويصير الذى بازاء حج بهذا المقدار ۱۵۳

واذا جمعنا مربعيهما واخذنا جذرهما كان هح بهذا المقدار اكثر من ۵۹۱ وثمن

وايضا ننصف زاوية حهج على ط بخط هط ويكون كما تقدم نسبة حه هج الى حج كنسبة هج

Left column:

the ratio **eg** to **gT** is greater than the ratio $1162^{1}/_{8}$ to 153, so that the ratio **Te** to **Tg** is greater than the ratio $1172^{1}/_{8}$ to 153.

Again, let the angle **Teg** be divided in halves by the line **ek**. Then the ratio **eg** to **gk** is greater than the ratio 2334¼ to 153, so that the ratio **gk** to **hk** is greater than the ratio 2339¼ to 153.

Again, let the angle **keg** be divided in halves by line **le**. Then the ratio **eg** to **gl** is greater than the ratio 4673½ to 153.

Because the angle **zeg** was a third of a right angle, it is necessary that angle **leg** is $^{1}/_{48}$ of a right angle.
We make at point **e** an

Middle column:

he
[plus] **eg** to **hg** is as the ratio **eg** to **gT**. If we add up the numbers [that belong to] **he** and **eg** and we place the two opposite to **eg**, [then] **eg** is more than $1162^{1}/_{8}$ and **Tg** is in this measure 153. As has been explained before **eT** is in this measure more than $1172^{1}/_{8}$.

We bisect also the angle **Teg** at **k** by the line **e k**. The ratio **Te** [plus] **eg** to **Tg** is as the ratio **eg** to **gk**. This change makes [the number that] is opposite to **eg** more than $2334^{3}/_{8}$ and [the number] that is opposite to **gk** [becomes] 153. **ek** is in this measure more than $2339^{3}/_{8}$.

We bisect also the angle **keg** at **l** by the line **el**. By the above-mentioned analogy, [the number] that is opposite to **eg** becomes more than 4673¾, **gl** is in this measure 153.

Because of the angle **z eg** being a third of a right [angle], the angle **leg** is $^{1}/_{48}$ of a right [angle]. We construct on the point **e** of the

Right column (Arabic):

الی جط واذا جمعنا عددی حه هج وجعلناهما بازاء هج کان هج اکثر من ۱۱۶۲ وثمن وطج بذلک المقدار ۱۵۳ ویکون بمثل ما مر هط بذلک المقدار اکثر من ۱۱۷۲ وثمن

وننصف ایضا زاویة طهج علی ک بخط هک وتکون نسبة طه هج الی طج ۲۰ کنسبة هج الی جک فتصیر هذه النوبة بازاء هج اکثر من ۲۳۳۴ وربع وثمن وبازاء جک ۱۵۳ ویکون هک بهذا المقدار اکثر من ۲۳۳۹ وربع وثمن

وننصف ایضا زاویة کهج علی ل بخط هل ویصیر علی القیاس المذکور بازاء هج اکثر من ۴۶۷۳ ونصف وربع ویکون جل بهذا المقدار ۱۵۳

فلکون زاویة زهج ثلث قائمة تکون زاویة لهج جزءا من ثمانیة واربعین جزءا من قائمة ونعمل علی نقطة ه من خط هج زاویة جهم مثل زاویة

angle equal to angle **leg**, being the angle **gem**, so that angle **lem** is $^1/_{24}$ of a right angle.

The straight line **lm** is therefore the side of a polygon containing the circle, having 96 equal angles.

Since we have proved that the ratio **eg** to **gl** is greater than the ratio 4673½ to 153, and double of **eg** is the line **ag** and double **gl** is the line **lm**, it is necessary that the ratio of **ag** to the perimeter of the polygon with 96 angles is greater than the ratio of 4673½ to 14688; which is greater than its triple by 667½, whose ratio to 4673½ is less than $^1/_7$.

Thus it is necessary that the polygon containing the circle be greater than triple its diameter by less than $^1/_7$ of the diameter.

$3^1/_7$ times the diameter is therefore more than

line **ge** the angle **gem** equal to the angle **gel**, then the angle **lem** is $^1/_{24}$ of a right [angle], and the side **lm** is a side of a regular figure consisting of 96 sides which circumscribe the circle.

Now if we multiply the number that is opposite to **lm** 96 times, the product of this number reaches 14688. The diameter is in this measure 4673½, so that [number] which is opposite to the perimeter of the figure is more than 3 times that [number] which is opposite to the diameter with [an excess of] 667½, to which the ratio to the number of the diameter is less than $^1/_7$. So the perimeter of the [above-] mentioned figure is longer than 3 times the diameter of the circle by less than $^1/_7$ times the diameter.

And the difference between the circumference of the circle and $3^1/_7$ times the diameter is greater than that difference necessarily.

چهل فزاویة لهم جزء من اربعة وعشرين جزءا من قائمة

ويكون ضلع لم ضلع الشكل المتساوى الاضلاع والزوايا ذى الستة والتسعين ضلعا المحيط بالدائرة

فاذا ضربنا العدد الذى بازاء لـ م فى ستة وتسعين بلغ ضعف هذا العدد ۱۴۶۸۸ ²¹ ويكون القطر بذلك المقدار ضعف ۴۶۷۳ ونصف فالذى ²² بازاء محيط الشكل اعظم من ثلاثة امثال الذى بازاء القطر بست مائة وسبعة وستين ونصف التى نسبتها الى عدد القطر اقل من السبع فاذا محيط الشكل المذكور اطول من ثلاثة امثال قطر دائرة بانقص من سبع القطر

و يكون نقصان محيط الدائرة من ثلاثة امثال القطر وسبعه اكثر من ذلك النقصان لامحالة

the deficit of the circle. Let there be a circle with diameter **ag** and the angle **bag** a third of a right angle. Then the ratio of **ab** to **bg** is less than the ratio 1351:780, while the ratio of **ag** to **gb** is equal to the ratio 1560:780, because **ag** is double **gb**.

We repeat the circle. Its diameter is **ag** and we draw on it the angle **gab**, a third of a right [angle]. Let the ratio **ag** to **gb**, which is 2:1, be as the ratio 1560:780. Then **ab** is in this measure less than 1351.

ونعيد الدائرة على قطرها اج ونرسم عليه زاوية جاب ثلث قائمة ولتكن نسبة اج الى جب التى هى نسبة الاثنين الى الواحد كنسبة ١٥٦٠ الى ٧٨٠ فيكون اب بذلك المقدار اقل من ١٣٥١

Let angle **bag** be divided in halves by line **ah**. Since angle **bah** is equal to angle **hgb** and angle **bag** has been divided into halves by line **ah**, it is necessary that angle **hgb** is equal to angle **hag**; angle **ah g** is common,

We bisect the angle **bag** by the line **ah** and we connect **gh**. So in the triangles **ahg**, **ghz** and **abz**, the angles **hag**, **hgz** and **baz** are equal and the two angles [in] **h** [and] **b** are right.

وننصف زاوية²³ باج بخط اح ونصل جح ولان فى مثلثات احج حجز زاب²⁴ زوايا حاج حجز باز متساوية وزاويتا²⁵ حب قائمة

so the angles of triangle **ahg** are equal to the angles of triangle **ghz**. The ratio of **ah** to **hg** is as the ratio of **gh** to **hz** and as the ratio **ag** to **gz** and as the ratio of **ga** and **ab** to **bg** and as the ratio of **ah** to **hg**.

[Then] the triangles are similar and because of that the ratio **ah** to **hg** is as the ratio **hg** to **hz** and as the ratio **ag** to **gz** and as the ratio **ab** to **bz**. Indeed, [it] is as the ratio of **ga** and **ab** combined to **gb**. The ratio of **ga** and **ab** combined to **gb** is as the ratio of **ah** to **hg.**

فتكون²⁶ المثلثات متشابهة وتكون لذلك نسبة اح الى حج كنسبة حج²⁷ الى حز وكنسبة اج الى جز وكنسبة اب الى بز بل كنسبة جا اب جميعاً²⁸ الى جب ونسبة جا اب جميعاً²⁹ الى جب كنسبة اح الى حج

From this it is clear that the ratio of **ah** to **hg** is less than the ratio 2911

The number of **ag** and **ab** combined is less than 2911 and the

وعددا جا³⁰ اب جميعاً³¹ اقل من ٢٩١١ وعدد جب ٧٨٠ فاذا جعلناهما

to 780, and that the ratio of **ag** to **gh** is less than the ratio 3013¾ to 780.

Let the angle **gah** be divided in halves by line **aT**, then it is clear from what we said that the ratio of **aT** to **Tg** is less than the ratio of 5924¾ to 780,
which is as the ratio of 1823 to 240,

because the ratio of each of the first two numbers to its corresponding one of the other numbers is as the ratio of 3¼:1
Therefore the ratio **ag** to **gT** becomes less than the ratio $1838^9/_{11}$ to 240.

Further, let the angle **Tag** be divided in halves by line **ak**. Then the ratio **ak** to **kG** is less than the ratio of $3661^9/_{11}$ to 240, which is as the ratio of 1007 to 66, since the ratio of each of the first two numbers to its corresponding one of the other numbers is as the ratio of 40 to 11.

number of **gb** is 780. So if we add up the two [numbers] which are opposite to **ah** and **eg**, then **ag** is in this measure less than 3013¾.

We bisect the angle **hag** by the line **aT** and we connect **Tg**. According to the above-mentioned analogy is [the number] which is opposite to **aT** less than 5924 and [the number] opposite to **Tg** is 780. That is in the ratio 1823:240, because the ratio of each of the first numbers [i.e. 5924 and 780] to the corresponding numbers [i.e. 1823 and 240] is the ratio of 3¼:1. **ag** is in this measure less than $1838^9/_{11}$, corresponding to the one [i.e. after division by 3¼].

We bisect the angle **Tag** by the line **ak**, then [the number] which is opposite to **ak** is less than $3661^9/_{11}$ and [the number] opposite to **kg** is 240, that is in the ratio of 1007:66. That is because the ratio of each of the two [i.e. $3661^9/_{11}$ and 240] to

بازاء اح حج کان اج بذلک المقدار اقل من ۳۰۱۳ ۳۲ ونصف وربع

وننصف زاویة حاج بخط اط ونصل طج فیکون علی قیاس ما مر بازاء اط اقل من ۵۹۲۴ وبازاء طج ۷۸۰ ویکون ذلک علی نسبة ۱۸۲۳ الی ۲۴۰

لأن نسبة کل واحد من العددین الاولین الی نظیره من هذین العددین نسبة ثلاثة وربع الی واحد ویکون اج بهذا المقدار اقل من ۱۸۳۸ وتسعة اجزاء من احد عشر جزءا من الواحد

وننصف زاویة طاج بخط اک فیکون بازاء اک اصغر من ۳۶۶۱ وتسعة اجزاء الی احد عشر وبازاء کج ۲۴۰ ویکون علی نسبة ۱۰۰۷ الی ۶۶ لأن نسبة کل واحد منهما الی نظیره من هذین نسبة اربعین الی احد عشر

Therefore the ratio **ag** to **kg** is as the ratio $1009^1/_6$ to 66.

Further, let the angle **kag** be divided in halves by line **la**. Then the ratio of **al** to **lg** is less than the ratio $2016^1/_6$ to 66.

Therefore the ratio of **ag** to **gl** is less than the ratio 2017¼ to 66. Inverted, the ratio of the perimeter of the polygon, each side being equal to line **gl**, to the diameter becomes greater than the ratio of 6336 to 2017¼.

But 6336 is greater than 3 times 2017¼ by more than $^{10}/_{71}$.

Therefore the perimeter of the 96-sided polygon contained in the circle is greater than $3^{10}/_{71}$ times the diameter. The circumference of the circle therefore becomes more than $3^{10}/_{71}$ times the diameter; its addition over this quantity is greater than the addition of the sides of the polygon.

The perimeter of the circle is therefore more than three times the diameter, exceeding by a quantity less than $^1/_7$ part but greater than

the corresponding of the two [i.e. 1007 and 66] is the ratio 40:11.

We bisect the angle **lag** by the line **al**, then [the number] which is opposite to **al** is greater than $2016^1/_6$ and [the number] opposite to **lg** is 66 and **ag** is in this measure 2017¼.

So the ratio **ag** to **gl** is less than the ratio 2017¼:66. If we multiply 66 by 96, then all the sides of the polygon of 96 sides inscribed in the circle is 6336.

That is more than 3 times 2017¼ with a remainder of $^{10}/_{71}$ of one [i.e. $^{10}/_{71}$ of 2017¼].

So the perimeter of the [above-] mentioned regular figure that is inscribed the circle is greater than 3 times its diameter with a remainder of $^{10}/_{71}$ [times its diameter]. The perimeter of the circle is greater than that so the perimeter of the circle is greater than 3 times its diameter by less than $^1/_7$ [times its diameter] and more than $^{10}/_{71}$ [times its diameter] and that is what we wanted.

وننصف زاوية كاج بخط ال فيكون بازاء ال اقل من ۲۰۱۶ وسدس وبازاء لج ۶۶ ويكون اج بذلك المقدار ۲۰۱۷ وربع فنسبة اج الى جل اصغر من نسبة ۲۰۱۷ وربع الى ۶۶ واذا ضربنا ستة وستين فى ستة وتسعين صار جميع اضلاع الشكل ذى الستة والتسعين ضلعا الذى على الدائرة ۶۳۳۶ وهو اكثر

من ثلاثة اضعاف الفين وسبعة عشر وربع باكثر من عشرة اجزاء من احد وسبعين جزءاً٣٣ من واحد

فمحيط الشكل المتساوى الاضلاع والزوايا المذكور الذى٣٤ على الدائرة تزيد على ثلاثة اضعاف قطرها باكثر من عشرة اجزاء من احد وسبعين جزءا من واحد ومحيط الدائرة اعظم منه فاذن٣٥ محيط الدائرة يزيد على ثلاثة اضعاف قطرها باقل من سبعه واكثر من عشرة اجزاء الى احد وسبعين جزءا وذلك ما اردناه

$^{10}/_{71}$ parts, and that is what we wanted to prove.

I say that there is another method of the astronomers and that is that they obtain a chord of a small arc which is an integer part of the perimeter with the fundamental principles that have been explained in the *Almagest* and other books of them containing demonstrations.

They make it one side of a figure inscribed in the circle. Its ratio to the perpendicular [line] from the center of the circle to it [i.e. the created side] is as the ratio of the side of the similar figure that is circumscribed around the circle to half of the diameter.

So they also deduce this side. Then they obtain, by the computation of these two [quantities] the two quantities, such that the perimeter is greater than one of them and less than the other. So the perimeter is obtained with a close approximation.

The example of that is: let the circle be **ab**, its center is **g** and **ab**

اقول وللمنجمين طريق آخر وهو انهم يحصلون وتر قوس صغيرة يكون جزءا من محيط الدائرة بالاصول التى تبينت فى كتاب *المجسطى* وغيره من كتبهم البرهانية

ويجعلونه ضلعا من اضلاع الشكل الذى فى الدائرة وتكون نسبته الى العمود الواقع من مركز الدائرة عليه كنسبة ضلع الشكل الذى على الدائرة الشبيهة به الى نصف القطر

فيحصلون ذلك الضلع ايضا يحصلون بحسبهما المقدارين الذين يزيد المحيط على احدهما وينقص من احدهما فيتحصل المحيط باقرب تقريب

مثاله لتكن الدائرة اٮ ومركزها ج و اٮ منه جزء من سبع مائة وعشرين

is $^1/_{720}$ part of the perimeter. We connect the chord **ab**; then its measure, based on a calculation of Abū al-Wafā al-Būzjānī according to the [above-]mentioned principles with a very close approximation, is 0; 31, 24, 55, 54, 55. That is a chord of half a degree, if the diameter is made 120 parts.

And if we make it [i.e. the chord] as a side of a regular figure that has 720 sides inside the circle, then the perimeter of that figure is according to his calculation 376; 59, 10, 59. If we bisect the chord of half a degree [then] the measure of **ad** is 0; 15, 42, 27, 57, 27 and the square of it is 0; 4, 6, 44, 2, 4, 57, 25, 18, 30, 9.

The square of the radius, that is the line **ag**, is 3600 parts. We subtract the square of **ad** of it [so that] the square of **dg** remains, which is 3599; 55, 53, 15, 57, 55, 2, 34, 41, 29, 51.

The square root of it is the line **dg**, which is 59; 59, 57, 56, 37, 56, 51.

جزءا من³⁶ المحيط ونصل وتر اب فيكون مقداره بحساب ابى الوفا البوزجانى على الاصول المذكورة باقرب تقريب ٥لاكدنهندنه خامسة وهو وتر نصف درجة اذا جعل القطر مائة وعشرين جزءا

واذا جعلناه ضلع شكل ذى سبع مائة وعشرين ضلعا فى الدائرة يكون محيط ذلك الشكل بحسبه ٣٧٦نطى‌نط ثالثة واذا نصفنا وتر نصف درجة كان مقدار اد هو³⁷ ٥ يه‌مب‌كزنزكز خامسة و³⁸مربعه ٥دومدب‌دنزكه‌يح³⁹ ‌لط عاشرة

ومربع نصف القطر الذى هو خط اج ٣٦٠٠ جزءا نقصنا من مربعة اد منه بقى مربعة دج٣٥٩٩نه‌نج⁴⁰يه⁴¹نزنه‌بلد⁴²ماكط⁴³ نا⁴⁴ جذره هو خط دجنط‌نطنزنو⁴⁵لز⁴⁶نونا سادسة

We multiply **ad** with **gh**, the radius, and we divide it by **dg**. The measure of **he** results which

is 0; 15, 42, 28, 29, 45. We double it and it becomes

0; 31, 24, 56, 59, 31. which is the measure of **ez**, which is a side of a figure that has 720 sides circumscribed around a circle that is similar to the former circle.

The perimeter of the figure is after calculation

376; 16, 59, 23, 54, 12. So if we make the diameter 120, the perimeter [i.e. of the circle] is 376 parts and a fraction that is greater than

0; 59, 10, 59, 0 and less than 0; 59, 23, 54, 12, and if we change the two to the measure that Archimedes mentioned, the perimeter is greater than 3 times the diameter plus something that is greater than $^{10}/_{70;\ 38,\ 41,\ 21}$, and less than $^{10}/_{70;\ 37,\ 47,\ 37}$, which is approximately

$^{10}/_{70;\ 38,\ 14,\ 29}$.

3) The ratio of the area of every circle to the

If the perimeter of the circle is $3^1/_7$ times the

ضربنا اد فى جح نصف القطر وقسمناه على دج خرج مقدار⁴⁷ جه⁴⁸ ه⁴⁹ يهمبكج⁵كطمه خامسة

ضعفناه بلغ ه لاكدنو⁵¹نطلا خامسة وهو مقدار هز وهو ضلع شكل ذى سبع مائة وعشرين ضلعا على الدائرة شبيهة بالاول

ومحيط الشكل بحسبه يكون ٣٧٦يو⁵²جـ نطكجنديب خامسة ايضا فاذا جعلنا القطر مائه وعشرين كان المحيط ٣٧٦ جزءا وكسرا اكثر من نطينطه رابعة واقل من نطكجنديب رابعة واذا حولناهما الى المقدار الذى ذكره ارشميدس كان المحيط يزيد على ثلاثة امثال القطر بما هو اكثر من عشرة اجزاء من سبعين جزءا ولحماكا ثالثة واقل من عشرة اجزاء من سبعين جزءا ولز⁵³مزلز⁵⁴ ثالثة ويكون بالتقريب عشرة اجزاء من سبعين جزءا ولحيد⁵⁵كط ثالثة

اذا كان محيط الدائرة ثلاثة امثال

square of its diameter is as the ratio 11 to 14.

Let the line **ab** be a diameter of the circle, and let there be made a square **gh** around it, and let **dg** behalf of **de**, and let **ez** be $^1/_7$ of **gd**. Then, since the ratio of **age** to **agd** is as the ratio of 21 to 7, while the ratio of **agd** to **aez** is as the ratio 7 to 1, it follows that the ratio of triangle **agz** to triangle **agd** becomes as the ratio of 22 to 7.

But the square **gh** is 4 times **adg**, while the triangle **agz** is equal to the circle **ab**; because the height **ag** is equal to this circle's radius. The base **gz** is equal to its circumference, because the circumference of

the circle is bigger than 3 times its diameter by about $^1/_7$ of it.

What we claimed is thus established; that the ratio of the circle **ab** to

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diameter, and this is an approximate ratio used by the surveyors then the ratio of the surface area of the circle to the square of its diameter is [the] ratio 11 to 14 according to this computation.

So let the diameter of the circle be **ab** and draw around it the square **gh**. Let **gd** behalf of **de** and **ez** one-seventh of **gd**. Because the ratio of the triangle **age** to the triangle **agd** is the ratio of 21 to 7, and the ratio of the triangle **agd** to the triangle **ahz** is the ratio of 7 to 1, the ratio of the triangle **agz** to the triangle **agd** is the ratio of 22 to 7.

The square of **gh** is 4 times the triangle **agd**, and the triangle **agz** is equal to the surface area of the circle because **ag** is equal to half of the diameter and **gz** is approximately equal to the [i.e. the circle's] perimeter [of the circle].

So the ratio of the square of the diameter to the surface area of

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القطر وسبعه وهی نسبة تقریبة اصطلح[56] المساحون کانت نسبة سطح الدائرة الی مربع قطرها نسبة احد عشر الی اربعة عشر بحسب ذلک

وليكن قطر الدائرة اب ونرسم عليه مربع جح وليكن جد نصف ده وهز سبع جد فلأن نسبة مثلث اجه الی مثلث اجد نسبة احد وعشرين الی سبعة ونسبة مثلث اجد الی مثلث اهز نسبة سبعة الی واحد تكون نسبة مثلث اجز الی مثلث اجد نسبة اثنين وعشرين الی سبعة

ومربع جح اربعة امثال مثلث اجد ومثلث اجز مساو لسطح الدائرة لان اج مساو لنصف القطر وجز مساو بالتقریب للمحیط

فنسبة مربع القطر الی سطح الدائرة نسبة ثمانية وعشرين الی اثنين

| the square **gh** is as the ratio of 11 to 14, and that is what we wanted to prove. | the circle is [as] the ratio of 28 to 22, that is, as the ratio of 14 to 11 and that is what we wanted.

This is the completion of the account on the measurement of the circle. Let us finish the treatise by praising God, the most exalted; the good result is because of Him. | وعشرين بل نسبة اربعة عشر الى احد عشر وذلک ما اردناه

وهذا اتمام القول فى تکسیر الدائرة ولنقطع الکلام حامدین للّه تعالى على حسن توفیقه |

<div align="center">***</div>

[1] H: مساویا	[20] H: adds إلى خط	[39] T: لح
[2] H: تساوى	[21] H: ۱۴۴۸۸	[40] H: کج, T: ambiguous
[3] H: دایرة	[22] H: فالسذى	[41] H: نه
[4] H: ابج	[23] H: زوایة	[42] T: possibly لو
[5] T: "ولیکن المرکز ... الاضلاع عمودا" missing	[24] H: ابز	[43] H: omitted
[6] H: المتساوى	[25] متساویة وزوایا	[44] H: omitted
[7] H: ینصف	[26] H: تکون	[45] H: omitted
[8] H: مماسا	[27] H: حج	[46] H: omitted
[9] H: معا	[28] H: جمیعا	[47] H: مقداره
[10] H: فقط	[29] H: جمیعا	[48] T: هح
[11] H: البواقى	[30] H: اج	[49] H: omitted
[12] H: فالمثلثات	[31] H: جمیعا	[50] T: omitted
[13] H: تنصف	[32] T: ۳۱۰۳	[51] H: نز
[14] T: لیکن	[33] H: جزءا	[52] H: بو
[15] H: اذا	[34] H: المذکورة التى	[53] T: نز
[16] H: مساویا	[35] H: فاذا	[54] H: کز
[17] H: مماسا	[36] H and T: هى, scribal error	[55] T: possibly ند
[18] H: الفنیا	[37] H: omitted	[56] H: adds اصطلح علیه
[19] H: واذارکبنا	[38] H: omitted	

Figures from Ṭūsī's text
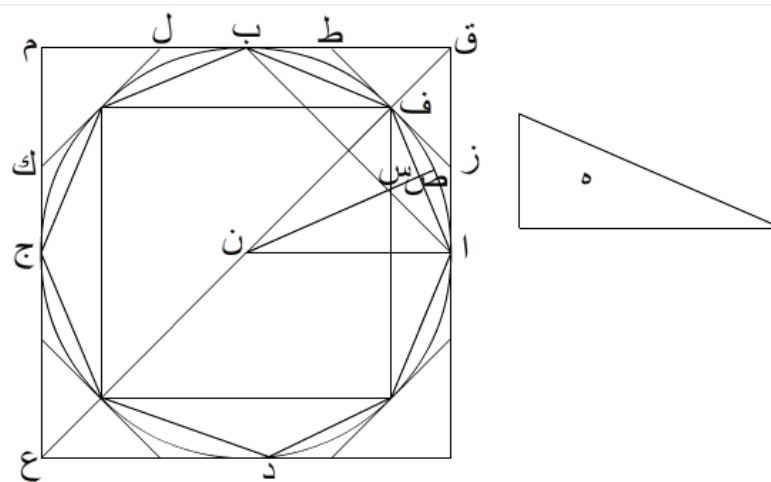
Figure 1-1
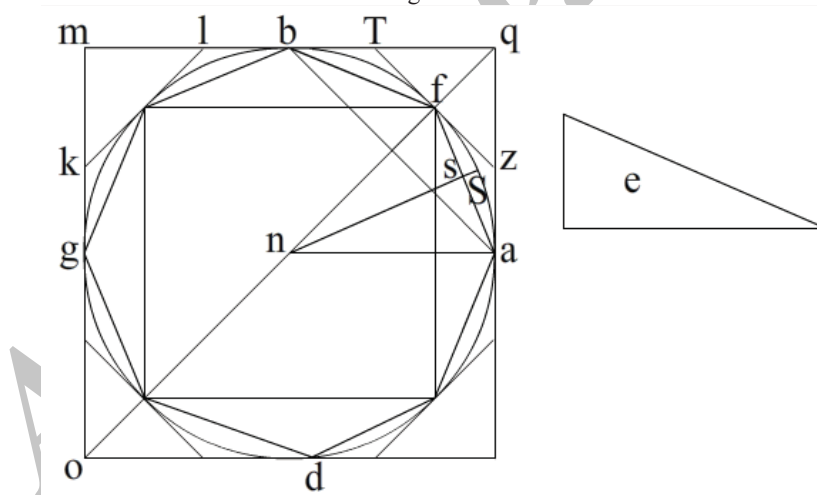
Figure 1-2

Figure 1: Ṭūsī's first proposition
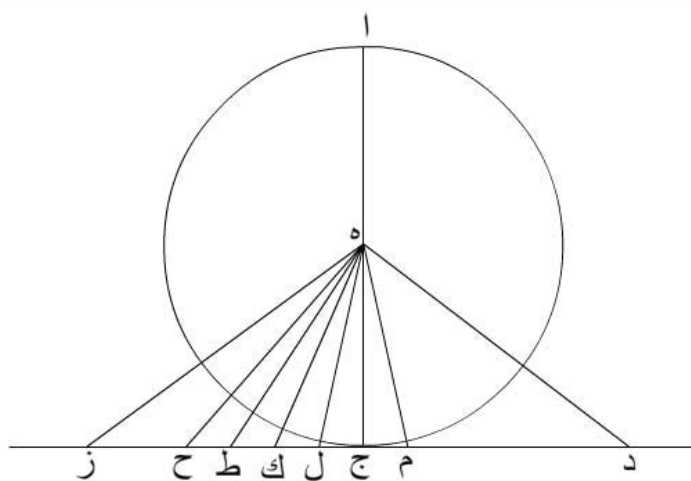
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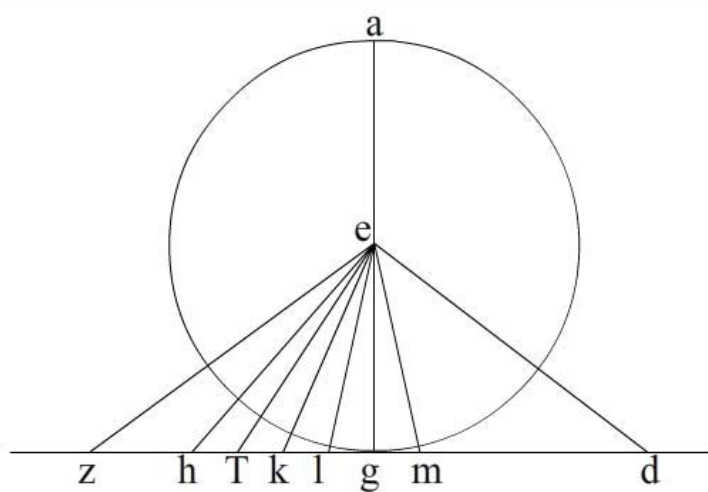
Figure 2-2

Figure 2: Ṭūsī's second proposition, circumscription
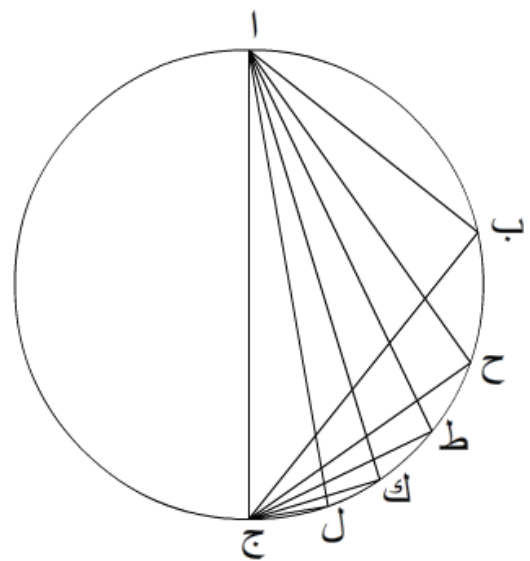
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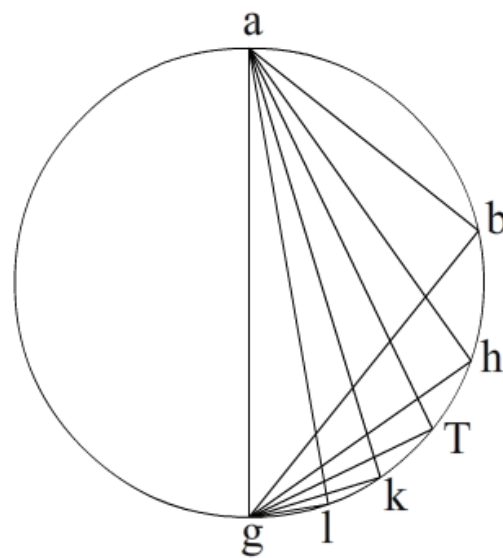
Figure 3-2

Figure 3: Ṭūsī's second proposition, inscription
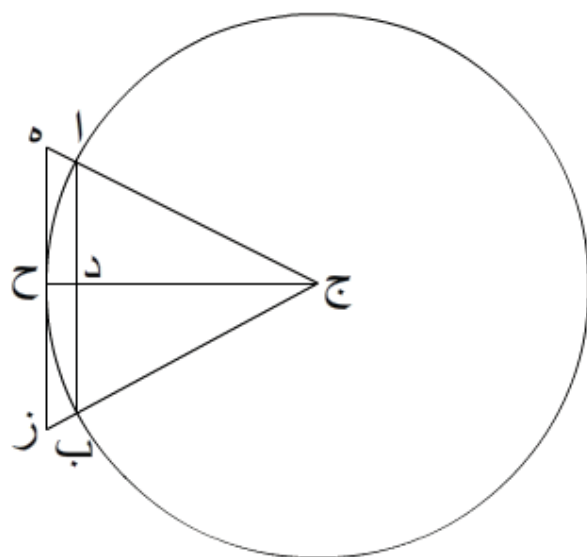
Figure 4-1
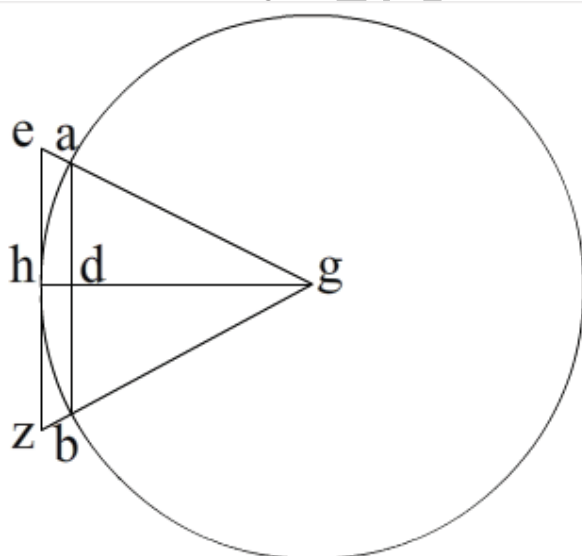
Figure 4-2

Figure 4: Ṭūsī's second proposition, another proof from the astronomers
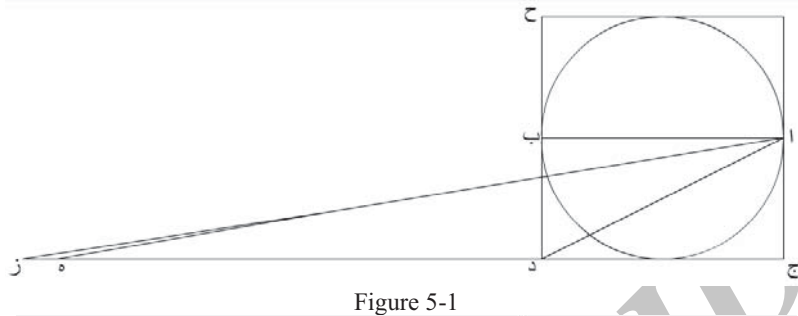
Figure 5-1
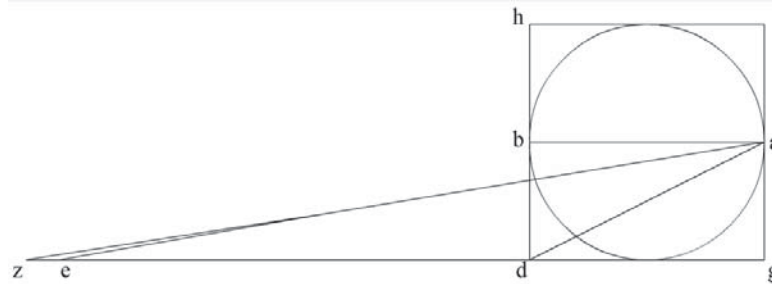
Figure 5-2

Figure 5: Ṭūsī's third proposition

The argument of the first proposition
In the first proposition, it is proved that the surface area of a circle is equal to the surface area of a right triangle of which one of the two right-angled sides is equal to the radius and the other the perimeter. Using modern formulas it is easy to see that this is correct: the surface of a triangle is $\frac{1}{2} \times h \times w$ with $h=$ height and $w=$ width. In a right triangle the two right-angled sides are the height and the width and in this special case they are r (the radius) and $2\pi r$ (the perimeter) so the surface area becomes $\frac{1}{2} \times r \times 2\pi r = \pi r^2$ and as we know that is exactly the surface of a circle.

The proof consists of investigating what happens if we assume the circle to be either greater or less than the triangle. In both cases a regular polygon can be constructed such that its surface area is apparently both bigger and smaller, resulting in a contradiction, and so, via reductio ad absurdum,we come to the conclusion that we must

drop the assumption and instead state that the surface area of the circle must be equal to the surface area of the triangle. A brief elaboration on the hypothesis if the circle were bigger should suffice to understand the process; a similar process could be constructed for the hypothesis that the circle is smaller than the triangle. First assume the circle to be greater than the triangle. An inscribed regular polygon is constructed by repeatedly bisecting the chords and connecting the points. See figure 1 and figure 6; if you draw a square inside the circle and outside the circle, the points at which the inside-square touches the circle can be connected with the points at which the outside-square touches the circle, and a regular octagon is constructed. This process can be repeated of course to get a regular 16-sided polygon, 32-sided polygon etc. If the process is repeated, the figure can come indefinitely close to the circle, making sure that it leaves not even an enough room for the circle to be equal to the (supposed) excess of the circle over the triangle (i.e. the regular polygon can be made such that it is bigger than the triangle). That this is the process that should be followed is not explicitly stated by Archimedes who merely states the construction of the inscribed square and the bisection of the arcs. That this process, if repeated, can separate any small amount you like between the circle and the regular polygon is explicitly proved in Euclid's *Elements* XII, proposition 2. For this case, however, it in principle goes back to the fifth postulate from Archimedes' *On the sphere and cylinder*, Book I, which states that if there are two quantities of the same kind, then there is a multiple of the difference between these quantities which is greater than the greatest of the two quantities[1] (Archimedes, p. 36 and p. 40). This means that a difference cannot be infinitesimal. Where Eutocius' commentary remains silent on this postulate, Ṭūsī does comment on it. He says that this postulate is in turn based on a definition from *Elements* V which states that "magnitudes which have a ratio to one another are those for who it is possible to exceed one another by multiplication" (which is definition 4; see Aṭ-Ṭūsī, pp. 23-24). He then combines this with *Elements* X, proposition 1, concluding that "the smallest of two comparable magnitudes can, by multiplication, become bigger than the biggest of

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1. Also known as 'Archimides' Postulate'.

the two" (Ibid, p. 24). *Elements* X.1, in turn, is a building block for proposition 2 in *Elements* XII, though this last step is not mentioned by Ṭūsī. Nevertheless, *Elements* XII.2 tells us that any small amount may be separated from a circle by a regular polygon, and since the difference of the circle and the triangle may not be infinitesimal, we know it is possible, by repeatedly bisecting the chords and connection on the points, to make a regular polygon that leaves for the circle less than the difference of the circle and the triangle: the polygon must thus be bigger than the triangle. Ṭūsī gives this proof by saying that 'the resulting triangles separate more than half of the sections' adding that this has been proved before (it is not clear what he is referring to). Ṭūsī also states that this should be repeated until the result is that segments between the circle and the regular polygon are smaller than the excess of the circle over the triangle. The extant Greek text is much more succinct than this, leaving out completely a remark about the repetition of this process. In the early Arabic translation this is touched upon by the comment that "If we have done like that according to what follows..." But it is only with the additions of Ṭūsī that the proof becomes entirely clear and unambiguous, when he changes that into "This is repeated until...".
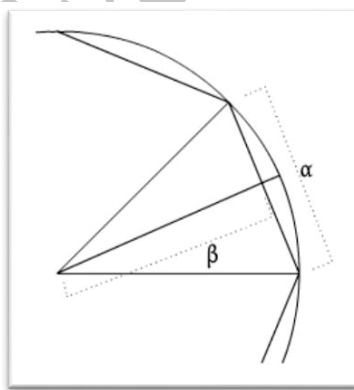

Figure 6

We now know that the polygon is greater than the triangle. What remains to be proven is that it is simultaneously less than the triangle, arriving at a contradiction. In modern notation this is done by

comparing the formulas of the surface areas of the regular polygon and the triangle. The surface area of the triangle is of course $\frac{1}{2} \times r \times p$ with r for radius and p for perimeter (of the circle). The surface area of the regular polygon is obtained by adding all the surface areas of the triangles that are made (i.e. multiplying the surface area of one triangle by n, for an n-sided polygon). So this becomes $n \times \frac{1}{2} \times \alpha \times \beta$ with α the width of a side and β the height of a triangle (see Figure 6). From the figure one immediately sees that $\beta < r$ and $n \times \alpha < p$. So $\frac{1}{2} \times \beta \times n \times \alpha < \frac{1}{2} \times r \times p$, so the surface area of the polygon is less than the surface area of the triangle. Ṭūsī, in comparison to the Arabic translation, leaves out certain steps in his proof that he probably deemed obvious. First he makes the step from unequal sides to unequal surface areas. He states that the rectangle contained by **ns** [the altitude] times the perimeter of the figure is twice the surface area of the regular polygon (which is easily seen to be correct using modern notation) and then states that this is smaller than twice the surface area of the triangle (i.e. the rectangle contained by the radius and the perimeter of the circle). From this he then concludes that the surface area of the regular polygon is smaller than the triangle. The polygon cannot be greater and less at the same time, so we have to drop the assumption that the circle is bigger than the triangle. The proof works likewise if the circle were to be imagined to be less than the triangle. Now the polygon is circumscribed instead of inscribed.

Already in the first paragraph Ṭūsī makes an elaboration, where he states that the result is that a circle is equal to a rectangle (*saṭḥ*) with sides equal to half of the circle's diameter and half of its perimeter. This seems a comment to make the proposition more precise, and it is this precision that we see come up when comparing the Arabic translation with Ṭūsī's revision. For example, in the second part of the first proposition Archimedes wants to say that after creating a regular polygon around the circle, there will still remain some space left in between the circle and the polygon. The Arabic translation and Ṭūsī's text both say "remaining segments outside the circle", but the Arabic translation uses the somewhat vague *tafḍalu ʿalā* while Ṭūsī uses the unambiguous *khārijah min*. We may also notice that at other occasions Ṭūsī's text is shorter, though usually this does not decrease

the comprehensibility. For example, the somewhat tedious repetition of 'and those similar to ...' (*wa-naẓāʾ iruhā min…*) of the old translation is dropped by Ṭūsī, who offers various alternatives, depending on the context. A rigorous study of the contributions that Ṭusī made in this respect, especially for the technical vocabulary, will only be possible after we have critical editions of both the early translation(s) of texts like the *Measurement of the circle* and Ṭūsī's 'revisions' of them.

The argument of the second proposition
Ṭūsī's second proposition is Archimedes' third proposition. Ṭūsī reorders it probably since in the original the second proposition relies on the result of the third proposition. He follows Archimedes' reasoning closely and gives no additional information on why the (at first sight) strange numerical ratios are used. In the ratio of **ez**, **zg** and **eg** he does however explain the Pythagorean theorem and even states that **eg** is 265 and a fraction.
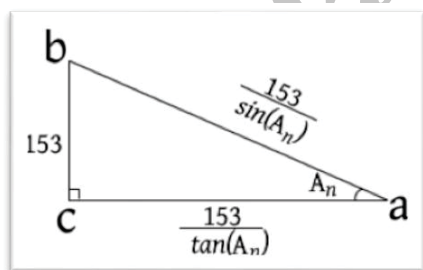
Figure 7

This proposition gives a lower and upper bound for π with a rather sophisticated proof. Before Archimedes, approximations of π were all in a single rational fraction. Here, Archimedes does not give a single rational fraction for π, but rather computes a lower and upper bound. As before in the first proposition, understanding one of the two approaches, means also an understanding of the other one as the procedure is the same. It will thus suffice to focus on the upper bound. A regular polygon is constructed circumscribing the circle. The difference with the first proposition is that already at the beginning

valuable information on the size of the constructed side of the regular polygon is taken into account. Since the angle is known (30°) the sides can be calculated. As was said before, we do not know exactly how Archimedes came up with such convenient approximations of √3, and likewise we do not know exactly how he made all the following computations. In modern day mathematics it is rather easy to follow. One first starts by constructing a triangle with $A_0=30°$. Then another triangle is formed with $A_{n+1} = {}^1\!/_2 \times A_n$. This can be repeated as many times as desired. In the table below all the numbers are given, with the numbers between brackets not given by Archimedes (nor Ṭūsī). The numbers are with regard to Figure 7 in the order **ab**, **ac**, **bc**. It is noteworthy to see that the numbers of **ab** and **ac** come closer and closer to each other. In fact, they can come indefinitely close to each other as in reality only points **a** and **c** are static (they correspond with points **e** and **g** respectively). Point **b** moves closer and closer to **c** (see Figure 2).

	start	first bisection	second	third	fourth
ez	306				
eg	265				
zg	153				
eh		$591^1\!/_8$			
eg		$(571^1\!/_7)$			
hg		153			
eT			$1172^1\!/_8$		
eg			$(1162^1\!/_7)$		
Tg			153		
ek				2339¼	
eg				$(2334^1\!/_3)$	
kg				153	
el					$(4673^2\!/_3)$
eg					4673½
lg					153

If we formalize the procedure we can come up with the following

inequality:

$$s_n = 6 \times 2^n \times \sin A_n < \pi < S_n = 6 \times 2^n \times \tan A_n$$

Where s_n is the perimeter of the inscribed polygon divided by the diameter of the circle, S_n is the perimeter of the circumscribed polygon divided by the diameter of the circle and A_n is the angle needed to make one side of the polygon. A_n is then defined as $A_0 = 30$, $A_n = \frac{1}{2} \times A_{n-1}$. This inequality defines a smaller interval as n increases because the sine and tangent look more and more alike when the angle approaches zero, thus it shows that you can get an estimation of π with an arbitrary small approximation with this method.

In Ṭūsī's text, after the third intersection the numbers go wrong. Instead of 2339¼ it reads 2339³/₈, off by ¹/₈ (*thumn*). This is then continued through the next numbers. This is not a recalculation by Ṭūsī because in the conclusion he returns to the Archimedian value 4673½ for the diameter. Maybe this erroneous *thumn* was added only after Ṭūsī but it might as well have been there already before Ṭūsī. From the comments Ṭūsī makes it is obvious that he did not recalculate them. It could be that the scribe was so used to write a ¹/₈ after a number (as with most of the numbers before) that he just erroneously continued with it. Another mismatch with the numbers happens when he described the inscribing of the circle. After the second intersection he states the value 5924 where Archimedes states 5924¾. His explanation on why the ratio 5924(¾):780 is equal to 1823:240 is a different approach than Archimedes but is rather obscurely written.

There are not a lot of peculiarities in the text but two are worth mentioning here, both pertaining to this section. The first one is where the translation reads "The diameter is in this measure 4673½ …" The Arabic literally reads "The diameter is in this measure twice 4673½ …". This 'twice' (*ḍiʿf*) seems out of place and that is also why it is not included in the translation. The other peculiarity is that Ṭūsī consistently uses *fī* ('in') to define a polygon inscribing the circle and *ʿalā* ('on') to define a polygon circumscribing the circle. However, where the translation reads "the polygon of 96 sides inscribed in the circle is 6336 …", *ʿalā* is used where *fī* is expected. Again there is no reasonable explanation and so we can merely neglect it and read it as

if it indicates that the polygon is inscribed in the circle. It is not clear whether we should attribute these slips of the pen to Ṭūsī or to a scribe early in the manuscript tradition.

The addition to the second proposition

After the recession of Archimedes' third proposition, Ṭūsī begins by stating "I say". This seems to introduce a longer remark, one that is more distant from the original text. That this part of the text does not come from Archimedes may already be clear from the use of sine and cosine functions, nor did he use sexigesimal notation which only after him was introduced in Greek science, coming from Babylonian astronomy. There are other clues as well, already in the first sentence of Ṭūsī'. Here, Knorr translates: "I say, and for the two results <there is> another method,…" (Knorr, p. 581). He translates 'the two results' adding in a footnote "literally: "sources", *manjamaini*, that is, the two bounds just computed" (Knorr, p. 584). I propose to read *munajjimīni* instead, being the genitive of the plural of *munajjim*, meaning, of course, astronomer. Later in the text this is confirmed when Ṭūsī states that this other proof is backed "with the fundamental principles that have been clarified in the *Almagest* and other certifying books of them." The *Almagest* being a book on astronomy, it confirms that Ṭūsī is speaking of a proof that was known among astronomers. It seems that this is a rather crucial correction, as it says something about Ṭūsī's editorial work and also about the nature of the '*Middle books*'. Saying that this addition relies on astronomers and the *Almagest* would seem to confirm that indeed most readers were working on the '*Middle books*' to get to the *Almagest* and become an astronomer themselves.

The basic idea laid out in this section is to define a chord that is $^1/_{720}^{th}$ part of an equilateral figure that is inscribed in the circle. All calculations are now done in the sexagesimal system. The whole calculation is based on a number from a trigonometric table of which the chord of ½° is taken. Ṭūsī takes this number from Abu al-Wafā' Būzjānī, who created a table with trigonometric numbers. From this number, the rest of the construction is calculated via some simple geometrical constructions. First the inscribed polygon is properly

described, then from the numbers of the inscribed polygon, the size of the outer chord is computed (according to Euclid's *Elements* VI.3). From this number a circumscribed polygon can be constructed and so an upper and lower bound can be established for the circumference of the circle. The conclusion of the proof is similar to the proof of Archimedes, the inner and outer polygons are compared to the diameter of the circle and in addition the mean of the two values is used as a close approximation of π.

Unfortunately, Ṭūsī is already not correct by assuming the chord of ½° to be 0; 31, 24, 55, 54, 55. The correct sexagesimal number is 0; 31, 24, 56, 58, 36, … and although this is not very far off (by approximately 0.0000046), as Luckey shows the value Ṭūsī states is probably mistakenly swapped with the value of Sin(½°), which is close to 0; 31, 24, 55, 54, 0 (Luckey, p. 44). This last value is only $\frac{55}{60}^5$ ($\approx 7\times10^{-8}$) off from the value Ṭūsī states. Although the error is very small, it could have been better and this would have definitely improved the concluding approximation of π. Besides this mistake, the numbers are sometimes hard to read from the manuscript and edition. They can only be confirmed when we calculate ourselves what the numbers should be. For example, the square of **dg** (the square of the apothem ,in figure 4) is 3599; 55, 53, 15, 57, 55, 2, 34, 41, 29, 51. However, if we would follow all the possibilities from the footnotes we could also obtain the value 3599; 55, 23, 55, 57, 55, 2, 36, 41, deviating by more than 0,0083. We know the first number is correct as it should add up to 3600 if we add the square of **ad** (using the Pythagorean theorem), so it shows that we cannot trust the text from the manuscript or edition alone.[1] For the apothem of the inscribed polygon, Ṭūsī gives 59; 59, 57, 56, 37, 56, 51 which is again really close to the true value of ($60\times\cos¼ =$) 59; 59, 57, 56, 37, 45,… This time off by approximately 0.000000014. Now Ṭūsī states that as the side of the inner-polygon stands to the apothem, the side of the outer-polygon stands to the radius. For Ṭūsī this makes 0; 31, 24, 56, 59, 31. It it is actually ($120\times\tan¼ =$) 0; 31, 24, 58, 3,… From these values, the

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1. For example, Knorr gives the erroneous value of 3599; 55, 23, 55, 57, 55, 2, 34, 41. He even gives another erroneous value in a footnote: 3599; 55, 13, 55, 57, 55, 2, 34, 41, 29, 51. It seems that he did not check his numbers. See Knorr, p. 582 and p. 584 note 10.

perimeter must be computed and Ṭūsī gives 376; 59, 10, 59. If we would multiply Ṭūsī's calculated side by 720 we would get 376; 59, 10, 58, 59 while the actual number must be 376; 59, 23, 41,… which is off by approximately 0.0036.

A bigger mistake is made in the computation of the perimeter of the outer-polygon, for which Ṭūsī gives 376; 16, 59, 23, 54, 12. A first correction on this number is to neglect the 16 which both Luckey and Knorr do. Actually, just a few lines after this value, Ṭūsī states it again but this time as 376; 59, 23, 54, 12, correcting his earlier mistake himself. Both Luckey and Knorr do not comment on their emendation. In both the Hyderabad print and the Tehran facsimile it clearly states *khāmisah* ('fifth'). This word after the number indicates what the power of sixty is for the last number, so it also indicates how many fractions are to be expected. Also Luckey and Knorr confirm this (Luckey even states that also Woepcke has read this; see Luckey, p. 44). Simply deleting the $^{16}/_{60}$ and stating *rābiʿah*, although it is correct and is even used this way by Ṭūsī a couple of lines later, does not seem to render the text as it was once written by Ṭūsī. In fact, we can show that it was the author's intention to write *khāmisah* in this case, instead of assuming a scribal error (or whatever assumption Luckey and Knorr made in silence). For this, we need to do the calculation of the circumference ourselves. We saw that after some calculation Ṭūsī gets 0; 31, 24, 56, 59, 31 for one side of a 720-sided polygon that circumscribes the circle. The perimeter of the polygon is then calculated by multiplying this side by 720. So we get 720 × 0; 31, 24, 56, 59, 31 = 12 × 31; 24, 56, 59, 31 (we divide 720 by 60 so we can move the ; one place). We repeat this and obtain:
12 × 31; 24, 56, 59, 31 = $^1/_5$ × 31, 24; 56, 59, 31 = 6, 16; 59, 23, 54, 12 = 360 + 16; 59, 23, 54, 12.

Here we see what has happened. While 360 + 16 is of course 376 and this would give us the (correct) number 376; 59, 23, 54, 12, Ṭūsī must have accidentally copied the 16 from his scrap paper into his manuscript. The *khāmisah* can only be explained as a quick count of the number of fractions on the side of Ṭūsī. As Ṭūsī used the correct number just shortly after, it could be argued that Ṭūsī was not even consciously aware of his error. This of course does not make it legit to

emendate the text in silence.

One point on the difficulty of these sexagesimal numbers needs to be made before we can move on. This is the number 0; 31, 24, 56, 59, 31 for **hz**, a side of a 720-sided circumscribing polygon. Knorr thinks it actually reads 0; 31, 24, 57, 59, 31 and elaborates in a footnote that this value, if multiplied by 720 gives a different value (namely, 376; 59, 35, 54, …) from what Ṭūsī gives. He then even goes so far as to say that if one would calculate backwards, 376; 59, 23, 54, 12 (the number Ṭūsī gives for the perimeter of the circumscribing polygon) would imply a starting value of 0; 31, 24, 56, 59, 31. The difference between six and seven is very ambiguous in manuscripts, since they are represented respectively by *wāw* (و) and *zā* (ز) and can thus easily be miscopied, especially given the inconsistent use of diacritics. If evidence shows that one of the two works better than the other, we may assume that it is that value that is intended by the author. Here Knorr recalculates the value to 56 instead of 57 but insists that this is due to a scribal error by the computist and even goes so far as reprimanding Luckey and Woepcke for simply transmitting the values and not checking them (Knorr, pp. 593-594, note 57). It is in fact quite simple to check whether it should be 56 or 57 and that is to reexamine the computation where the number of **hz** is based on. Ṭūsī computes the number of **hz** by doubling the magnitude of **he**. The magnitude of **he** is 0; 15, 42, 28, 29, 45. We can double each individual value and then readjusting the values to the sexagesimal system. Doing this, we get 0; 30, 84, 56, 58, 90. Readjusting gives 0; 31, 24, 56, 59, 30. This gives us a second clue that the value should read 56. First it was noted that when Knorr calculated **hz** from the value of the perimeter (calculating backwards) it gave the number 56. Then we saw that if we calculate **hz** from **he** (calculating forward) it also gave the number 56. It is therefore safe to say that, bearing the ambiguity of the reading of the numbers six and seven in mind, the text really reads 56. On a side note, it is peculiar that the edition and facsimile read 31 as the last number while our computation gave 30. In this case, we can attribute it to a scribal error, but because the two numbers only differ by approximately 1×10^{-9} this is not an issue.

Near the end Ṭūsī makes from these numbers an approximation that

looks very similar to the approximation of Archimedes. This is in an unusual form of $3+{}^{10}/_{70;38,41,21} \approx 3,141553196379305$, and $3+{}^{10}/_{70;37,47,37} \approx 3,141583110236151$. Ṭūsī gives as an average $3+{}^{10}/_{70;38,14,29} \approx 3,141568151727502$. All these numbers are of the form $3+{}^{10}/_{70;x}$, with x a fraction. The fraction of the last number is actually the average of the other two fractions (this makes the final approximation not the exact middle value of Ṭūsī's upper and lower bound). This last value that Ṭūsī states is approximately 2.45×10^{-5} (0,0000245) less than the actual value of π. That means that the first four decimals Ṭūsī gives are correct. However, with the combination of a couple of scribal (and possibly calculation) errors and the fact that his results rely on a wrong value of the chord of ½°, we can assume that Ṭūsī was capable of even more accurate approximations of π.

The argument of the third proposition

Here Ṭūsī returns to the original text of Archimedes, following it closely but with one important change. Archimedes opens his proposition by stating that π is equal to $3^{1}/_{7}$ and works the proposition out from this value. Ṭūsī however, merely states that if π is equal to $3^{1}/_{7}$ then the following proposition holds true. This is important as π is actually not really equal to $3^{1}/_{7}$ (as was just concluded in Ṭūsī's second proposition), rather it is (as Ṭūsī states) a ratio commonly used by surveyors. He also elaborates more on details that Archimedes skips. For example, he does not state the result (that the ratio circle: square is 11:14), but also states the ratio 22:28, which is the ratio actually obtained from the proof.

Concluding remarks

Presenting the evidence through a comparison of Archimedes' original text and Ṭūsī's revision, the difference between the two becomes abundantly clear. Already a cursory view of the translations shows that. In the elaboration of the arguments we came across other aspects of Ṭūsī's text that may not be so obvious from skimming the surface, but which do contribute to the unique character of his revisions and show that these texts are in themeselves an original contribution to

mathematics and are therefore not mere imitations of their Greek originals.

Of these aspects, four stand out. First, there is of course the rearrangement of the propositions. This is significant because Ṭūsī chose not to make an edition that would be merely imitating Archimedes' text, but one that was logically correct (in Archimedes' text, the second proposition logically relies on the third). Secondly, it is obvious that Ṭūsī made a conscious effort to streamline the reasoning, adding steps where elucidation was needed and reducing steps where it seemed plain what was meant. Third, and perhaps most important, is of course the highly original addition of Ṭūsī to his second proposition. Here, Ṭūsī uses a number from a trigonometric table to compute the circumference of a regular polygon that is inscribed in a circle and one that is circumscribed, making two 720-sided polygons. From these two values Ṭūsī deduces an estimate of π that is approximately 0.0000245 off of the real value of π. In fact, we noticed that the manuscripts contain scribal errors so Ṭūsī's real computed value could have been an even closer estimation. Fourth, Ṭūsī seemed to sense the error in the statement that π is equal to $3^{1}/_{7}$ (this is asserted in Archimedes' third proposition and is needed to complete the proof of the proposition) and he corrected this into an if clause, remarking that this is an approximation regularly used by surveyors.

If the small treatise on the measurement of the circle is any indication of the nature of all of the *Taḥrīr al-mutawassiṭāt*, 'Revision of the middle books', then we can conclude that these revisions almost function as a commentary to their original versions, indeed, exceeding earlier commentaries. They give evidence to Ṭūsī's mathematical acumen, and further give a unique insight in the state of knowledge of Ṭūsī's time, not only of mathematics but also for example of astronomy and land surveying. Given the impact these treatises had in the centuries after Ṭūsī, we can only hope that the study on these important documents will be continued in the future.

References

Archimedes, Netz, R. (transl.), *The works of Archimedes*, vol. 1: The Two Books *On the sphere and cylinder*, Cambridge: Cambridge University Press, 2004.

Brockelmann, C., *Geschichte der arabischen Litteratur*, 2nd ed., Brill, Leiden, 1943–49.

Clagett, M., *Archimedes in the Middle Ages*, vol. I The Arabo-Latin Tradition, The University of Wisconsin Press, Madison, 1964.

Heath, T., *A History of Greek Mathematics*, vol. II, Clarendon Press, Oxford, 1921.

Heiberg, J.L., *Archimedis opera omnia cum commentariis Eutocii*, Leipzig: B. G, Teubner, 1880.

Kheirandish, E., *A Report on Iran's 'Jewel' Codices of Ṭūsī's* Kutub al-Mutawassiṭāt, in Pourjavady, N. Vesel, Ž., *Naṣīr al-Dīn Ṭūsī Philosophe et Savant du XIIIe Siècle*, Presses Universitaires d'Iran, Téhéran, 2000.

Knorr, W., "The Textual Tradition of Archimedes' Dimension of the Circle" in *Textual Studies in Ancient and Medieval Geometry*, Boston, Birkhaüser, 1989.

Krause, M., "Stambuler Handschriften islamischer Mathematiker", *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik* (Berlin), Abteilung B: Studien 3. 1936, pp. 437-532. (Reprinted in: Sezgin, F., *Miscellaneous Texts and Studies on Islamic Mathematics and Astronomy VIII*, *Islamic Mathematics and Astronomy*, vol. 83, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1998, pp.237-332).

Lorch, R., "The Arabic Transmission of Archimedes' Sphere and Cylinder and Eutocius' Commentary", *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften*, 5. 1989, pp. 94-114.

Luckey, P., *Der Lehrbrief über den Kreisumfang (ar-Risāla Muḥīṭīya) von Gamšid b. Masᶜūd Kāšī übersetzt und erläutert*. Herausgegeben von Alfred Siggel, Akademie Verlag, Berlin, 1953. (Reprinted in: Sezgin, F., *Al-Kāshī Texts and Studies*, *Islamic Mathematics and Astronomy*, vol. 56, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1998, pp. 227-329).

Rashed, R., "Al-Kindī's Commentary on Archimedes' 'The Measurement of the Circle' ", *Arabic Sciences and Philosophy*, vol. 3, Cambridge University Press, Cambridge, 1993, pp. 7-54.

Rosenfeld, B. A., Ihsanoglu, E., *Mathematicians, Astronomers, and other scholars of Islamic civilization and their works (7th-19thc.)*, Research Center for Islamic History Art and Culture, Istanbul, 2003.

Sezgin, F., *Geschichte des arabischen Schrifttums*, Band V, Mathematik bis ca. 430 H, Brill, Leiden, 1974.

Steinschneider, M., "Die mittleren Bücher der Araber und ihre Bearbeiter", *Zeitschrift für Mathematik und Physik*, nr. 10, Leipzig, 1865, pp. 456-498. (Reprinted in: Sezgin, F., *Euclid in the Arabic Tradition Text and Studies I*, *Islamic Mathematics and Astronomy*, vol. 17, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1997, pp. 54-97).

Aṭ-Ṭūsī, Naṣīraddīn, 'Kitāb al-Kura wa-l-usṭuwāna li-Arkhimīdis bi-taḥrīr', *Majmūʿ al-rasāʾ il* (2 vols.), Ḥaydarābād, 1359/1940, vol. 2, pp. 127-133. (Reprinted in: Sezgin, F., *A Collection of Mathematical and Astronomical Treatises as Revised by Naṣīraddīn aṭ-Ṭūsī, Islamic Mathematics and Astronomy*, vol. 48, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1998, pp. 377-389).