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# $\alpha$-projectable and laterally $\alpha$-complete Archimedean lattice-ordered groups with weak unit via topology

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Dedicated to Themba Dube on the occasion of his 65th birthday

**Abstract.** Let **W** be the category of Archimedean lattice-ordered groups with weak order unit, **Comp** the category of compact Hausdorff spaces, and $\mathbf{W} \xrightarrow{Y} \mathbf{Comp}$ the Yosida functor, which represents a **W**-object $A$ as consisting of extended real-valued functions $A \leq D(YA)$ and uniquely for various features. This yields topological mirrors for various algebraic (**W**-theoretic) properties providing close analysis of the latter. We apply this to the subclasses of $\alpha$-projectable, and laterally $\alpha$-complete objects, denoted $P(\alpha)$ and $L(\alpha)$, where $\alpha$ is a regular infinite cardinal or $\infty$. Each **W**-object $A$ has unique minimum essential extensions $A \leq p(\alpha)A \leq l(\alpha)A$ in the classes $P(\alpha)$ and $L(\alpha)$, respectively, and the spaces $Yp(\alpha)A$ and $Yl(\alpha)A$ are recognizable (for the most part); then we write down what $p(\alpha)A$ and $l(\alpha)A$ are as functions on these spaces. The operators $p(\alpha)$ and $l(\alpha)$ are compared: we show that both preserve closure under all implicit functorial operations which are finitary. The cases of $A = C(X)$ receive special attention. In particular, if $(\omega < \alpha)$ $l(\alpha)C(X) = C(Yl(\alpha)C(X))$, then $X$ is finite. But $(\omega \leq \alpha)$ for

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infinite $X$, $p(\alpha)C(X)$ sometimes is, and sometimes is not, $C(Yp(\alpha)C(X))$.

## 1 Introduction and Preliminaries

We begin with basic definitions, etc., trying to make the Abstract quickly comprehensible. More detail is in the introductions to [23] and [24], of which this paper is a loose continuation. General references for $\ell$-groups and vector lattices are [1], [4], [12], and [32].

**Definition 1.1.** In an $\ell$-group $A$ (always assumed Archimedean, thus Abelian):

For $S \subseteq A$, $S^{\perp} \equiv \{a \in A \mid |a| \wedge |s| = 0 \, \forall s \in S\}$ is an *ideal* (convex sub-$\ell$-group). Ideals $S^{\perp\perp}$ are called *polars*.

Let $\alpha$ be a regular infinite cardinal or the symbol $\infty$; we write $\omega \leq \alpha \leq \infty$. $|S|$ is the cardinal of the set $S$, and $|S| < \infty$ means $S$ is of any size.

An *$\alpha$-polar* in $A$ is an $S^{\perp\perp}$ for $|S| < \alpha$.

$A \in P(\alpha)$ ($A$ is *$\alpha$-projectable*) means that each $\alpha$-polar $S^{\perp\perp}$ is an $\ell$-group direct summand, i.e., each $a \in A$ can be written uniquely $a = a_1 + a_2$ with $a_1 \in S^{\perp\perp}$ and $a_2 \in S^{\perp}$.

The following terms are used in the literature: if $A \in P(\omega)$ (resp., $P(\infty)$), $A$ is called *projectable* (resp., *strongly projectable*). For vector lattices, the terminology "principal projection property" (resp., "projection property") is sometimes used.

$A \in L(\alpha)$ ($A$ is *laterally $\alpha$-complete*) if each disjoint $S \subseteq A^+$ ("disjoint" means for all $s_1, s_2 \in S$, if $s_1 \neq s_2$, then $s_1 \wedge s_2 = 0$) with $|S| < \alpha$, the supremum $\bigvee\{s \mid s \in S\}$ exists in $A$. Note that any $A \in L(\omega)$ (since $A$ is a lattice). [23, 3.2] shows that $L(\alpha) \subseteq P(\alpha)$ in $\mathbf{W}$.

We turn to $\mathbf{W}$ and the Yosida Theorem, and now restrict our $\ell$-groups to $\mathbf{W}$: $(A, u_A) \in \mathbf{W}$ means $A$ is an Archimedean $\ell$-group (thus Abelian) and $u_A$ is a distinguished weak unit (meaning $\{u_A\}^{\perp} = \{0\}$), positive unless $A = \{0\}$, where $u_A = 0$.

A $\mathbf{W}$-homomorphism $(A, u_A) \xrightarrow{\varphi} (B, u_B)$ is an $\ell$-group homomorphism with $\varphi(u_A) = u_B$. With these as morphisms, $\mathbf{W}$ is a category.

The "interval" $\mathbb{R} \cup \{\pm\infty\} = [-\infty, +\infty]$ is given the obvious topology and order. For a space $X$ (always Tychonoff, frequently compact Hausdorff), $D(X)$ is the set of continuous $X \xrightarrow{f} [-\infty, +\infty]$ for which $f^{-1}(\mathbb{R})$ is dense in $X$. This is a lattice containing the constant function with value 1 as a weak unit, with addition partially defined by $f+g=h$ meaning $f(x)+g(x)=h(x)$ for every $x \in f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R}) \cap h^{-1}(\mathbb{R})$. A **W**-object in $D(X)$ is an $A \in \mathbf{W}$ which is a sublattice, with the partial addition of $D(X)$ fully defined on $A$, and with the constant function with value 1 contained in $A$ and serving as the distinguished weak unit; we express all that succinctly by writing $A \leq D(X)$. For arbitrary $A, B \in \mathbf{W}$, $A \leq B$ means there is a **W**-embedding of $A$ into $B$.

Let **Comp** denote the category of compact Hausdorff spaces and continuous maps.

**The Yosida Representation 1.2.** The functor $Y$.

(a) (of objects) If $(A, u_A) \in \mathbf{W}$, there is a $YA \in \mathbf{Comp}$ and a **W**-isomorphism $(A, u_A) \xrightarrow{\eta_A} \eta_A(A) \leq D(YA)$ with $\eta_A(A)$ separating the points of $YA$. $YA$ is unique up to homeomorphism for that data.

(b) (of morphisms) If $(A, u_A) \xrightarrow{\varphi} (B, u_B)$ is a **W**-morphism, there is a unique continuous $YA \xleftarrow{Y\varphi} YB$ for which $\eta_B(\varphi(a)) = \eta_A(a) \circ Y\varphi$ for all $a \in A$. Moreover, $\varphi$ is one-to-one if and only if $Y\varphi$ is onto. While $\varphi$ onto implies $Y\varphi$ is one-to-one, the converse does not hold. ([40] exhibits $(A, u_A) \to D(YA)$, the rest is from [26].)

For example, consider $A = C(X) \equiv \{f\colon X \to \mathbb{R} \mid f \text{ is continuous}\}$ and $A^* = C^*(X) = \{f \in C(X) \mid f \text{ is bounded}\}$, where $X$ is any Tychonoff space. Since the natural maps $A^* \to A \to D(YA)$ and $A^* \cong C(\beta X) \to D(\beta X)$ both separate points, it follows from the uniqueness of the Yosida representation that $YA = YA^* = \beta X$ (the Čech-Stone compactification of $X$).

We now view all **W**-objects $(A, u_A)$ as being in their Yosida representation, and just write $A \leq D(YA)$ ($u_A = 1$ being understood when $A \neq \{0\}$).

We work toward combining $P(\alpha), L(\alpha)$ with Yosida.

**Theorem 1.3.** ([24, 2.9]) *In* **W**, *$P(\alpha)$ (resp., $L(\alpha)$) is a hull class, i.e., for each $A \in$* **W**, *there is an extension $p(\alpha)A$ (resp., $l(\alpha)A$) minimum among essential extensions to $P(\alpha)$-objects (resp., $L(\alpha)$-objects), and it is unique up to* **W***-isomorphism.*

Referring to "essential" in Theorem 1.3, in any category a monic $m$ is called *essential* if $f \circ m$ monic implies $f$ monic. In **W**, monic means one-to-one, and essential can be said several ways ([4]).

**Lemma 1.4.** ([26]) *In* **W**, *the embedding* $A \xrightarrow{\varphi} B$ *is essential if and only if the surjection* $YA \xleftarrow{Y\varphi} YB$ *is irreducible (the image of a proper closed set is proper), and in that case we call* $Y\varphi$ *a* cover.

Thus, for every $A \in \mathbf{W}$, the map $YA \xleftarrow{\sigma} Yp(\alpha)A$ is a cover.

There is a related theory of "covering properties", to a fragment of which we shall allude.

**Definition 1.5.** (a) If $X$ is a space and $f \in D(X)$, then $\text{coz}_X(f) = \{x \in X \mid f(x) \neq 0\}$. If $S \subseteq D(X)$, then let $\text{coz}_X(S) = \bigcup\{\text{coz}_X(f) \mid f \in S\}$ and, for $\omega \leq \alpha \leq \infty$, let $\alpha\text{coz}_X(C(X)) = \{\text{coz}_X(S) \mid S \subseteq C(X) \text{ and } |S| < \alpha\}$.

(b) If $A \in \mathbf{W}$, $f \in A$, and $S \subseteq A$, then let $\text{coz}(f) = \text{coz}_{YA}(f)$ and $\text{coz}(S) = \bigcup\{\text{coz}(f) \mid f \in S\}$. And for $\omega \leq \alpha \leq \infty$, let $\alpha\text{coz}(A) = \{\text{coz}(S) \mid S \subseteq A \text{ and } |S| < \alpha\}$. Note the difference: if $U \in \alpha\text{coz}_X(C(X))$, then $U \subseteq X$, but if $U \in \alpha\text{coz}(C(X))$, then $U \subseteq YC(X) = \beta X$.

(c) For $\omega \leq \alpha \leq \infty$, a space $X$ is called *$\alpha$-disconnected* if $\{U_i\}_{i\in I} \subseteq \omega\text{coz}_X(C(X))$ and $|I| < \alpha$ imply that the closure of $\bigcup_{i\in I} U_i$ in $X$ is open. $D(\alpha)$ denotes the class of such spaces. Then: $X \in D(\alpha)$ if and only if $\beta X \in D(\alpha)$; spaces in $D(\omega) = D(\omega_1)$ are called *basically disconnected* (BD), and spaces in $D(\infty)$ are called *extremally disconnected* (ED). Note that $X \in D(\alpha)$ implies $X$ is zero-dimensional (ZD), meaning $X$ has a base of clopen sets. (See [15] for some of this.)

It is known that $D(\alpha)$ is a "covering class" in **Comp**, which means that for every space $X \in$ **Comp** there is $X \xleftarrow{\sigma} d(\alpha)X$ minimum for covers of $X$ by spaces in $D(\alpha)$ (See [19], [36], [39]).

One may suspect something like **W** versus **Comp** as: minimum essential extensions $A \xrightarrow{\varphi} B$ to **W**-objects with a property $\mathcal{P}$ are associated (as $YA \xleftarrow{Y\varphi} YB$) to minimum covers with a property $\mathcal{L}$. There is a literature on that ([7], [34], inter alia). Here we have cases in points, which we turn to.

## 2 Yosida spaces of the $P(\alpha)$ and $L(\alpha)$ hulls

We explain now what $Yp(\alpha)A$ and $Yl(\alpha)A$ are, then show constructions of the $p(\alpha)A$ and $l(\alpha)A$ in the next section.

If $X$ is a space and $S \subseteq X$, then we write $\chi(S)$ for the characteristic function of $S$ on $X$. We note: If $U \in \text{clop}(YA)$, then $\chi(U) \in A$ (from the "point-separating" in Theorem 1.2(a), and a little arithmetic). Recall that $\alpha\text{coz}(A)$ consists of subsets of $YA$.

**Theorem 2.1.** ([23, 2.2 and 2.4]) *$A \in P(\alpha)$ if and only if both of the following hold.*

(a) *For every $U \in \alpha\text{coz}(A)$, $\overline{U}$ is open.*

(b) *For every $a \in A$ and every $U \in \text{clop}(YA)$, $a\chi(U) \in A$.*

**Remark 2.2.** The condition in Theorem 2.1(a) can be called *weakly* $P(\alpha)$ ($wP(\alpha)$). $wP(\alpha)$ is also a hull class. $wP(\omega)$ is (defined and) shown to be a hull class in [22]. The extension to general $\alpha$ will be evident; the $wP(\alpha)$ hull operator is denoted $wp(\alpha)$. $A$ is called *local* ($A \in Loc$) if $f \in D(YA)$ and $f$ locally in $A$ implies $f \in A$, where "locally in $A$" is in the topological sense of local as functions on $YG$. $Loc$ is also a hull class, indeed an essential reflection, and the associated hull operator is "loc". That $Y$ loc $A = YA$ is not hard ([26]).

**Lemma 2.3.** *Suppose $YA$ is ZD. Then, $A$ is local if and only if for every $a \in A$ and every $U \in \text{clop}(YA)$, one has $a\chi(U) \in A$ ([23]).*

Thus, Theorem 2.1 says $P(\alpha) = wP(\alpha) \cap Loc$. Also, it's easy to see that $p(\alpha) = \text{loc} \circ wp(\alpha)$ (the case $\alpha = \omega$ is in [22]).

**Corollary 2.4.** (a) *If $A \in P(\alpha)$, then $YA$ is ZD.*

(b) ([23, 2.4]) *For $\omega < \alpha$:*

- *$A \in wP(\alpha)$ if and only if $YA \in D(\alpha)$.*
- *$A \in P(\alpha)$ if and only if $YA \in D(\alpha)$ and $A \in Loc$.*

(c) *For $\omega < \alpha$: If $A \in L(\alpha)$, then $YA \in D(\alpha)$ and $A \in Loc$.*

*Proof.* (a) and (b) follow from Theorem 2.1(a).
(c) is just because $L(\alpha) \subseteq P(\alpha)$ (noted in Definition 1.1). □

**Corollary 2.5.** *$C(X) \in P(\alpha)$ if and only if $X \in D(\alpha)$.*

*Proof.* $C(X)$ is a ring, thus $C(X) \in Loc$ ( [26]) and $X \in D(\alpha)$ if and only if $YC(X) = \beta X \in D(\alpha)$. For $\alpha = \omega$, use that $P(\alpha) = wP(\alpha) \cap Loc$. For $\alpha > \omega$, apply Corollary 2.4(b). □

In Corollary 2.5, the cases $\alpha = \omega, \infty$ appear in [32, Section 43].

We require (now and later) some more information about spaces in $D(\alpha)$, and the following "over-class".

**Definition 2.6.** (Analogous to $D(\omega) \subseteq F$, where "$F$" denotes the class of $F$-spaces as in [15, Chapter 14]).

$X$ is a $F(\alpha)$-space ($X \in F(\alpha)$) just in case all disjoint $U, V \in \alpha\text{coz}_X(C(X))$ are completely separated.

If $X$ is any topological space with subspace $S$, then $S$ is *$C^*$-embedded in $X$* if every $f \in C^*(S)$ extends to some $\overline{f} \in C^*(X)$.

**Lemma 2.7.** (a) *$X \in F(\alpha)$ if and only if every $U \in \alpha\text{coz}_X(C(X))$ is $C^*$-embedded.*

(b) *$F = F(\omega) = F(\omega_1)$, and $F(\infty) = D(\infty)$.*

(c) *If $X \in F(\alpha)$, then dense $U \in \omega\text{coz}_X(C(X))$ are $C^*$-embedded (called "$X$ is quasi-$F$"), and the last if and only if $D(X)$ is a* **W***-object.*

(d) *$D(\alpha) \subseteq F(\alpha)$.*

*Proof.* We prove (b)–(d) assuming that (a) holds, then prove (a).

For (b), see [15].

For (c), note that if $U \in \omega\text{coz}_X(C(X))$, then $U \in \alpha\text{coz}_X(C(X))$. The term "quasi-$F$" is from [13], and the "iff" here is proved in [30].

For (d), if $U, V \in \alpha\text{coz}_X(C(X))$ are disjoint, then $\overline{U}$ and $\overline{V}$ are open, so $\overline{U} \cap \overline{V} = \emptyset$, and $\chi(\overline{U})$ separates $U$ and $V$.

Finally, to establish (a) we use the version of the Urysohn Extension Theorem in [15, 1.15 and 1.17]: a subspace $S$ of a Tychonoff $X$ is $C^*$-embedded in $X$ if and only if disjoint zero-sets of $S$ are completely separated in $X$. Suppose $S \in \alpha\text{coz}_X(C(X))$ with $X \in F(\alpha)$ and $Z_1, Z_2$ are disjoint zero-sets of $S$. There are disjoint cozero-sets $C_1, C_2$ of $S$ with $Z_i \subseteq C_i$ for $i \in \{1, 2\}$. A cozero-set in an $\alpha$-cozero-set is an $\alpha$-cozero-set, so $C_1$ and $C_2$ are completely separated in $X$. □

**Theorem 2.8.** *If $X \in D(\alpha)$, then $D(X) \in L(\alpha)$ ($\subseteq P(\alpha)$).*

*Proof.* By Lemma 2.7, $D(X)$ is a **W**-object.

Suppose $\{f_i\}_{i\in I} \subseteq D(X)^+$ is disjoint and $|I| < \alpha$. Each $\overline{\mathrm{coz}_X(f_i)}$ is open, and $U \equiv \overline{\bigcup_{i\in I} \mathrm{coz}_X(f_i)}$ is open. Let

$$S = \left(\bigcup_{i\in I} \overline{\mathrm{coz}_X(f_i)}\right) \cup (X - U).$$

Since $X \in F(\alpha)$, $S$ is $C^*$-embedded (each by Lemma 2.7).

Let $f \in D(S)$ be such that $f|_{\overline{\mathrm{coz}_X(f_i)}} = f_i$ for $i \in I$ and $f|_{X-U} = 0$. Since $S$ is dense and $C^*$-embedded (and $[-\infty, +\infty]$ is compact), $f$ extends to $\overline{f} \in D(X)$ ([15, 6.4]). One sees that $\overline{f} = \bigvee_{i\in I} f_i$. □

**Theorem 2.9.** $(\omega < \alpha)$ *For every* $A \in \mathbf{W}$,

$$Yp(\alpha)A = Yl(\alpha)A = d(\alpha)YA.$$

*Proof.* Given $A$, we have the cover $YA \overset{\sigma}{\leftarrow} d(\alpha)YA \equiv X$. Since $X \in D(\alpha)$, we have $D(X) \in L(\alpha)$ by Theorem 2.8.

Since $\sigma$ is a cover, $A \approx A \circ \sigma \leq D(X)$ is an essential extension with codomain in $P(\alpha)$. Hence $p(\alpha)A \leq D(X)$ by the minimality of $p(\alpha)A$. Then $Yp(\alpha)A \leq YD(X) = X$ (as covers), from the Yosida functor. But $Yp(\alpha)A \in D(\alpha)$ by Theorem 2.4. Thus $Yp(\alpha)A = X$ by the minimality of $d(\alpha)YA$.

Since $A \leq p(\alpha)A \leq l(\alpha)A \leq D(X)$, we see too that $Yl(\alpha)A = X$. □

Note, Theorem 2.9 assumes $\omega < \alpha$. The case $\omega = \alpha$ is less purely topological and more complicated.

**Theorem 2.10.** *Let* $A \in \mathbf{W}$.

(a) ([22]) $YA \leftarrow Yp(\omega)A$ *is the minimum among covers* $YA \overset{\sigma}{\leftarrow} X$ *for which the closure of* $\sigma^{-1}(\mathrm{coz}(a))$ *is open in* $X$ *for all* $a \in A$.

(b) ([6], [25]) $Yp(\omega)A$ *is the Stone space of the Boolean subalgebra generated by* $\{\{P \in \mathrm{Min}(A) \mid a \notin P\}\}_{a\in A}$ *in the power set of* $\mathrm{Min}(A)$ (*here* $\mathrm{Min}(A)$ *is the collection of minimal prime subgroups of* $A$).

Two related questions arise: What about Theorem 2.10(a) (mutatis mutandis) for $\omega < \alpha$? For compact $X$, is there/what is the minimum among covers $X \overset{\sigma}{\leftarrow} Z$ which have $\overline{\sigma^{-1}(U)}^Z$ open for all $U \in \alpha\mathrm{coz}_X(C(X))$ (for

$\omega \le \alpha$, here)? To the first, [22, 3.7(b)] says "it's the same". For the second, [25, 6.3] (and a little thought) says such a minimum exists. Then, what is it? $d(\alpha)X$? For $\alpha = \infty$, it's easy to see that this minimum is in $D(\infty)$ (=ED), and thus is $d(\infty)X$ (the Gleason cover).

But, for $\omega = \alpha$, Vermeer [39] has constructed this minimum called $\Lambda_1 X$ ($= Yp(\omega)C(X)$, by Theorem 2.10(a)), and shown that $d(\omega)X$ is achieved by transfinite iteration of $\Lambda_1$, and presented the example $\Lambda_1 X < d(\omega)X$ (qua covers) in Corollary 2.11(b) following.

**Corollary 2.11.** (*Of* Theorem 2.10 *and the literature*)

(a) *For every $A$, $Yp(\omega)A \le d(\omega)YA$ (qua covers of $YA$, which means $Yp(\omega)A$ is covered by $d(\omega)YA$).*

(b) *If $Z = \beta\mathbb{N} - \mathbb{N}$, then $Yp(\omega)C(Z) < d(\omega)Z$, i.e., $Yp(\omega)C(Z) \notin D(\omega)$ (by the minimality of $d(\omega)Z$, see* Definition 1.5*).*

(c) *Suppose $Z$ compact (so $YC(Z) = Z$). If every open set in $Z$ is a cozero-set (e.g., $Z$ compact metrizable), then $Yp(\omega)C(Z) = Yp(\alpha)C(Z) = d(\infty)Z$ for every $\alpha$ (a fortiori, $= d(\omega)Z$).*

*Proof.* (a) $X = d(\omega)YA$ has $\overline{U}$ open for all $U \in \omega\mathrm{coz}_X(C(X))$, not just the $\sigma^{-1}(\mathrm{coz}(a))$.

(b) [39, Theorem 3.6].

(c) By Theorem 2.10 and the remark above that $d(\omega)Z$ is the minimum cover making preimages of opens in $Z$, open in the cover. □

**Remark 2.12.** What "really is" $d(\alpha)X$ ($\omega < \alpha$)?

Since $d(\alpha)X$ is ZD, it is the Stone space of $\mathrm{clop}(d(\alpha)X)$, of course. The question is to be interpreted with the addition "in terms of $X$", thus "What is $\mathrm{clop}(d(\alpha)X)$, in terms of $X$?".

From various details of Stone Duality between Boolean Algebras and compact ZD spaces, and the discussion in [20] and [21], the following suspect/conjecture emerges: $\mathrm{clop}(d(\alpha)X) = \alpha\mathcal{B}X/\alpha M$ (where $\alpha\mathcal{B}X$ is the $\sigma$-algebra generated by the $\alpha$-cozero sets in the power set of $X$, and $\alpha M$ is its $\sigma$-ideal of meagre sets).

This is true if and only if $\alpha\mathcal{B}X/\alpha M$ is an $\alpha$-complete Boolean Algebra, which is true in at least these three cases:

(i) $\alpha = \infty$. This is because $d(\infty)X$ is the Stone space of the regular open algebra, which algebra is $\infty\mathcal{B}X/\infty M$ ([16], [37]).

(ii) $\alpha = \omega_1$. Some people surely know this, but in any event it follows from the discussion in [2].

(iii) $X$ is $\alpha$-cozero-complemented (i.e., for every $U \in \alpha\mathrm{coz}_X(C(X))$ there is a disjoint $V \in \alpha\mathrm{coz}_X(C(X))$ with $U \cup V$ dense in $X$). This is a slight extension of [21, 3.2].

We depart the subject.

## 3 Representation of the hulls $p(\alpha)A$ and $l(\alpha)A$

Our descriptions are the main results of the paper. We make two constructions: $A_X$ in Theorem 3.1 and $\overline{A_X}$ in Theorem 3.3. Note that these constructions depend on $\alpha$, but the notation will suppress that for the sake of simplicity.

A frequently used notation (for emphasis) is: $\dot{\bigcup} U_i$ for the union of disjoint sets $\{U_i\}$, $\dot{\sum} f_i$ for a sum of disjoint elements $\{f_i\}$ in an $\ell$-group or in a $D(X)$.

**Theorem 3.1.** ([22, 2.5 and 2.6]) *Suppose $X$ is compact and ZD and $A \leq D(X)$. Define $A_X$ to be the set of all $\dot{\sum}_{i\in I} a_i\chi(U_i) \in D(X)$ such that $|I| < \omega$, each $a_i \in A$, and $\{U_i\}_{i\in I} \subseteq \mathrm{clop}(X)$ is a disjoint family. (Note, we could enlarge $\{U_i\}_{i\in I}$ to $\{U_i\}_{i\in I} \cup (X - \bigcup_{i\in I} U_i)$ and on $X - \bigcup_{i\in I} U_i$, let the function be 0; so we could suppose that $\bigcup_{i\in I} U_i = X$.) Then $A_X \leq D(X)$, $YA_X = X$, $A_X \in Loc$, and $(*)$ $A_X \in P(\omega)$ if and only if $\overline{\mathrm{coz}(a)}$ is open for every $a \in A$.*

*Proof.* See the reference given. The last assertion does not appear there, but is obvious. □

**Corollary 3.2.** ([22, 2.6]) *Suppose $A \in$ **W**, and take $X = Yp(\omega)A$ in* Theorem 3.1. *Then $A_X = p(\omega)A$.*

Note here that, as described in Section 2, $X = Yp(\omega)A$ need not be $\omega$-disconnected (BD, $D(\omega)$). Toward the representation especially for $l(\alpha)A$, $\omega < \alpha$, we extend the ideas in Theorem 3.1 as follows.

Let $X \in D(\alpha)$ and consider $A \leq D(X)$, $|I| < \alpha$, $\{a_i\}_{i\in I} \subseteq A$, and $\{U_i\}_{i\in I}$ a disjoint family in $\mathrm{clop}(X)$, where $X = YA$. Then $(*)$ $f \equiv \dot{\sum}_{i\in I} a_i\chi(U_i)$ is a priori just defined on $U \equiv \bigcup_{i\in I} U_i$, and $U \in \alpha\mathrm{coz}(A)$, so $\overline{U}$ is open. Then, we extend the definition of $f$ to $U \cup (X - \overline{U}) \equiv S$, which is dense and in $\alpha\text{coz}(A)$, by letting $f \equiv 0$ on $X - \overline{U}$. Then, $S$ is $C^*$-embedded in $X$ (by (c) and (d) of Lemma 2.7), and so (by [15, 6.4]) $f$ extends further to a function in $D(X)$.

So, we can understand an expression $(*)$ to include "$\bigcup_{i \in I} U_i$ is dense in $X$" – we say "$\{U_i\}_{i \in I}$ is a clopen quasi-partition of $X$" – and $f \in D(X)$. Thus, one obtains an analogue $A_{X,\alpha}$ of $A_X$ for $\alpha > \omega$ (and note that $A_{X,\omega} = A_X$).

Then our extension of Theorem 3.1 is

**Theorem 3.3.** *Suppose compact $X \in D(\alpha)$ and $A \leq D(X)$. Then:*

(a) $A_X \in P(\alpha)$.

(b) *Let $\overline{A_{X,\alpha}}$ be the set of all $\dot{\sum}_{i \in I} a_i \chi(U_i)$ such that $|I| < \alpha$, $\{a_i\}_{i \in I} \subseteq A$, and $\{U_i\}_{i \in I}$ is a disjoint family in* clop$(X)$. *Then $\overline{A_{X,\alpha}} \leq D(X)$, $\overline{A_{X,\alpha}} \in Loc$, and $\overline{A_{X,\alpha}} \in L(\alpha)$.*

*Proof.* (a) In Theorem 3.1, $(*)$ is satisfied.

(b) (This goes as the proof of Theorem 3.1, mutatis mutandis, but we write down some details.)

We expressed that $\overline{A_{X,\alpha}} \subseteq D(X)$. To see that $\overline{A_{X,\alpha}} \in \mathbf{W}$, take $f = \dot{\sum}_{i \in I} a_i \chi(U_i)$, $g = \dot{\sum}_{j \in J} b_j \chi(V_j)$ in $\overline{A_{X,\alpha}}$, where we assume $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ are clopen quasi-partitions in $X$, and consider $\otimes = +, -, \vee, \wedge$. Then, one sees that $\{U_i \cap V_j\}_{i,j}$ is a clopen quasi-partition of $X$ and

$$f \otimes g = \dot{\sum_{i,j}} (a_i \otimes b_j) \chi(U_i \cap V_j) \in \overline{A_X}.$$

So $\overline{A_{X,\alpha}} \in \mathbf{W}$.

Since $\chi(U) \in \overline{A_{X,\alpha}}$ whenever $U \in \text{clop}(X)$, we see that $\overline{A_{X,\alpha}}$ separates points of $X$, so $Y\overline{A_{X,\alpha}} = X$. Since any $a\chi(U) \in \overline{A_{X,\alpha}}$ and $X$ is ZD, we have $\overline{A_{X,\alpha}} \in Loc$ (see Theorem 5.1).

Finally, $\overline{A_{X,\alpha}} \in L(\alpha)$ is shown much as $D(X) \in L(\alpha)$ was shown, to wit. Let $\{f_\gamma\}_{\gamma \in \Gamma}$ be a disjoint family in $\overline{A_{X,\alpha}}$ with $|\Gamma| < \alpha$. Let $\mathcal{U}_\gamma$ be the set of $U_i$'s in the expression for $f_\gamma$ with $\text{coz}(f_\gamma) \cap U_i \neq \emptyset$. Then, since $\{\text{coz}(f_\gamma)\}_{\gamma \in \Gamma}$ is a disjoint family, $\bigcup_{\gamma \in \Gamma} \mathcal{U}_\gamma \equiv \mathcal{V}$ is a disjoint clopen family with $|\mathcal{V}| < \alpha$, and one may let $f$ be defined as $f_\gamma$ on each $U \in \mathcal{U}_\gamma$ and 0 on $X - \overline{\bigcup \mathcal{V}}$. Then $f$ extended over $X$ realizes $\bigvee_{\gamma \in \Gamma} f_\gamma$ in $\overline{A_{X,\alpha}}$. □

To be completely explicit about the hulls:

**Theorem 3.4.** *Suppose $\omega < \alpha$. Let $A \in \mathbf{W}$, and let $X = Yp(\alpha)A = Yl(\alpha)A = d(\alpha)YA$ (recalling* Theorem 2.9*), denoted qua cover as $YA \overset{\sigma}{\leftarrow} X$. Identify $A$ with its isomorph $A \circ \sigma \leq D(X)$. Then, $p(\alpha)A = (A \circ \sigma)_X$ and $l(\alpha)A = \overline{(A \circ \sigma)_{X,\alpha}}$. Explicitly for reference later, about the elements:*

*The elements of $p(\alpha)A$ are exactly the $f \in D(X)$ of the form $f = \dot{\sum}_{i\in I}(a_i \circ \sigma)\chi(U_i)$, where $I$ is finite, $a_i \in A$ for $i \in I$, and $\{U_i\}_{i\in I}$ is a clopen partition of $X$.*

*The elements of $l(\alpha)A$ are exactly the $f \in D(X)$ of the form $f = \dot{\sum}_{i\in I}(a_i \circ \sigma)\chi(U_i)$, where $|I| < \alpha$, $a_i \in A$ for $i \in I$, and $\{U_i\}_{i\in I}$ is a clopen quasi-partition in $X$.*

Both [5] and [11] construct $p(\omega)A$ and $p(\infty)A$ for a representable $\ell$-group $A$ in ways which have elements in common with the method of the present paper. [5] remarks that [11] fails to leave the reader with a "concrete feeling for these hulls". Our method, which is restricted to $\mathbf{W}$, of course, considerably enhances concreteness.

Note too, that [11] shows that, via the construction there, if $A$ is an $f$-ring, so too are the hulls. In $\mathbf{W}$, that is considerably extended by Theorem 4.1 here.

The history of these hulls, and others, is complicated. See the references in [5], [11], and in [23] and [24], inter alia.

We apologize to neglected authors.

## 4 Some features of $p(\alpha)$ and $l(\alpha)$

The "features" involve: If $A$ has additional algebraic properties, then $p(\alpha)A$ and $l(\alpha)A$ do/do not possess those properties. And, how $p(\alpha)$ and $l(\alpha)$ treat boundedness. Our representations of the hulls informs these issues.

The "additional algebraic properties" are closures under sets of functorial implicit operations of $\mathbf{W}$. Such an operation is an $o \in C(\mathbb{R}^{\mathbb{N}})$, and $A$ is *$o$-closed* means: if $\{a_n\}_{n\in\mathbb{N}} \subseteq A$, then $o \circ \langle a_n \rangle \in A$ in the following sense. Let $S = \bigcap_{n\in\mathbb{N}} a_n^{-1}(\mathbb{R})$, dense in $YA$ by the Baire Category Theorem. $\langle a_n \rangle \colon S \to \mathbb{R}^{\mathbb{N}}$ is $\langle a_n \rangle(x) = (a_1(x), a_2(x), \dots) \in \mathbb{R}^{\mathbb{N}}$, so $o \circ \langle a_n \rangle \in C(S)$, and if this extends over $YA$ (automatically uniquely), we write "$o \circ \langle a_n \rangle \in A$". Then, for $\mathcal{O} \subseteq C(\mathbb{R}^{\mathbb{N}})$, $A$ is $\mathcal{O}$-closed if $A$ is $o$-closed for every $o \in \mathcal{O}$.

The classes $\mathcal{O}$-closed in **W** comprise exactly the full monoreflective subcategories $\mathcal{R}$ in **W** for which each reflection map is essential, and $\mathcal{R} = H\mathcal{R}$ (i.e., $\mathcal{R}$ is closed under **W**-homomorphic images).

An $o \in C(\mathbb{R}^{\mathbb{N}})$ is $n$-ary ($n < \omega$) if $o = \overline{o} \circ P_n$ for some $\overline{o} \in C(\mathbb{R}^n)$, where $P_n\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$ is projection onto the first $n$ coordinates, and finitary if $n$-ary for some $n$; $\mathcal{O} \subseteq C(\mathbb{R}^{\mathbb{N}})$ is finitary if each $o \in \mathcal{O}$ is finitary.

Examples of many 1-ary's are: For $p$ a prime, let $d(p)\colon \mathbb{R} \to \mathbb{R}$ be given by $d(p)(x) = \frac{x}{p}$. Then $A$ is divisible if $A$ is $\mathcal{O}$-closed for $\mathcal{O} = \{d(p) \mid p \text{ prime}\}$. Also, for $r \in \mathbb{R}$, let $m(r)\colon \mathbb{R} \to \mathbb{R}$ be given by $m(r)(x) = rx$. Then $A$ is a vector lattice if $A$ is $\mathcal{O}$-closed, where $\mathcal{O} = \{m(r) \mid r \in \mathbb{R}\}$.

The property "$A$ is an $f$-ring" is binary.

The property "$A$ is uniformly complete" is infinitary: This property is $u$-closed, for $u\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ given by $u((x_n)) = \sum_{n\in\mathbb{N}} \frac{1}{2^n}(|x_n| \wedge 1)$.

The largest $\mathcal{O}$-closed class is **W**, which has $\mathcal{O} = F(\omega)$, the **W**-object in $C(\mathbb{R}^{\mathbb{N}})$ generated by 1 and all projections $\mathbb{R}^{\mathbb{N}} \equiv \prod_{k\in\mathbb{N}} \mathbb{R}_k \xrightarrow{\pi(n)} \mathbb{R}_n$. This is the **W**-object free with respect to the functor $F$ from **W** to pointed sets, which has $F(u_A)$ the distinguished point in $F(A)$.

The smallest $\mathcal{O}$-closed class has $\mathcal{O} = C(\mathbb{R}^{\mathbb{N}})$, and this gives the class of "$\Phi$-algebras closed under countable composition" from [29], which coincides with the class $\{C(\mathcal{F}) \mid \mathcal{F} \text{ a frame}\}$ ([31], [33]).

All this is discussed in detail, for **W**, in [17], and in an abstract setting in [18].

**Theorem 4.1.** (a) *Suppose $\mathcal{O} \subseteq C(\mathbb{R}^{\mathbb{N}})$ is finitary. Then, if $A$ is $\mathcal{O}$-closed, so are $p(\alpha)A$ and $l(\alpha)A$.*

(b) *There are $A$ $u$-closed with $p(\omega)A = p(\infty)A$ and $l(\omega_1)A = l(\infty)A$, and these are not $u$-closed.*

*Proof.* (a) We suppose that $A \leq D(X)$ with $X$ ZD, and $A$ is $\mathcal{O}$-closed. We show that $A_X$ is too. This gives the result for $p(\alpha)A$ in (a) by virtue of Section 3.

To suppress tedious typography, we take liberties with the notation.

Let $o \in \mathcal{O}$. Since $\mathcal{O}$ is finitary, we may view $o \in C(\mathbb{R}^n)$ for some $n < \omega$. We need to show $o \circ \langle f_i \rangle \in A_X$ for $\{f_1, \dots, f_n\} \subseteq A_X$.

$f \in A_X$ means $f = \dot{\sum}_{U\in\mathcal{U}} a_U \chi(U)$, with $\mathcal{U}$ a finite clopen partition, and $a_U \in A$ for $U \in \mathcal{U}$. Write $f/\mathcal{U}$.

Given $f_i \in A_X$, $i \in \{1,\ldots,n\}$, and $f_i/\mathcal{U}_i$, let $\mathcal{V} = \bigwedge_{i\in I}\mathcal{U}_i$ ($\mathcal{V}$ is all $U_1 \cap \cdots \cap U_n$, $U_i \in \mathcal{U}_i$) and rewrite $f_i$ expressing $f_i/\mathcal{V}$ for each $i \in \{1,\ldots,n\}$ as $f_i = \sum_{V\in\mathcal{V}} a_{V,i}\chi(V)$, where $a_{V,i} = a_U$ if $U \in \mathcal{U}_i$ with $V \subseteq U$.

Then, $o \circ \langle f_i\rangle = \sum_{V\in\mathcal{V}}(o \circ \langle a_{V,i}\rangle)\chi(V)$, where $\langle a_{V,i}\rangle = \langle a_{V,1},\ldots,a_{V,n}\rangle$ (by the "liberties with notation"), and $o \circ \langle a_{V,i}\rangle \in A$, so $o \circ \langle f_i\rangle \in A_X$.

We turn to $l(\alpha)$. This goes as for $p(\alpha)$, with the necessary modification of replacing the finite partitions with quasi-partitions of size less than $\alpha$.

Suppose that $A \leq D(X)$ with $X \in D(\alpha)$, and $A$ is $\mathcal{O}$-closed. We show that $\overline{A_{X,\alpha}}$ is too. This gives the desired conclusion by virtue of Section 3.

We continue the "liberties with notation".

Again, let $o \in \mathcal{O}$ so $o \in C(\mathbb{R}^n)$, $n < \omega$. Now, $f \in \overline{A_{X,\alpha}}$ means $f = \dot{\sum}_{U\in\mathcal{U}} a_U\chi(U)$, with $\mathcal{U}$ a clopen quasi-partition in $X$, $|\mathcal{U}| < \alpha$, and $a_U \in A$ for $U \in \mathcal{U}$. Write $f/\mathcal{U}$.

Given $\{f_1,\ldots,f_n\} \subseteq \overline{A_X}$ with $f_i/\mathcal{U}_i$, $\bigwedge_{i=1}^n \mathcal{U}_i = \mathcal{V}$ as before are again a quasi-partition with $|\mathcal{V}| < \alpha$, and we rewrite $f_i$ expressing $f_i/\mathcal{V}$. Then, as in the last paragraph of the proof for $p(\alpha)$ above, $o \circ \langle f_i\rangle \in \overline{A_{X,\alpha}}$.

(b) An example is $A = C([0,1])$, the important features being from [35], as we now explain.

[35] says: Suppose $X$ compact metrizable. The maximum ring of quotients (in the general sense of Johnson-Utumi) of $C(X)$, called $Q(X)$, is uniformly complete if and only if the set $\mathrm{isol}(X)$ of isolated points of $X$ is dense in $X$ (in which case $Q(X) = C(\mathrm{isol}(X))$). (And for $X$ compact metrizable, this $Q(X)$ is just the usual "ring of fractions" (called $Q_{\mathrm{cl}}(X)$). See [14] about $Q(X)$, $Q_{\mathrm{cl}}(X)$, etc.)

Suppose $X$ is compact metrizable. Then for all $\alpha \geq \omega$, one has $d(\alpha)X = d(\infty)X = gX$ (the Gleason cover) by Lemma 2.11(c) (or [39, Theorem 3.5]), hence $Yp(\alpha)C(X) = gX$ for $\alpha \geq \omega$ and $Yl(\alpha)C(X) = gX$ for $\alpha > \omega$.

Now for all $G \in \mathbf{W}$, we have $YG = YBG$, and

$$(*)\ p(\omega)G \leq p(\alpha)G \leq l(\alpha)G \leq l(\infty)G.$$

Also, for any $X$, $Q(X) = l(\infty)C(X)$ ([38]).

Thus, for any compact metrizable $X$ and $G = C(X)$, uniform completeness of any item $A$ in $(*)$ means that $BA = C(gX)$ (since $BC(X)$ is a vector lattice, the Stone-Weierstrass Theorem yields that $l(\infty)C(X)$ is uniformly complete too). But, for $X = [0,1]$, that fails by [35, Theorem 2.6]. □

[8], [9], and precursor articles examine and classify hull operators $h$ by the equations that are satisfied by $h$ together with $B$ (the bounded coreflection in $\mathbf{W}$, $BA = \{a \in A \mid \exists n \in \mathbb{N}\, |a| \leq n \cdot u_A\}$). The cases in point here are: *$h$ commutes with $B$* (i.e., $hB = Bh$; we say *$h$ is CB*); *$h$ is antithesis of preserving boundedness* if, by definition, $h = hB$; we say $h$ is *anti-PB*. It is not hard to see that no $h$ is both (written CB$\,\cap\,$anti-PB$\,=\emptyset$).

The following is part of data exhibited in the Hasse Diagram [8, p.167]. We don't know if a full proof has been published; we present one now.

**Theorem 4.2.** [8]

(a) $(\omega \leq \alpha \leq \infty)$ *$p(\alpha)$ is CB.*

(b) $(\omega < \alpha \leq \infty)$ *$l(\alpha)$ is anti-PB.*

*Proof.* We note first the cases $\omega = \alpha$. For $p(\omega)$, the result is explicit in [25] (we prove it again, the same way below). And $l(\omega)$ is just the identity, and this fails anti-PB.

Now, keep in mind the representation of elements of $p(\alpha)A$ and $l(\alpha)A$ as of the form $\dot{\sum}_I$ as discussed in Section 3.

Suppose now $\omega < \alpha$.

Now, for all $A \in \mathbf{W}$, we know $YA \overset{\sigma}{\leftarrow} Yp(\alpha)A$ or $Yl(\alpha)A$ and the elements of $p(\alpha)A/l(\alpha)A$ are of the form $(*)$ $f = \dot{\sum}_{i\in I}(a_i \circ \sigma)\chi(U_i)$, for appropriate $I$ and $\{U_i\}_{i\in I}$.

(a) Since $BA$ is essential in $A$, $p(\alpha)BA \leq p(\alpha)A$. First, $p(\alpha)BA = Bp(\alpha)BA$ (called "$p(\alpha)$ preserves boundedness", and written *$p(\alpha)$ is PB*), because in $(*)$, if the finitely many $a_i$ are bounded, so is the finite sum $\dot{\sum}_{i\in I}(a_i \circ \sigma)\chi(U_i)$. Thus $p(\alpha)BA \leq Bp(\alpha)A$.

Reversely, if in $(*)$ the $f$ is bounded, say $|f| \leq n$, then if the $a_i$ are replaced by $(a_i \wedge n) \vee (-n) \in BA$, we get the same $f$, showing $f \in p(\alpha)BA$.

(b) Here, $\omega < \alpha$ and $Yl(\alpha)A = d(\alpha)YA \equiv X$.

Again $BA \leq A$ essential yields $l(\alpha)BA \leq l(\alpha)A$. For the reverse, take $f \in D(X)^+$. Then, some arithmetic shows $f^{-1}(\mathbb{R}) = \dot{\bigcup}_{n\in\mathbb{N}}U_n$, where $U_n \in$ clop$(X)$ for $n \in \mathbb{N}$ and $f|_{U_n} \leq n$ because $X \in D(\omega)$. Then $\{U_n\}_{n\in\mathbb{N}}$ is a quasi-partition in $X$ and

$$f = \dot{\sum_{n\in\mathbb{N}}} f\chi(U_n) = \dot{\sum_{n\in\mathbb{N}}}(f \wedge n)\chi(U_n).$$

Applying this to $f = a \circ \sigma$ yields $(**)$ $a \circ \sigma = \dot{\sum}_{n\in\mathbb{N}}((a \wedge n) \circ \sigma)\chi(U_n)$.

Now take $f \in (l(\alpha)A)^+$, per $(*)$ as $f = \dot{\sum}_{i\in I}(a_i \circ \sigma)\chi(U_i)$ and for each $i \in I$ insert $(**)$, obtaining

$$\begin{aligned} f &= \sum_{i\in I}\left(\sum_{i,n}((a_i \wedge n) \circ \sigma)\chi(U_n^i)\right)\chi(U_i) \\ &= \dot{\sum_{I\times\mathbb{N}}}((a_i \wedge n) \circ \sigma)\chi(U_n^i \cap U_i). \end{aligned}$$

Take note that the index set $I \times \mathbb{N}$ is of size less than $\alpha$. Thus $f \in l(\alpha)BA$. □

## 5 About $C(X)$

We first characterize $C(X) \in P(\alpha)$ (resp., $L(\alpha)$), then consider the inclusion $p(\alpha)C(X) \leq C(d(\alpha)X)$ for $X$ compact. Here it is understood that the distinguished weak unit of $C(X)$ is the constant function with value 1.

First, we summarize the general situation.

**Theorem 5.1.** $(\omega \leq \alpha \leq \infty)$

(a) *$A \in wP(\alpha)$ if and only if $BA \in wP(\alpha)$.*

(b) *If $A \in P(\alpha)$, then $BA \in P(\alpha)$.*

(c) *$[A \in P(\alpha) \Longleftarrow BA \in P(\alpha)]$ if and only if $A \in Loc$.*

(d) *If $A \in Loc$ and $\omega\mathrm{coz}(A) = \omega\mathrm{coz}(C(YA))$, then $[A \in P(\alpha)$ if and only if $YA \in D(\alpha)]$.*

*Proof.* (a) $YA = YBA$, so $\mathrm{coz}(A) = \mathrm{coz}(BA)$.

(b) If $A \in P(\alpha)$, then $A \in wP(\alpha)$, so $BA \in wP(\alpha)$ by (a); and $BA \in Loc$.

(c) Again, use (a). ([23, Remarks 2.3(a)] contains an example of a non-local $A$ with $YA = \beta\mathbb{N}$, which makes $BA \in P(\infty)$.)

(d) The case $\alpha > \omega$ is immediate from Corollary 2.4(b). The case $\alpha = \omega$ uses that $\mathrm{coz}(A) = \omega\mathrm{coz}(C(YA))$ (also, recall from Remark 2.2 that $P(\alpha) = wP(\alpha) \cap Loc$). □

The following adds some information to Corollary 2.5.

**Corollary 5.2.** ($\omega \leq \alpha \leq \infty$) *The following are equivalent*:

(a) $C(X) \in P(\alpha)$ (*or* $wP(\alpha)$).

(b) $BC(X) \in P(\alpha)$ (*or* $wP(\alpha)$).

(c) $X \in D(\alpha)$ (*and/or* $\beta X \in D(\alpha)$).

*Proof.* The "(or $wP(\alpha)$)" in (a) and (b) are because $C(X) \in Loc$. In (c), $X \in D(\alpha)$ if and only if $\beta X \in D(\alpha)$. The rest follows from Theorem 5.1. □

We turn to the question: When is $C(X) \in L(\alpha)$? First, we treat the case $\alpha = \omega_1$, due to Buskes ([6]), with a small elaboration of his result.

The space $X$ is called a $P$-space if all cozero sets are closed (see [15, 4J]).

**Theorem 5.3.** *These are equivalent.*

(a) $C(X) \in L(\omega_1)$.

(b) *Each countable disjoint family in* $C(X)^+$ *has an upper bound in* $C(X)$, *and* $\beta X$ *is ZD.*

(c) $X$ *is a* $P$*-space.*

*Proof.* (a) $\Longrightarrow$ (b). We need only that $\beta X$ is ZD, which follows since $L(\omega_1) \subseteq P(\omega_1)$ and for every $A \in \mathbf{W}$, if $A \in P(\omega_1)$, then $YA$ is ZD (use Corollary 2.5 and Definition 1.5).

(b) $\Longrightarrow$ (c). Take $U \in \omega\mathrm{coz}_X(C(X))$. We show $U$ is closed. Now $U = \mathrm{coz}_X(u)$ for $u \in C^*(X)$. Let $\tilde{u} \in C(\beta X)$ extend $u$, and let $\tilde{U} = \mathrm{coz}_X(\tilde{u})$. Then, $\tilde{U} \cap X = U$. Since $\beta X$ is ZD, $\tilde{U} = \bigcup_{n<\omega} U_n$ for $U_n \in \mathrm{clop}(\beta X)$, and then $U = \tilde{U} \cap X = \dot{\bigcup}_{n<\omega}(U_n \cap X)$ and $V_n \equiv U_n \cap X \in \mathrm{clop}(X)$. Here each $V_n \neq \emptyset$ (unless $U$ is already clopen, in which case we're done), so if $x \in \overline{U} - U$, any neighborhood $W$ of $x$ meets infinitely many $V_n$'s.

Now suppose (b), so there is $f \in C(X)$ with $f \geq n\chi(V_n)$ (pointwise) for all $n < \omega$. If $x \in \overline{U} - U$, then there is a neighborhood $W$ of $x$ with $f(x) - 1 \leq f(y) \leq f(x) + 1$ for every $y \in W$, so $W$ can meet only finitely many of the $V_n$'s. Thus there does not exist $x \in \overline{U} - U$.

(c) $\Longrightarrow$ (a). Given disjoint $\{f_n\}_{n<\omega} \subseteq C(X)^+$, $U = \dot{\bigcup}_{n<\omega}\mathrm{coz}_X(f_n) \in \omega\mathrm{coz}_X(C(X))$, so $U$ is closed (since $X$ is a $P$-space). Then $f$ defined as

$f|_{\mathrm{coz}_X(f_n)} = f_n|_{\mathrm{coz}_X(f_n)}$, and $f|_{X-U} = 0$ is locally continuous, thus continuous, and $f$ is the pointwise supremum of $\{f_n\}_{n<\omega}$. Thus $f = \bigvee_{n<\omega} f_n$ in $C(X)$. □

The preceding (Theorem 5.3) is a lemma to, and a special case of, the following.

**Theorem 5.4.** $(\omega < \alpha \leq \infty)$. *The following are equivalent.*

(a) $C(X) \in L(\alpha)$.

(b) $X \in D(\alpha)$ *(and/or* $\beta X \in D(\alpha)$*) and* $X$ *is a* $P$*-space.*

(c) $C(X) \in P(\alpha) \cap L(\omega_1)$.

*Proof.* (b) $\Longrightarrow$ (c), by Corollary 5.2 and Theorem 5.3.

(a) $\Longrightarrow$ (c). $L(\alpha) \subseteq P(\alpha)$ and $L(\alpha) \subseteq L(\omega_1)$.

(b)/(c) $\Longrightarrow$ (a). Recall from Lemma 2.7 that $D(\alpha) \subsetneq F(\alpha)$.

Suppose (b)/(c), and consider disjoint $\{f_i\}_{i\in I} \subseteq C(X)^+$, with $|I| < \alpha$. Since $X$ is a $P$-space, the $\mathrm{coz}_X(f_i)$ are clopen, and since $X \in D(\alpha)$, one sees $V \equiv \overline{\bigcup_{i\in I} \mathrm{coz}_X(f_i)}$ is clopen. Thus, $(\bigcup_{i\in I} \mathrm{coz}_X(f_i)) \cup (X - V) \equiv S$ is $C^*$-embedded, and dense. Define $f \in C(S)$ as $f|_{\mathrm{coz}_X(f_i)} = f_i|_{\mathrm{coz}_X(f_i)}$ for $i \in I$, and $f|_{X-V} = 0$. Extend $f$ to $\hat{f} \in D(X)$. Now, $\hat{f}^{-1}(\infty)$ is a zero-set, thus open (since $X$ is a $P$-space). Moreover, $\hat{f}^{-1}(\infty)$ intersects the dense set $S$ and therefore is empty. Now, one easily sees that $f = \bigvee_{i\in I} f_i$ in $C(X)$ (in fact, is the pointwise join). □

**Example 5.5.** (a) $(\alpha = \infty)$ Theorem 5.4 for $\alpha = \infty$ (also noted in [6]) says $C(X) \in L(\infty)$ if and only if $X$ is ED and a $P$-space. Isbell ([31]) has shown that if $X$ is ED and a $P$-space, and if the cardinal $|X|$ is non-measurable, then $X$ is discrete. But, if $Y$ is discrete and $|Y|$ is measurable, then the Hewitt realcompactification $\upsilon Y$ $(\supsetneq Y)$ is ED and a $P$-space (and not discrete). See [15].

(b) $(\omega < \alpha < \infty)$ This witnesses the other cases of Theorem 5.4 and will be used later. Let $D$ be discrete with $|D| \geq \alpha$, and let $X(\alpha) = D \cup \{x(\alpha)\}$ be $D$ with the point $x(\alpha)$ adjoined, where neighborhoods $U$ of $x(\alpha)$ have $|D - U| < \alpha$. Then $x(\alpha)$ is a $P$-point of $X(\alpha)$, so $X(\alpha)$ is an $\alpha$-disconnected $P$-space. Of course $YC(X(\alpha)) = \beta X(\alpha)$. Let $\dot{D}$ denote the one-point compactification of $D$ (caution: this use of the dot notation should not be confused with the earlier use of the dot to denote disjoint sums and unions).

Then $\beta X(\alpha) = d(\alpha)\dot{D}$, because (one can show) that $\beta X(\alpha)$ is the minimum $D(\alpha)$ cover of $\dot{D}$.

(c) ($\omega < \alpha \leq \infty$) One may wonder if Theorem 5.4(c) holds for any $A \in \mathbf{W}$, i.e., if $L(\alpha) = P(\alpha) \cap L(\omega_1)$ in $\mathbf{W}$. Here are examples to the contrary.

Let $X \in D(\alpha)$ be compact with a clopen quasi-partition of size $\omega_1$ (e.g., the $\beta X(\alpha)$ in (b)).

Let $F(X, \mathbb{R}) = \{f \in C(X) \mid |f(X)| < \omega\}$ and $A = l(\omega_1)F(X, \mathbb{R})$. We show $A \notin L(\alpha)$. Any $a \in A$ is of the form $a = \dot{\sum}_{i \in I} r_i \chi(U_i)$ with $\{U_i\}_{i \in I}$ a countable clopen quasi-partition in $X$. Put $U(a) = \bigcup_{i \in I} U_i$, which is dense, and evidently $|a(U(a))| \leq \omega$.

Let $\{V_j\}_{j \in J}$ be a clopen quasi-partition in $X$ with $|J| = \omega_1$. Take distinct $r_j \in \mathbb{R}$ ($j \in J$). Then $f = \bigvee_{j \in J} v_j \chi(V_j)$, extended over $X$, is in $l(\alpha)F(X, \mathbb{R})$ (recall that $\bigcup_{j \in J} V_j$ is $C^*$-embedded in $X$).

Supposing $f \in A$, we have $U(f)$ as above. But every $V_j \cap U(f) \neq \emptyset$, so $r_j \in f(U(f))$. Thus $|f(U(f))| \geq \omega_1$. The contradiction shows $f \notin A$, so $A \notin L(\alpha)$.

If $X$ is compact and $\omega \leq \alpha \leq \infty$, then $p(\alpha)C(X) \leq C(Yp(\alpha)C(X))$. We illustrate instances of $<$, and of $=$.

**Example 5.6.** (a) $Yp(\omega)C(\dot{\mathbb{N}}) = \beta\mathbb{N}$ (which is ED) and $p(\omega)C(\dot{\mathbb{N}}) < C(\beta\mathbb{N})$.

The first is easily checked. The second is shown much as Example 5.5(c): Any $a \in p(\omega)C(\dot{\mathbb{N}})$ has $|a(\beta\mathbb{N})| \leq \omega$, while there are $f \in C(\beta\mathbb{N})$ with $|f(\beta\mathbb{N})| = c$.

(b) ($\omega < \alpha \leq \infty$) We exhibit compact $X$ which is not $\omega$-disconnected so $C(X) < p(\omega)C(X)$, for which, for all $\omega < \alpha \leq \infty$, we have:

$$Yp(\omega)C(X) = Yp(\alpha)C(X) = d(\infty)X,$$

and

$$p(\omega)C(X) = p(\alpha)C(X) = C(d(\infty)X).$$

We recall a construction from [28], which see for details. Suppose $E$ is compact ED, and $p \neq q$ in $E$ are non-$P$-points. Let $\gamma$ be the quotient map identifying $p$ and $q$, let $E_\gamma = E - \{p, q\}$, and denote the resulting surjection $\dot{E}_\gamma \xleftarrow{\gamma} E$ (since the image under $\gamma$ is the one-point compactification of $E_\gamma$).

Then: $\dot{E}_\gamma \notin D(\omega)$ (because $p, q$ are not $P$-points), and $E$ is the unique proper cover of $\dot{E}_\gamma$.

To make our $X$: let $Y$ be infinite compact ED, $Y'$ a copy of $Y$, for $y \in Y$ denote the corresponding point in $Y'$ as $y'$. Let $E = Y + Y'$ (topological sum). Take $y$ a non-$P$-point of $Y$, with corresponding $y' \in Y'$, and identify $y$ with $y'$ in $E = Y + Y'$, per the construction outlined above. We now have $X = \dot{E}_\gamma \overset{\gamma}{\leftarrow} E$ with the properties mentioned, which result in:

$C(\dot{E}_\gamma) < p(\omega)C(\dot{E}_\gamma)$, $Yp(\omega)C(\dot{E}_\gamma) = E$, then using the material in Section 2, $p(\omega)C(\dot{E}_\gamma) = p(\alpha)C(\dot{E}_\gamma)$ for all $\omega \leq \alpha \leq \infty$.

We now show $p(\omega)C(\dot{E}_\gamma) = C(E)$.

By Theorem 3.4, $p(\omega)C(\dot{E}_\gamma)$ is comprised of all $\sum_{i\in I}(g_i \circ \gamma)\chi(U_i)$, where $|I| < \omega$, $\{g_i\}_{i\in I} \subseteq C(\dot{E}_\gamma)$, and $\{U_i\}_{i\in I}$ is a clopen partition of $E$. Now, any $f \in C(E)$ takes this form, indeed as

$$(*)\ f = (g \circ \gamma)\chi(Y) + (g' \circ \gamma)\chi(Y')$$

with $g, g' \in C(\dot{E}_\gamma)$, defined as follows.

For $y \in Y$, $g(y) \equiv f(y) \equiv g(y')$, and $g'(y) \equiv f(y') \equiv g'(y')$. These $g, g'$ factor through $\gamma$, which is quotient, thus we construct $g, g' \in C(\dot{E}_\gamma)$.

One checks $(*)$.

## 6 $bL(\alpha)$ (in W)

We consider application of our methods to one more family of hull classes, $bL(\alpha)$, which has received attention the literature, beginning with [38] for $\alpha = \infty$. We simply summarize the situation, with references and a few indications of proof.

**Definition 6.1.** ($\omega < \alpha \leq \infty$) $A$ is *boundedly laterally $\alpha$-complete* ($A \in bL(\alpha)$) if for every $|I| < \alpha$, $\{a_i\}_{i\in I} \subseteq A^+$ disjoint and $A$-bounded (i.e., there is $a \in A$ such that $a_i \leq a$ for every $i \in I$), the join $\bigvee_{i\in I} a_i$ exists in $A$.

**Theorem 6.2.** *In* **W**:

(a) ([23, 3.2], *with credits to* [38]) $L(\alpha) \subseteq bL(\alpha) \subseteq P(\alpha)$. *So* $A \in bL(\alpha)$ *implies* $YA \in D(\alpha)$.

(b) ([23, 2.9], *with credits to* [38]) $bL(\alpha)$ *is a hull class in* **W**. *Denoting the hull operator* $bl(\alpha)$, $p(\alpha) \leq bl(\alpha) \leq l(\alpha)$.

(c) *Suppose* $A \in \mathbf{W}$. *Denote the cover* $YA \overset{\sigma}{\leftarrow} d(\alpha)YA$. *Then* $Ybl(\alpha)A = d(\alpha)YA$, *and*

$$bl(\alpha)A = \{f \in l(\alpha)A \mid \exists a \in A \text{ with } |f| \leq a \circ \sigma\}.$$

(d) $bl(\alpha)$ *is* $CB$ (*commutes with the bounded coreflection* $B$).

(e) $C(X) \in bL(\alpha)$ *if and only if* $X \in D(\alpha)$ *if and only if* $C(X) \in P(\alpha)$.

*Proof.* (Sketch)

(a) and (b): See the references above.

(c) As with $l(\alpha)$ earlier in this paper.

(d) The last item in (c) shows that $bl(\alpha)$ is the "convex modification" of $l(\alpha)$, called $\overline{c}(l(\alpha))$ in [10]; since $l(\alpha)$ is anti-PB (Theorem 4.2), $\overline{c}(l(\alpha))$ is $CB$ by ([10]). (This can be shown directly from (c).)

(e) Apply Theorem 5.3 and (c) here, observing that, there, the condition "$X$ is a $P$-space" disappears because of the now added condition "$\{a_i\}$ is $A$-bounded". □

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