



# S-metrizability and the Wallman basis of a frame

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Dedicated to Themba Dube on the occasion of his 65<sup>th</sup> birthday

**Abstract.** The Wallman basis of a frame and the corresponding induced compactification was first investigated by Baboolal [2]. In this paper, we provide an intrinsic characterisation of S-metrizability in terms of the Wallman basis of a frame. Particularly, we show that a connected, locally connected frame is S-metrizable if and only if it has a countable locally connected and uniformly connected Wallman basis.

## 1 Introduction and Preliminaries

In [7], García-Máynez utilised the Wallman basis to construct locally connected compactifications and characterise S-metrizable spaces. The purpose of this paper is to generalise García-Máynez's characterisation of S-metrizable spaces. Thus we present a study of the Wallman basis of a frame, which was introduced by Baboolal in [2], and the corresponding construction of the Wallman compactification of frame. We present an isomorphism theorem for the Wallman compactification of dense metric sublocales of a

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metric frame. This together with Baboolal's work on insular ideals of a Wallman compactification (see [2]), leads to obtaining a generalization of García-Máynez's intrinsic characterisation of S-metrizability in terms of the Wallman basis of a frame.

We will first recall relevant material which will be required. A *frame*  $L$  is a complete lattice which satisfies the infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\},$$

for all  $x \in L, S \subseteq L$ , where  $\bigvee S$  denotes  $\bigvee \{s \mid s \in S\}$ . The top element of a frame  $L$  is denoted by  $1_L$  and the bottom element by  $0_L$ . If no ambiguity is caused then we simply use 0 and 1. A map  $h : L \longrightarrow M$  between frames is called a *frame homomorphism*, if  $h$  preserves all finite meets, including the top element, and all arbitrary joins, including the bottom element.  $h$  is *dense* if whenever  $h(x) = 0_M$  then  $x = 0_L$ .  $h$  is an *onto* frame homomorphism if for every  $y \in M$  there is an  $x \in L$  such that  $h(x) = y$ , and  $h$  is *one-to-one* if whenever  $h(a) = h(b)$ , then  $a = b$  for  $a, b \in L$ .  $h$  is a *frame isomorphism* if and only if  $h$  is onto, one-to-one.  $h$  has a *right adjoint*  $h_* : M \longrightarrow L$  satisfying the property that for all  $x \in M$  and for all  $y \in L$ ,  $x \leq h_*(y)$  iff  $h(x) \leq y$ .

Given a topological space  $X$ ,  $\mathcal{O}X = \{U \subseteq X \mid U \text{ is open}\}$  is a frame. For any continuous map  $f : X \longrightarrow Y$ , from the topological space  $X$  to a topological space  $Y$ , we have a frame homomorphism,

$$\begin{aligned} \mathcal{O}(f) : \mathcal{O}(Y) &\longrightarrow \mathcal{O}(X), \\ U &\mapsto f^{-1}(U). \end{aligned}$$

$\mathcal{O} : \mathbf{Top} \longrightarrow \mathbf{Frm}$  is a contravariant functor, where **Top** denotes the category of topological spaces and continuous maps, and **Frm** denotes the category of frames and frame homomorphisms. The contravariant functor is given by

$$\begin{aligned} \Sigma : \mathbf{Frm} &\longrightarrow \mathbf{Top}, \\ L &\mapsto \Sigma L. \end{aligned}$$

$\Sigma L$ , called the spectrum of  $L$ , is the space of all frame homomorphisms  $\psi : L \longrightarrow \underline{2}$ , where  $\underline{2}$  denotes the two element frame  $\{0, 1\}$ .  $\Sigma L$  has open sets  $\Sigma_a = \{\psi \in \Sigma L \mid \psi(a) = 1\}$ , for  $a \in L$ , and  $\{\Sigma_a \mid a \in L\}$  is a topology on  $\Sigma L$ . For any frame homomorphism  $h : L \longrightarrow M$ , we have  $\Sigma h : \Sigma L \longrightarrow \Sigma M$  which is defined by composing a frame homomorphism from  $\Sigma M$  with  $h$ , that is,  $\Sigma h(\psi) = \psi \cdot h$ , for  $\psi \in \Sigma M$ .

We now recall definitions of corresponding topological concepts for frames. The *pseudocomplement* of  $a$  is denoted  $a^*$  and is characterized by the following formula

$$a^* = \bigvee \{x \in L \mid a \wedge x = 0\}.$$

For elements  $a, b$  in a frame  $L$ , we say that  $a$  is *rather below*  $b$ , written  $a \prec b$ , if there exists an element  $c$  in  $L$  such that  $a \wedge c = 0$  and  $b \vee c = 1$ . A frame  $L$  is said to be *regular* if

$$a = \bigvee \{x \in L \mid x \prec a\}, \text{ for every } a \text{ in } L.$$

A frame  $L$  is *compact* if whenever  $\bigvee S = 1$ , then there exists a finite subset  $F$  of  $S$  such that  $\bigvee F = 1$ . An element  $x$  in a frame  $L$  is said to be *connected* if whenever  $x = b \vee c$  with  $b \wedge c = 0$  we have either  $b = 0$  or  $c = 0$ . Furthermore, a frame  $L$  is *connected* if its top element  $1$  is connected, and it is said to be *locally connected* provided each element in the frame can be written as the join of connected elements.

A *compactification* of a frame  $M$  is a compact regular frame  $L$  together with a dense onto homomorphism  $h : L \longrightarrow M$ , denoted by  $(L, h)$ . A compactification  $(L, h)$  is said to be *perfect* with respect to an element  $u \in M$ , if

$$h_*(u \vee u^*) = h_*(u) \vee h_*(u^*),$$

where  $h_* : M \longrightarrow L$  is the right adjoint of  $h$ . The compactification  $(L, h)$  is said to be a *perfect compactification* of  $M$ , if it is perfect with respect to every element  $u \in M$ .

We recall the following from Banaschewski [4]. A *strong inclusion* on a frame  $M$  is a binary relation  $\blacktriangleleft$  on  $M$  such that:

1. if  $x \leq a \blacktriangleleft b \leq y$  then  $x \blacktriangleleft y$ ,
2.  $\blacktriangleleft$  is a sublattice of  $M \times M$ ,

3.  $a \blacktriangleleft b \implies a \prec b$ ,
4.  $a \blacktriangleleft b \implies a \prec c \prec b$ , for some  $c \in M$ ,
5.  $a \blacktriangleleft b \implies b^* \blacktriangleleft a^*$ ,
6. for each  $a \in M$ ,  $a = \bigvee \{x \in M \mid x \blacktriangleleft a\}$ .

Let  $S(M)$  be the set of all strong inclusions on  $M$ . Let  $K(M)$  be the set of all compactifications of  $M$ , partially ordered by  $(L, h) \leq (N, f)$  if and only if there exists a frame homomorphism  $g : L \longrightarrow N$  making the following diagram commute.

$$\begin{array}{ccc} L & \xrightarrow{g} & N \\ h \downarrow & & \downarrow f \\ M & \xlongequal{\quad} & M \end{array}$$

Banaschewski [4] showed that  $K(M)$  is isomorphic to  $S(M)$  by defining maps  $K(M) \longrightarrow S(M)$  and  $S(M) \longrightarrow K(M)$ , which are inverses of each other and are order preserving. For the map  $S(M) \longrightarrow K(M)$ , let  $\blacktriangleleft$  be any strong inclusion on  $M$ . Let  $\gamma M$  be the set of all strongly regular ideals of  $M$  (That is, the ideals  $J$  of  $M$  such that  $x \in J$  implies there exists  $y \in J$  with  $x \blacktriangleleft y$ ). Then the join map  $\bigvee : \gamma M \longrightarrow M$  is dense and onto and  $\gamma M$  is a regular subframe of the frame of ideals of  $M$ ,  $\mathcal{I}(M)$ . Hence  $\bigvee : \gamma M \longrightarrow M$  is a compactification of  $M$  associated with the given  $\blacktriangleleft$ .

We will be concerned with metric frames [10], which are defined as follows: A *diameter* on a frame  $L$  is a map  $d : L \longrightarrow \mathbb{R}^+$  (the non-negative reals including  $\infty$ ) such that:

- (M1)  $d(0) = 0$ .
- (M2) If  $a \leq b$  then  $d(a) \leq d(b)$ .
- (M3) If  $a \wedge b \neq 0$  then  $d(a \vee b) \leq d(a) + d(b)$ .
- (M4) For each  $\varepsilon > 0$ ,  $U_\varepsilon^d = \{u \in L \mid d(u) < \varepsilon\}$  is a cover.

A diameter  $d$  is called *compatible* if

- (M5) For each  $a \in L$ ,  $a = \bigvee \{x \in L \mid x \triangleleft_d a\}$ , where  $x \triangleleft_d a$  means there exists  $U_\varepsilon^d$  such that

$$U_\varepsilon^d x = \bigvee \{u \in U_\varepsilon^d \mid u \wedge x \neq 0\} \leq a.$$

A diameter  $d$  is called a *metric diameter* if

- (M6) For each  $a \in L$  with  $d(a) < \infty$ , and  $\varepsilon > 0$  there exist  $u, v \leq a$ ,

$d(u), d(v) < \varepsilon$  such that  
 $d(a) - \varepsilon < d(u \vee v).$

A frame  $L$  with a specified compatible metric diameter  $d$  is called a *metric frame* and is denoted by  $(L, d)$ .  $(L, d)$  is said to be uniformly locally connected (abbreviated ulc) if given any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $d(a) < \delta$  then there exists a connected  $c$ ,  $a \leq c$  and  $d(c) < \varepsilon$ .

## 2 The Wallman compactification and dense sublocales of compact metric frames

Our first aim is to show that every compact metric frame is a Wallman compactification of each of its dense sublocales. In order to do so, we will generalise a result of Steiner [13]. The Wallman compactification for frames was first introduced by Johnstone [8]. We begin by defining the Wallman compactification of a frame  $M$ .

**Definition 2.1.** For any frame  $M$ ,  $B \subseteq M$  is called a *Wallman basis* of  $M$  if:

1. The bottom and top elements of  $M$  are in  $B$ , and  $a, b \in B$  implies that  $a \vee b \in B$  and  $a \wedge b \in B$ .
2. For every  $a \in M$ ,  $a = \bigvee \{b \in B \mid b \prec_B a\}$ , where  $b \prec_B a$  means that there exists  $c \in B$  such that  $b \wedge c = 0$  and  $c \vee a = 1$ .
3. For  $a, b \in B$  such that  $a \vee b = 1$ , there exist  $c, d \in B$  such that  $c \wedge d = 0$  and  $a \vee c = b \vee d = 1$ .

**Proposition 2.2** ([2]). *Let  $M$  be a regular frame and  $B$  a Wallman basis for  $M$ . Define  $a \blacktriangleleft_B b$  in  $M$  by*

$$a \blacktriangleleft_B b \text{ iff there exists } c \in B \text{ such that } a \prec_B c \prec_B b.$$

*Then  $\blacktriangleleft_B$  is a strong inclusion on  $M$ .*

From Proposition 2.2, the corresponding compactification associated with this Wallman basis  $B$ , denoted by  $\gamma_B M$ , is called the *Wallman compactification* of  $M$ . Here  $\gamma_B M$  consists of all strongly regular ideals of  $M$  associated with  $\blacktriangleleft_B$  and we have the join map  $\bigvee : \gamma_B M \longrightarrow M$ .

Baboolal [2] also showed how using the Wallman basis of a frame, one could obtain a Wallman basis for the corresponding Wallman compactification, using the join map.

**Proposition 2.3** ([2]). *Let  $B$  be a Wallman basis of  $M$ , then  $k(B)$  is a basis for  $\gamma_B M$  where  $k : M \longrightarrow \gamma_B M$  is the right adjoint of  $\bigvee : \gamma_B M \longrightarrow M$ .*

We now recall a result of Steiner [13], in spaces. Before generalising the result in frames, we also recall the statement of the Boolean Ultrafilter Theorem which is required in the next proof we present.

**Proposition 2.4** ([13]). *If  $(X, d)$  is a compact metric space, then it has a base  $\mathcal{B}$  of open regular sets which satisfies the following:  $B_1, B_2 \in \mathcal{B}$  implies that  $B_1 \cap B_2 \in \mathcal{B}$  and  $B_1 \cup B_2 \in \mathcal{B}$ . We say that  $\mathcal{B}$  is a ring consisting of regular open sets.*

**Definition 2.5.** An element  $a$  of a frame  $M$  is called *regular* if  $a = a^{**}$ .

**Remark 2.6.** We note the following:

1. If  $X$  is a topological space, then an open set  $U$  is said to be regular open if  $U = \text{int}(\overline{U})$ .
2. It can be shown that an open set  $U \in \mathcal{O}X$  is regular open if and only if  $U = U^{**}$ , where  $U^*$  refers to the pseudocomplement of  $U$  in the frame  $\mathcal{O}X$ .  
Thus an open set  $U$  is *regular open* if and only if  $U \in \mathcal{O}X$  is a regular element.

**Definition 2.7.** Let  $M$  be a frame and  $B \subseteq M$ .  $B$  is called a *ring* in  $M$ , if  $b_1, b_2 \in B$  implies that  $b_1 \wedge b_2 \in B$  and  $b_1 \vee b_2 \in B$ .

**Theorem 2.8** ([5], (**Boolean ultrafilter theorem**)). *Every non trivial Boolean algebra contains an ultrafilter (That is, a maximal proper filter).*

**Lemma 2.9** ([5]). *The following are equivalent:*

1. *Every non trivial Boolean algebra contains an ultrafilter.*
2. *Every compact regular frame  $M$  is spatial.*

3.  $\Sigma M \neq \emptyset$ , for every non-trivial, compact regular  $M$ .

In the next proposition we provide a generalisation Steiner's result.

**Proposition 2.10.** *If  $(M, d)$  is a compact metric frame, then  $M$  has a base  $B$  of regular elements, and  $B$  is a ring.*

*Proof.* If  $(M, d)$  is a compact metric frame then  $(M, d)$  is compact regular, since every metric frame is regular. If we assume the Boolean ultrafilter theorem, then by Lemma 2.9,  $M$  is spatial. Thus

$$\eta : M \longrightarrow \mathcal{O}\Sigma M, \text{ given by } \eta(a) = \Sigma_a = \{\psi : M \longrightarrow \underline{2} \mid \psi(a) = 1\},$$

for  $a \in M$ , is an isomorphism. From [6],  $(\Sigma M, \rho)$  is a metric space with metric given by

$$\rho(\xi, \eta) = \inf\{d(a) \mid \xi(a) = 1 = \eta(a)\}, \text{ for } \xi, \eta \in \Sigma M,$$

and  $\tau_\rho$  (the topology on  $\Sigma M$  generated by  $\rho$ ) is exactly  $\mathcal{O}\Sigma M$ . Furthermore, since  $M$  is compact,  $\mathcal{O}\Sigma M$  is compact and therefore  $\Sigma M$  is compact. So  $(\Sigma M, \rho)$  is a compact metric space and by Proposition 2.4, has a ring base  $\mathcal{B}$  consisting of regular open sets of  $\Sigma M$ . Each  $\Sigma_a \in \mathcal{B}$  is regular open in  $\Sigma M$ , so  $\Sigma_a \in \mathcal{O}\Sigma M$  is a regular element of the frame  $\mathcal{O}\Sigma M$ . Since  $\eta$  is an isomorphism,  $\eta^{-1}(\mathcal{B}) = B$  is a ring base for  $M$  consisting of regular elements. We can assume that  $0_M, 1_M$  is also in  $B$ , without loss of generality, since  $B \cup \{0_M, 1_M\}$  is still a ring base for  $M$ .  $\square$

The existence of a ring basis  $B$  of regular elements for a compact frame  $L$ , is now guaranteed by Proposition 2.10. Utilizing this, we can show that for any dense onto frame homomorphism  $h : L \rightarrow M$  where  $L$  is compact, the image of  $B$  under  $h$  is a Wallman basis.

**Proposition 2.11.** *Let  $h : L \longrightarrow M$  be a dense onto frame homomorphism. Suppose that  $L$  is compact and let  $B$  be a ring basis of regular elements of  $L$ . Then  $h(B)$  is a Wallman basis of  $M$ .*

*Proof.* (1): Take any  $h(b_1), h(b_2) \in h(B)$ , for  $b_1, b_2 \in B$ . Then  $h(b_1) \wedge h(b_2) = h(b_1 \wedge b_2)$ , and since  $B$  is a ring,  $h(b_1 \wedge b_2) \in h(B)$ . Now  $h(b_1) \vee h(b_2) = h(b_1 \vee b_2) \in h(B)$ , since  $B$  is a ring. Also,  $0_M = h(0_L) \in h(B)$  and  $1_M = h(1_L) \in h(B)$ .

(2): Take any  $w \in M$ . We will show that  $w = \bigvee \{h(b) \mid b \in B, h(b) \prec_{h(B)} w\}$ . Now  $w = h(a)$ , for some  $a \in L$  since  $h$  is onto, and  $a = \bigvee \{b \mid b \in B, b \prec a\}$ , since  $L$  is regular and  $B$  is a basis of  $L$ .

Claim 1:  $b \prec a \iff b \prec_B a$ . (2.1)

For  $b \prec a$ , we have  $b^* \vee a = 1_L$ . Now  $b^* = \bigvee \{c \mid c \in B, c \leq b^*\}$ , so by the compactness of  $L$ , we have  $c_1 \vee c_2 \vee \dots \vee c_n \vee a = 1_L$ , for suitable  $c_i \leq b^*$  and  $c_i \in B$  for  $i = 1, \dots, n$ . Since  $B$  is closed under finite joins, then  $c = c_1 \vee c_2 \vee \dots \vee c_n \in B$ , and so  $c \vee a = 1_L$  with  $c \in B$  and  $c \leq b^*$ . Hence  $c \wedge b = 0_L$ . Thus for  $b \prec a$ , we have shown that there exists  $c \in B$  such that  $b \wedge c = 0_L$  and  $c \vee a = 1_L$ . Hence  $b \prec_B a$ .

Now  $b \prec_B a$  implies  $b \prec a$  is immediate, hence  $b \prec a$  if and only if  $b \prec_B a$ .

We also note that  $b \prec_B a$  implies  $h(b) \prec_{h(B)} h(a)$ , since for  $c \in B$  such that  $b \wedge c = 0_L$  and  $c \vee a = 1_L$ , we have  $h(b) \wedge h(c) = 0_M$ ,  $h(c) \vee h(a) = 1_M$  and  $h(c) \in h(B)$ . Thus

$$\begin{aligned} w = h(a) &= h\left(\bigvee \{b \in B \mid b \prec a\}\right) \\ &= h\left(\bigvee \{b \in B \mid b \prec_B a\}\right) \\ &= \bigvee \{h(b) \mid b \in B, b \prec_B a\} \\ &\leq \bigvee \{h(b) \mid b \in B, h(b) \prec_{h(B)} h(a)\} \\ &= \bigvee \{h(b) \mid b \in B, h(b) \prec_{h(B)} w\} \\ &\leq w. \end{aligned}$$

So  $w = \bigvee \{h(b) \mid b \in B, h(b) \prec_{h(B)} w\}$ , as required.

(3): Take any  $h(a), h(b) \in h(B)$  with  $a, b \in B$ , such that  $h(a) \vee h(b) = 1_M$ . Then  $h(a \vee b) = 1_M$ . We have to show that there exist  $h(c), h(d) \in h(B)$  such that  $h(c) \wedge h(d) = 0_M$  and  $h(c) \vee h(a) = 1_M = h(d) \vee h(b)$ . Now  $a \vee b \in B$ , so  $a \vee b$  is regular.

Claim 2: If  $x \in L$  is regular and  $h(x) = 1_M$ , then  $x = 1_L$ . (2.2)

Assume that  $h(x) = 1_M$  where  $x$  is regular. Then,

$$\begin{aligned} (h(x))^* &= 0_M \\ \implies h(x^*) &= 0_M \end{aligned}$$



$$\begin{aligned} &\implies x^* = 0_L \quad (\text{since } h \text{ is dense}) \\ &\implies x^{**} = 1_L. \end{aligned}$$

Since  $x$  is regular,  $x = 1_L$ , as claimed.

Hence  $h(a \vee b) = 1_M$  implies  $a \vee b = 1_L$ . Now  $a = \bigvee \{x \mid x \in B, x \prec_B a\}$ , and  $b = \bigvee \{y \mid y \in B, y \prec_B b\}$ , therefore

$$\bigvee \{x \mid x \in B, x \prec_B a\} \vee \bigvee \{y \mid y \in B, y \prec_B b\} = 1_L.$$

Since  $M$  is compact, there exists  $x \in B$  with  $x \prec_B a$ , and there exists  $y \in B$  with  $y \prec_B b$  such that  $x \vee y = 1_L$ .  $x \prec_B a$  implies that there exists  $c \in B$ , such that  $x \wedge c = 0_L$  and  $c \vee a = 1_L$ , and  $y \prec_B b$  implies that there exists  $d \in B$  such that  $y \wedge d = 0_L$  and  $d \vee b = 1_L$ . Now,  $c \wedge d = (c \wedge d) \wedge (x \vee y) = (c \wedge d \wedge x) \vee (c \wedge d \wedge y) = 0_L$ . Hence  $h(c) \wedge h(d) = h(c \wedge d) = 0_M$ . Furthermore,  $h(c) \vee h(a) = 1_M$ , since  $c \vee a = 1_L$  and  $h(d) \vee h(b) = 1_M$ , since  $d \vee b = 1_L$ . Hence condition (3) is satisfied.

We have shown that  $h(B)$  is a Wallman basis of  $M$ . □

We briefly discuss an application of Proposition 2.11 to dense metric sublocales to guarantee the existence of a Wallman basis for all dense metric sublocales of compact frames. We recall the definition of a metric sublocale [9].

**Definition 2.12** ([9]). Let  $(L, \rho)$  be a metric frame and  $h : L \longrightarrow M$  be an onto frame homomorphism. For  $a \in M$ , let

$$d(a) = \inf \{ \rho(x) \mid a \leq h(x), x \in L \},$$

then  $d$  is a compatible metric diameter on  $M$ , and  $(M, d)$  is called a *metric sublocale* of  $(L, \rho)$ . Additionally, if  $h$  is a dense map, then we call  $(M, d)$  a *dense metric sublocale* of  $(L, \rho)$ .

**Corollary 2.13.** *Let  $(M, d)$  be a dense metric sublocale of  $(L, \rho)$ , with a dense onto homomorphism  $h : L \longrightarrow M$ . Suppose that  $L$  is compact and let  $B$  be a ring basis of regular elements of  $L$ . Then  $h(B)$  is a Wallman basis of  $M$ .*

*Proof.* Follows immediately from Proposition 2.11.  $\square$

We now recall a result that follows directly from the work of Banaschewski in [4].

**Theorem 2.14** ([4]). *Let  $M$  be a frame. Let  $(L, h)$  be a compactification of  $M$  associated with strong inclusion  $\blacktriangleleft_1$ , and let  $(N, f)$  be a compactification of  $M$  associated with strong inclusion  $\blacktriangleleft_2$ . If  $\blacktriangleleft_1 = \blacktriangleleft_2$ , then  $L \cong N$ .*

It is well-known in the literature that rather below relation,  $\prec$ , interpolates in a compact regular frame. We recall this fact below and then present an isomorphism theorem for the Wallman compactification of dense sublocales of a frame.

**Proposition 2.15** ([5]). *Let  $L$  be a compact regular frame. Then for any  $a, b \in L$ ,  $a \prec b$  implies that there exists  $c \in L$  such that  $a \prec c \prec b$ . We say that  $\prec$  interpolates in a compact regular frame.*

**Theorem 2.16.** *With the conditions as in Proposition 2.13, the Wallman compactification  $\gamma_{h(B)}M$  of  $M$  is isomorphic to  $L$  (as frames).*

*Proof.* By Proposition 2.2,  $h(B)$  determines a strong inclusion on  $M$  given by:  $x \blacktriangleleft y$  for  $x, y \in M$  if and only if there exists  $h(b)$  for  $b \in B$ , such that  $x \prec_{h(B)} h(b) \prec_{h(B)} y$ . Thus,  $\gamma_{h(B)}M = \{J \mid J \text{ is a strongly regular ideal}\}$ , where  $J$  is said to be strong regular if  $x \in J$  implies there exists  $y \in J$  such that  $x \blacktriangleleft y$ .  $\gamma_{h(B)}M$  is a compact regular frame and the join map

$$\begin{aligned} \bigvee : \gamma_{h(B)}M &\longrightarrow M \\ J &\mapsto \bigvee J \end{aligned}$$

makes  $\gamma_{h(B)}M$  a compactification of  $M$ . We will show that  $\gamma_{h(B)}M \cong L$ . Let  $h_*$  be the right adjoint of  $h$ . We note that  $h : L \longrightarrow M$  is a compactification of  $M$  (since  $L$  is a compact regular frame), and this induces a strong inclusion  $\blacktriangleleft_1$  on  $M$  given by

$$x \blacktriangleleft_1 y \iff h_*(x) \prec h_*(y).$$

It suffices to show that  $\blacktriangleleft = \blacktriangleleft_1$ , for then by Theorem 2.14,  $\gamma_{h(B)}M \cong L$ . So suppose that  $x \blacktriangleleft_1 y$ , for  $x, y \in M$ . Then  $h_*(x) \prec h_*(y)$  and therefore there exists  $z \in L$  such that  $h_*(x) \prec z \prec h_*(y)$ , since  $\prec$  interpolates in compact regular frames by Proposition 2.15. Now  $h_*(x) \prec z$  implies  $h_*(x)^* \vee z = 1_L$ , and so  $h_*(x)^* \vee \bigvee \{b \in B \mid b \leq z\} = 1_L$ . Since  $L$  is compact and  $B$  is closed under finite joins, it follows that  $h_*(x)^* \vee b = 1_L$ , for some  $b \in B$  with  $b \leq z$ . Now,

$$\begin{aligned} & h_*(x) \prec b \leq z \prec h_*(y) \\ \implies & h_*(x) \prec b \prec h_*(y) \quad (b \in B) \\ \implies & h_*(x) \prec_B b \prec_B h_*(y) \quad (\text{by equation (2.1)}) \\ \implies & hh_*(x) \prec_{h(B)} h(b) \prec_{h(B)} hh_*(y) \\ \implies & x \prec_{h(B)} h(b) \prec_{h(B)} y \\ \implies & x \blacktriangleleft y. \end{aligned}$$

Now suppose  $x \blacktriangleleft y$ , for  $x, y \in M$ . Then there exists  $b_1 \in B$  such that

$$x \prec_{h(B)} h(b_1) \prec_{h(B)} y.$$

$x \prec_{h(B)} h(b_1)$  implies there exists  $c_1 \in B$  such that  $x \wedge h(c_1) = 0_M$  and  $h(c_1) \vee h(b_1) = 1_M$ . Now  $h(h_*(x) \wedge c_1) = hh_*(x) \wedge h(c_1) = x \wedge h(c_1) = 0_M$ . So,  $h_*(x) \wedge c_1 = 0_L$ , since  $h$  is a dense map. Furthermore,  $c_1 \vee b_1 \in B$  and is therefore regular, so by equation (5.2), since  $h(c_1 \vee b_1) = h(c_1) \vee h(b_1) = 1_M$ , we must have  $c_1 \vee b_1 = 1_L$ . Hence we have shown that  $h_*(x) \prec b_1$ . Now, we observe that

$$\begin{aligned} & h(b_1) \leq y \\ \implies & b_1 \leq h_*(y) \\ \implies & h_*(x) \prec b_1 \prec h_*(y) \\ \implies & h_*(x) \prec h_*(y) \\ \implies & x \blacktriangleleft_1 y. \end{aligned}$$

Hence, we have shown that  $\gamma_{h(B)}M \cong L$ . □

### 3 S-metrizability and the Wallman basis

The purpose of this section is to provide one of the main results of this paper. We present a characterisation of S-metrizability in terms of the Wallman basis of a frame. S-metrizability of a frame is defined in terms of a connectedness property, called *Property S*, which is attributed to Sierpinski [12].

**Definition 3.1.** Let  $(L, d)$  be a metric frame.  $L$  is said to have *Property S* if, given any  $\varepsilon > 0$ , there exist  $a_1, a_2, \dots, a_n$  such that  $\bigvee_{i=1}^n a_i = 1$ , where  $a_i$  is connected and  $d(a_i) < \varepsilon$  for each  $i$ .

**Definition 3.2.** Let  $(L, d)$  be a metric frame. Then  $(L, d)$  is *S-metrizable* if  $L$  admits a metric diameter that has Property S.

In what remains, we will let  $M$  be a locally connected frame. We briefly state required theory from [2].

**Definition 3.3.** An element  $0 \neq c \in M$  is a *component* of an element  $u \in M$  if:

1.  $c$  is connected and  $c \leq u$ ,
2.  $c$  is maximally connected in  $u$  (that is, whenever  $c \leq x \leq u$  and  $x$  is connected in  $M$ , then  $c = x$ ).

**Remark 3.4.** We note that if  $c_\alpha$  and  $c_\beta$  are components of  $u \in M$ , and  $c_\alpha \neq c_\beta$ , then  $c_\alpha \wedge c_\beta = 0$

**Definition 3.5.** Let  $B \subseteq M$  be a Wallman basis. Then  $B$  is *locally connected* if each component of each element of  $B$  is also in  $B$ .

**Definition 3.6.** A basis  $B$  of  $M$  is *uniformly connected* if whenever  $A$  is finite,  $\bigvee A = 1$  and  $A \subseteq B$ , then there exists finite cover  $C \subseteq B$ , such that every  $c \in C$  is connected and  $C$  is a refinement of  $A$ , denoted by  $C \leq A$ .

**Definition 3.7.** Let  $\gamma_B M$  be the Wallman compactification associated with a Wallman basis  $B$ . An ideal  $J \in \gamma_B M$  is said to be *insular* if whenever  $x \in J$ , there exists  $y \in J$  having finitely many components, such that  $y \in B$  and  $x \blacktriangleleft y$ .

In [2], Baboolal obtained the following characterisation for insular ideals of the Wallman compactification associated with a locally connect Wallman basis on a locally connected frame. This result plays an important role in the main result of this paper.

**Theorem 3.8** ([2]). *Let  $B$  be a locally connected Wallman basis for the locally connected frame  $M$ . Then the following are equivalent:*

1.  $\bigvee : \gamma_B M \longrightarrow M$  is a perfect locally connected compactification of  $M$ .
2.  $B$  is uniformly connected.
3. Every ideal  $J$  in  $\gamma_B M$  is insular.

Although the following Lemma is known, it is difficult to find in the literature. We therefore, provide a proof for completeness.

**Lemma 3.9.** *Let  $M$  be a locally connected frame and  $c$  be a component of  $v \in M$ . Then  $v \leq c \vee c^*$ .*

*Proof.* By the local connectedness of  $M$ ,  $v = \bigvee_{\alpha \in I} c_\alpha$ , where  $c_\alpha$  are the components of  $v$ . Now  $c = c_\alpha$ , for some  $\alpha \in I$ . For  $\beta \neq \alpha$ ,  $c_\beta \wedge c_\alpha = 0_M$ , so  $c_\beta \leq c^*$ . This implies that  $\bigvee_{\beta \neq \alpha} c_\beta \leq c^*$ , therefore  $v = c \vee (\bigvee_{\beta \neq \alpha} c_\beta) \leq c \vee c^*$ .  $\square$

Next we shall show that S-metrizability of a locally connected frame ensures the existence of a countable locally connected and uniformly connected Wallman basis. Before doing this, we need the following two propositions on *countability*.

**Proposition 3.10.** *Every compact metric frame has a countable base.*

*Proof.* Let  $(M, d)$  be a compact metric frame. For each  $n \in \mathbb{N}$ ,  $U_{\frac{1}{n}}^d = \{x \in M \mid d(x) < \frac{1}{n}\}$  is a cover of  $M$ . So by compactness of  $M$ , there exists a finite cover  $F_n \subseteq U_{\frac{1}{n}}^d$ , of  $M$ .

Let  $B = \bigcup_{n=1}^{\infty} F_n$ . Then  $B$  is countable. We shall show that  $B$  is a base for  $M$ . Take any  $a \in M$ . Then  $a = \bigvee \{x \in M \mid x \triangleleft_d a\}$ . Now for any  $x \triangleleft_d a$ , there exists  $\varepsilon > 0$ , such that  $U_\varepsilon^d x \leq a$ . Take  $n \in \mathbb{N}$ , such that  $\frac{1}{n} < \varepsilon$ . Then  $U_{\frac{1}{n}}^d x \leq a$ . Since  $F_n$  is a cover of  $M$ ,

$$x = x \wedge \bigvee \{y \mid y \in F_n\} = \bigvee \{x \wedge y \mid y \in F_n, y \neq 0\}.$$

Now,  $y \in F_n$  and  $x \wedge y \neq 0$  imply that  $y \leq a$  and therefore

$$x \leq \bigvee \{y \in F_n \mid x \wedge y \neq 0\} \leq a.$$

Since  $a$  is a join of the  $x$ 's, it follows that  $a$  is a join of elements that come from  $B$ , since each  $y \in F_n$  is in  $B$ . So  $B$  is a countable base.  $\square$

**Proposition 3.11.** *If  $(M, d)$  is a compact locally connected metric frame, then each  $u \in M$  has only countably many components.*

*Proof.* Since  $M$  is locally connected,  $u = \bigvee_{\alpha \in I} c_\alpha$ , where  $c_\alpha$  are the components of  $u$ . Let  $B$  be a countable base of  $M$ . The existence of a countable base follows from Proposition 3.10. Each  $c_\alpha$  is a join of elements from  $B$ , so we can choose any  $b_\alpha \in B$  such that  $b_\alpha \leq c_\alpha$ . Whenever  $\alpha, \beta \in I$  and  $\alpha \neq \beta$ , then  $c_\alpha \wedge c_\beta = 0$ , therefore  $b_\alpha \neq b_\beta$ . Thus if  $I$  were uncountable, then  $\{b_\alpha\}_{\alpha \in I}$  would be uncountable. But  $\{b_\alpha\}_{\alpha \in I} \subseteq B$ , and  $B$  is countable. Hence  $\{b_\alpha\}_{\alpha \in I}$  is countable, which is a contradiction. Thus  $I$  is countable.  $\square$

**Theorem 3.12** ([11]). *Let  $(M, d)$  be a connected, locally connected metric frame. Then  $(M, d)$  is  $S$ -metrizable if and only if  $(M, d)$  has a perfect locally connected metrizable compactification.*

We are now ready to present the main result of this section:

**Proposition 3.13.** *Let  $(M, d)$  be a connected metric frame. If  $M$  is  $S$ -metrizable then  $M$  has a countable, locally connected and uniformly connected Wallman basis.*

*Proof.* Assume that  $(M, d)$  is  $S$ -metrizable. Then by Theorem 3.12,  $(M, d)$  has a perfect locally connected metrizable compactification (just take the completion of  $(M, d)$ ). Call it  $(L, \rho)$  and let  $h : (L, \rho) \rightarrow (M, d)$  be a dense surjection where  $\rho(a) = d(h(a))$ , for all  $a \in L$ . We know by Propositions 2.10 and 3.10, that whenever  $L$  is a compact metric frame, then  $L$  has a countable ring basis, call it  $B_0$ , consisting of regular elements. Let

$$C_0 = \{c \in L \mid c \text{ is a component of some } b \in B_0\},$$

and let  $B_1 = \langle B_0 \cup C_0 \rangle$ , where  $\langle B_0 \cup C_0 \rangle$  denotes the ring generated by  $B_0$  and  $C_0$ . We will now show that  $B_1$  is the smallest ring containing  $B_0$  and  $C_0$ . Since  $B_1 = \langle B_0 \cup C_0 \rangle$ , we have

$$B_1 = \{x \in L \mid x \text{ is a finite join of elements } y, y = \bigwedge_{i=1}^n t_i, t_i \in B_0 \cup C_0\}.$$

Take any  $x, y \in B_1$ . Then  $x = \bigvee_{i=1}^n x_i$ , where  $x_i = s_1^i \wedge \dots \wedge s_{k_i}^i$ , for  $s_j^i \in B_0 \cup C_0$ , and  $y = \bigvee_{i=1}^m y_i$ , where  $y_i = t_1^i \wedge \dots \wedge t_{q_i}^i$ , for  $t_{q_i}^i \in B_0 \cup C_0$ . Thus  $x \vee y = \bigvee_{i=1}^n x_i \vee \bigvee_{i=1}^m y_i$ , with  $x_i$  and  $y_i$  as described above, so  $x \vee y \in B_1$ . Now,  $x \wedge y = \bigvee_{i=1}^n \bigvee_{j=1}^m (x_i \wedge y_j)$ , where  $x_i \wedge y_j = s_1^i \wedge \dots \wedge s_{k_i}^i \wedge t_1^j \wedge \dots \wedge t_{q_j}^j$ . So  $x \wedge y \in B_1$ . Hence  $B_1$  is a ring containing  $B_0$  and  $C_0$ , and  $B_1$  is the smallest ring containing  $B_0$  and  $C_0$ .

We now show that  $B_1$  consists of regular elements. We first note that if  $x$  and  $y$  are regular then  $x \wedge y$  is regular. For if  $x = x^{**}$  and  $y = y^{**}$ , then  $(x \wedge y)^{**} = x^{**} \wedge y^{**} = x \wedge y$  and so  $x \wedge y$  is regular. If  $c \in C_0$ , then  $c$  is a component of some  $b \in B_0$ . Now  $c \leq b$  implies that  $c^{**} \leq b^{**} = b$ , so  $c \leq c^{**} \leq b$ . Now,  $c$  is connected therefore  $c^{**}$  is connected. Since  $c$  is a component we must have  $c = c^{**}$ . Hence  $c$  is regular. Thus  $B_0 \cup C_0$  consists of regular elements and finite meets of elements from  $B_0 \cup C_0$  is regular. Let

$$H_1 = \{x \in L \mid x \text{ is a finite meet of elements from } B_0 \cup C_0\}.$$

Then  $H_1$  consists of regular elements. For each  $m > 1$ , let

$$H_m = \{x \in L \mid x \text{ is a join of at most } m \text{ elements from } H_1\}.$$

We prove by induction that each  $H_m$  consists of regular elements. Let  $m > 1$  and assume  $H_{m-1}$  consists of regular elements. Let  $x \in H_m$ . Then there exist  $h_1, h_2, \dots, h_m \in H_1$  such that  $x = h_1 \vee h_2 \vee \dots \vee h_m$ . Take any  $h_k$  for  $1 \leq k \leq m$ . Now,

$$\begin{aligned} h_k &= b_1 \wedge \dots \wedge b_t \wedge c_1 \wedge \dots \wedge c_s \quad (\text{where } b_i \in B_0, c_j \in C_0) \\ &= b \wedge c_1 \wedge \dots \wedge c_s, \end{aligned}$$

where  $b = b_1 \wedge \dots \wedge b_t \in B_0$ , since  $B_0$  is a ring. Each  $c_i$  is a component of some  $v_i \in B_0$ , so

$$h_k = b \wedge c_1 \wedge \dots \wedge c_s$$

$$\leq b \wedge v_1 \wedge \dots \wedge v_s = d_k \in B_0.$$

Claim:  $d_k \leq h_k \vee h_k^*$ .

$h_k \vee h_k^* = (b \wedge c_1 \wedge \dots \wedge c_s) \vee (b \wedge c_1 \wedge \dots \wedge c_s)^*$ . Now  $h_k = b \wedge c_1 \wedge \dots \wedge c_s \leq c_i$ , for  $i = 1, \dots, s$ . So  $c_i^* \leq h_k^*$ , for each  $i$ , and thus  $c_1^* \vee \dots \vee c_s^* \leq h_k^*$ . Hence,

$$\begin{aligned} h_k \vee h_k^* &\geq (b \wedge c_1 \wedge \dots \wedge c_s) \vee (c_1^* \vee \dots \vee c_s^*) \\ &= (b \vee (c_1^* \vee \dots \vee c_s^*)) \wedge (c_1 \vee (c_1^* \vee \dots \vee c_s^*)) \wedge \dots \wedge (c_s \vee (c_1^* \vee \dots \vee c_s^*)) \\ &\geq b \wedge (c_1 \vee c_1^* \vee \dots \vee c_s^*) \wedge (c_2 \vee c_1^* \vee \dots \vee c_s^*) \wedge \dots \wedge (c_s \vee c_1^* \vee \dots \vee c_s^*) \\ &\geq b \wedge (c_1 \vee c_1^*) \wedge (c_2 \vee c_2^*) \wedge \dots \wedge (c_s \vee c_s^*) \quad (\text{By Lemma 3.9}) \\ &\geq b \wedge v_1 \wedge v_2 \wedge \dots \wedge v_s = d_k. \end{aligned}$$

Thus proving the claim that  $d_k \leq h_k \vee h_k^*$ .

We now show that  $x$  is regular. Firstly,  $x = h_1 \vee h_2 \vee \dots \vee h_m \leq d_1 \vee d_2 \vee \dots \vee d_m$ . Hence  $x^{**} \leq (d_1 \vee d_2 \vee \dots \vee d_m)^{**} = d_1 \vee d_2 \vee \dots \vee d_m$ , since  $d_i \in B_0$  and  $B_0$  is a ring of regular elements. Fix any  $i$ ,  $1 \leq i \leq m$ . Now  $x = h_i \vee \bigvee_{j \neq i} h_j$ , hence

$$\begin{aligned} x \wedge h_i^* &\leq \bigvee_{j \neq i} h_j \\ \implies (x \wedge h_i^*)^{**} &\leq (\bigvee_{j \neq i} h_j)^{**} = \bigvee_{j \neq i} h_j \quad (\text{by the induction hypothesis}) \\ \implies x^{**} \wedge h_i^{***} &\leq \bigvee_{j \neq i} h_j \\ \implies x^{**} \wedge h_i^* &\leq \bigvee_{j \neq i} h_j \end{aligned}$$

Hence for all  $i$ , we have  $x^{**} \wedge h_i^* \leq \bigvee_{j \neq i} h_j$ . Now,

$$\begin{aligned} x^{**} &\leq d_1 \vee d_2 \vee \dots \vee d_m \\ &\leq (h_1 \vee h_1^*) \vee (h_2 \vee h_2^*) \vee \dots \vee (h_m \vee h_m^*) \\ &= (h_1 \vee \dots \vee h_m) \vee (h_1^* \vee \dots \vee h_m^*) \\ &= x \vee h_1^* \vee h_2^* \dots \vee h_m^*. \end{aligned}$$

Therefore,

$$x^{**} = x^{**} \wedge (x \vee h_1^* \vee h_2^* \dots \vee h_m^*)$$



$$\begin{aligned}
 &= (x^{**} \wedge x) \vee (x^{**} \wedge h_1^*) \vee (x^{**} \wedge h_2^*) \vee \dots \vee (x^{**} \wedge h_m^*) \\
 &\leq x \vee \bigvee_{j \neq 1} h_j \vee \bigvee_{j \neq 2} h_j \vee \dots \vee \bigvee_{j \neq m} h_j \\
 &\leq x.
 \end{aligned}$$

Since  $x \leq x^{**}$ , we conclude that  $x = x^{**}$ , and so  $x$  is regular.

Thus by induction on  $m$ ,  $H_m$  consists of regular elements for every  $m > 1$ . Thus  $B_1 = \langle B_0 \cup C_0 \rangle$  consists of regular elements. Let  $B_2 = \langle B_1 \cup C_1 \rangle$ , where  $C_1$  consists of components of elements from  $B_1$ . By a similar argument in which we showed that  $B_1$  consists of regular elements, we can show that  $B_2$  consists of regular elements. Thus  $B = \bigcup_{n=0}^{\infty} B_n$ , consists of regular elements. Also,  $B$  is a ring basis since  $B_n \subseteq B_{n+1}$  and since each  $B_n$  is a ring basis. Hence by Proposition 2.13,  $h(B)$  is a Wallman basis for  $(M, d)$ .

Claim:  $h(B)$  is countable.

$B_0$  is countable and by Proposition 3.11, since  $(L, \rho)$  is compact and locally connected, it follows that  $C_0$  is countable. Thus the ring generated by  $B_0$  and  $C_0$  is countable. So  $B_1$  is countable. It follows that all  $B_n$ 's are countable. Hence  $B = \bigcup_{n=0}^{\infty} B_n$  is countable. In addition,  $h(B)$  would then be a countable base, as claimed.

We now show that  $h(B)$  is a locally connected base. Take any  $h(b) \in h(B)$ , where  $b \in B$ . Let  $w$  be a component of  $h(b)$ . We will show that  $w \in h(B)$ . Now,  $b \in B_n$  for some  $n$ . We know that  $b = \bigvee_{\alpha} \{c_{\alpha} \mid c_{\alpha} \text{ is a component of } b\}$ , therefore

$$h(b) = \bigvee_{\alpha} \{h(c_{\alpha}) \mid c_{\alpha} \text{ is a component of } b\}.$$

Since  $(L, \rho)$  is a perfect compactification, then each  $h(c_{\alpha})$  is connected in  $M$ . Now  $w \leq h(b)$  implies  $w \wedge h(c_{\alpha}) \neq 0_M$ , for some component  $c_{\alpha}$  of  $b$ . Therefore  $w \leq w \vee h(c_{\alpha}) \leq h(b)$ , with  $w \vee h(c_{\alpha})$  connected in  $M$ . Since  $w$  is a component of  $h(b)$ ,  $h(c_{\alpha}) \leq w$ . Also,

$$w = w \wedge h(b) = (w \wedge h(c_{\alpha})) \vee \bigvee_{\beta \neq \alpha} (w \wedge h(c_{\beta})).$$

Furthermore,

$$(w \wedge h(c_{\alpha})) \wedge \bigvee_{\beta \neq \alpha} (w \wedge h(c_{\beta})) = w \wedge (h(c_{\alpha}) \wedge \bigvee_{\beta \neq \alpha} h(c_{\beta})) = 0_M.$$

Whenever  $\beta \neq \alpha$ , then  $h(c_\alpha) \wedge h(c_\beta) = h(c_\alpha \wedge c_\beta) = h(0_L) = 0_M$ . So since  $w$  is connected and  $w \wedge h(c_\alpha) \neq 0_M$ , we must have  $\bigvee_{\beta \neq \alpha} (w \wedge h(c_\beta)) = 0_M$ . Hence  $w = w \wedge h(c_\alpha) \leq h(c_\alpha)$ , and therefore  $w = h(c_\alpha)$ . But  $c_\alpha$  is a component of  $b \in B_n$  for some  $n$ , so  $c_\alpha \in B_{n+1} \subseteq B$ . Thus  $w = h(c_\alpha)$  with  $c_\alpha \in B$ , showing that  $h(B)$  is a locally connected basis.

Lastly, we show that  $h(B)$  is a uniformly connected base. We have  $h : (L, \rho) \longrightarrow (M, d)$  is a perfect locally connected metrizable compactification of  $M$ , therefore by Proposition 2.16, the Wallman compactification  $\gamma_{h(B)}M \cong L$ , as frames. Thus  $\gamma_{h(B)}M$  is a perfect locally connected compactification of  $M$ . By Theorem 3.8,  $h(B)$  is uniformly connected. Thus  $h(B)$  is a countable, locally connected and uniformly connected Wallman base for  $M$ .  $\square$

## 4 The Main Result

The following metrization theory from [9], is required for our main result:

**Definition 4.1.** A subset  $X \subseteq M$  is said to be *locally finite* if there exists a cover  $W$  of  $M$  such that each  $w \in W$  meets only finitely many elements from  $X$ .

**Definition 4.2.** A basis  $B$  of  $M$  is said to be  $\sigma$ -*locally finite* if  $B = \bigcup_{n=1}^{\infty} B_n$  and each subset  $B_n$  is locally finite.

**Theorem 4.3** ([9]). *Let  $M$  be a regular frame.  $M$  is metrizable if and only if  $M$  has a  $\sigma$ -locally finite basis.*

We now establish our main result in this section, which is a generalisation of a result of García-Máynez [7].

**Theorem 4.4.** *Let  $M$  be a connected and locally connected frame. The following are equivalent:*

1.  $M$  is  $S$ -metrizable.
2.  $M$  has a countable locally connected and uniformly connected Wallman basis.
3.  $M$  has a countable locally connected Wallman basis  $B$  such that every ideal  $J$  of  $\gamma_B M$  is insular.

*Proof.*  $1 \implies 2$ : Follows from Proposition 3.13.

$2 \iff 3$ : Follows from Theorem 3.8.

$2 \implies 1$ : Suppose then that  $M$  has a countable locally connected and uniformly connected Wallman basis  $B$ . By Theorem 3.8,  $\bigvee : \gamma_B M \longrightarrow M$  is a perfect locally connected compactification of  $M$ . From Proposition 2.3,  $k(B)$  is a basis for  $\gamma_B M$ , where  $k : M \longrightarrow \gamma_B M$  is the right adjoint of  $\bigvee : \gamma_B M \longrightarrow M$ . Since  $B$  is countable, then  $k(B)$  is countable. Thus  $\gamma_B M$  has a countable basis and hence by Theorem 4.3  $\gamma_B M$  must be metrizable, since it is regular. So  $M$  has a perfect locally connected metrizable compactification and hence by Theorem 3.12 is  $S$ -metrizable.  $\square$

**Remark 4.5.** It should be noted that in [7], García-Máynez does not assume connectedness nor local connectedness. However, it is not expected that local connectedness could be relaxed in the point-free context.

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