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S-metrizability and the Wallman basis of a frame

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Dedicated to Themba Dube on the occasion of his 65^{th} birthday

Abstract. The Wallman basis of a frame and the corresponding induced compactification was first investigated by Baboolal [2]. In this paper, we provide an intrinsic characterisation of S-metrizability in terms of the Wallman basis of a frame. Particularly, we show that a connected, locally connected frame is S-metrizable if and only if it has a countable locally connected and uniformly connected Wallman basis.

1 Introduction and Preliminaries

In [7], García-Máynez utilised the Wallman basis to construct locally connected compactifications and characterise S-metrizable spaces. The purpose of this paper is to generalise García-Máynez's characterisation of S-metrizable spaces. Thus we present a study of the Wallman basis of a frame, which was introduced by Baboolal in [2], and the corresponding construction of the Wallman compactification of frame. We present an isomorphism theorem for the Wallman compactification of dense metric sublocales of a

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metric frame. This together with Baboolal's work on insular ideals of a Wallman compactification (see [2]), leads to obtaining a generalization of García-Máynez's intrinsic characterisation of S-metrizability in terms of the Wallman basis of a frame.

We will first recall relevant material which will be required. A *frame* L is a complete lattice which satisfies the infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s | s \in S\},$$

for all $x \in L, S \subseteq L$, where $\bigvee S$ denotes $\bigvee\{s \mid s \in S\}$. The top element of a frame L is denoted by 1_L and the bottom element by 0_L. If no ambiguity is caused then we simply use 0 and 1. A map $h : L \longrightarrow M$ between frames is called a *frame homomorphism*, if h preserves all finite meets, including the top element, and all arbitrary joins, including the bottom element. h is *dense* if whenever $h(x) = 0_M$ then $x = 0_L$. h is an *onto* frame homomorphism if for every $y \in L$ there is an $x \in M$ such that $h(x) = y$, and h is *one-to-one* if whenever $h(a) = h(b)$, then $a = b$ for $a, b \in L$. h is a frame *isomorphism* if and only if h is onto, one-to-one. h has a *right adjoint* $h_* : M \longrightarrow L$ satisfying the property that for all $x \in M$ and for all $y \in L$, $x \leq h_*(y)$ iff $h(x) \leq y$.

Given a topological space X, $\mathcal{O}X = \{U \subseteq X | \ U \text{ is open}\}$ is a frame. For any continuous map $f : X \longrightarrow Y$, from the topological space X to a topological space Y, we have a frame homomorphism,

$$\begin{aligned} \mathcal{O}(f) : \ \mathcal{O}(Y) &\longrightarrow \mathcal{O}(X), \\ U &\mapsto f^{-1}(U). \end{aligned}$$

$\mathcal{O} : \textbf{Top} \longrightarrow \textbf{Frm}$ is a contravariant functor, where **Top** denotes the category of topological spaces and continuous maps, and **Frm** denotes the category of frames and frame homomorphisms. The contravariant functor is given by

$$\begin{aligned} \Sigma : \textbf{Frm} &\longrightarrow \textbf{Top}, \\ L &\mapsto \Sigma L. \end{aligned}$$

ΣL, called the spectrum of L, is the space of all frame homomorphisms $\psi : L \longrightarrow \underline{2}$, where $\underline{2}$ denotes the two element frame $\{0,1\}$. ΣL has open sets $\Sigma_a = \{\psi \in \Sigma L \mid \psi(a) = 1\}$, for $a \in L$, and $\{\Sigma_a | \ a \in L\}$ is a topology on ΣL. For any frame homomorphism $h : L \longrightarrow M$, we have $\Sigma h : \Sigma M \longrightarrow \Sigma L$ which is defined by composing a frame homomorphism from ΣM with h, that is, $\Sigma h(\psi) = \psi \cdot h$, for $\psi \in \Sigma M$.

We now recall definitions of corresponding topological concepts for frames. The *pseudocomplement* of a is denoted a^* and is characterized by the following formula

$$a^* = \bigvee \{x \in L \mid a \wedge x = 0\}.$$

For elements a, b in a frame L, we say that a is *rather below* b, written $a \prec b$, if there exists an element c in L such that $a \wedge c = 0$ and $b \vee c = 1$. A frame L is said to be *regular* if

$$a = \bigvee \{x \in L \mid x \prec a\}, \text{ for every } a \text{ in } L.$$

A frame L is *compact* if whenever $\bigvee S = 1$, then there exists a finite subset F of S such that $\bigvee F = 1$. An element x in a frame L is said to be *connected* if whenever $x = b \vee c$ with $b \wedge c = 0$ we have either $b = 0$ or $c = 0$. Furthermore, a frame L is connected if its top element 1 is connected, and it is said to be *locally connected* provided each element in the frame can be written as the join of connected elements.

A *compactification* of a frame M is a compact regular frame L together with a dense onto homomorphism $h : L \longrightarrow M$, denoted by (L, h). A compactification (L, h) is said to be *perfect* with respect to an element $u \in M$, if

$$h_*(u \vee u^*) = h_*(u) \vee h_*(u^*),$$

where $h_* : M \longrightarrow L$ is the right adjoint of h. The compactification (L, h) is said to be a *perfect compactification* of M, if it is perfect with respect to every element $u \in M$.

We recall the following from Banaschewski [4]. A *strong inclusion* on a frame M is a binary relation $\blacktriangleleft$ on M such that:

1. if $x \leq a \blacktriangleleft b \leq y$ then $x \blacktriangleleft y$,
2. $\blacktriangleleft$ is a sublattice of $M \times M$,

3. $a \blacktriangleleft b \Longrightarrow a \prec b$,
4. $a \blacktriangleleft b \Longrightarrow a \prec c \prec b$, for some $c \in M$,
5. $a \blacktriangleleft b \Longrightarrow b^* \blacktriangleleft a^*$,
6. for each $a \in M$, $a = \bigvee\{x \in M \mid x \blacktriangleleft a\}$.

Let $S(M)$ be the set of all strong inclusions on M. Let $K(M)$ be the set of all compactifications of M, partially ordered by $(L, h) \leq (N, f)$ if and only if there exists a frame homomorphism $g : L \longrightarrow N$ making the following diagram commute.

$$\begin{array}{ccc} L & \xrightarrow{g} & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ M & = & M \end{array}$$

Banaschewski [4] showed that $K(M)$ is isomorphic to $S(M)$ by defining maps $K(M) \longrightarrow S(M)$ and $S(M) \longrightarrow K(M)$, which are inverses of each other and are order preserving. For the map $S(M) \longrightarrow K(M)$, let $\blacktriangleleft$ be any strong inclusion on M. Let γM be the set of all strongly regular ideals of M (That is, the ideals J of M such that $x \in J$ implies there exists $y \in J$ with $x \blacktriangleleft y$). Then the join map $\bigvee : \gamma M \longrightarrow M$ is dense and onto and γM is a regular subframe of the frame of ideals of M, $\mathcal{I}(M)$. Hence $\bigvee : \gamma M \longrightarrow M$ is a compactification of M associated with the given $\blacktriangleleft$.

We will be concerned with metric frames [10], which are defined as follows: A *diameter* on a frame L is a map $d : L \longrightarrow \mathbb{R}^+$ (the non-negative reals including ∞) such that:

(M1) $d(0) = 0$.
(M2) If $a \leq b$ then $d(a) \leq d(b)$.
(M3) If $a \wedge b \neq 0$ then $d(a \vee b) \leq d(a) + d(b)$.
(M4) For each $\varepsilon > 0$, $U^d_\varepsilon = \{u \in L \mid d(u) < \varepsilon\}$ is a cover.

A diameter d is called *compatible* if

(M5) For each $a \in L$, $a = \bigvee\{x \in L \mid x \lhd_d a\}$, where $x \lhd_d a$ means there exists U^d_ε such that

$$U^d_\varepsilon x = \bigvee\{u \in U^d_\varepsilon \mid u \wedge x \neq 0\} \leq a.$$

A diameter d is called a *metric diameter* if

(M6) For each $a \in L$ with $d(a) < \infty$, and $\varepsilon > 0$ there exist $u, v \leq a$,

$d(u), d(v) < \varepsilon$ such that

$$d(a) - \varepsilon < d(u \vee v).$$

A frame L with a specified compatible metric diameter d is called a *metric frame* and is denoted by (L, d). (L, d) is said to be uniformly locally connected (abbreviated ulc) if given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(a) < \delta$ then there exists a connected c, $a \leq c$ and $d(c) < \varepsilon$.

2 The Wallman compactification and dense sublocales of compact metric frames

Our first aim is to show that every compact metric frame is a Wallman compactification of each of its dense sublocales. In order to do so, we will generalise a result of Steiner [13]. The Wallman compactification for frames was first introduced by Johnstone [8]. We begin by defining the Wallman compactification of a frame M.

Definition 2.1. For any frame M, $B \subseteq M$ is called a *Wallman basis* of M if:

1. The bottom and top elements of M are in B, and $a, b \in B$ implies that $a \vee b \in B$ and $a \wedge b \in B$.
2. For every $a \in M$, $a = \bigvee\{b \in B \mid b \prec_B a\}$, where $b \prec_B a$ means that there exists $c \in B$ such that $b \wedge c = 0$ and $c \vee a = 1$.
3. For $a, b \in B$ such that $a \vee b = 1$, there exist $c, d \in B$ such that $c \wedge d = 0$ and $a \vee c = b \vee d = 1$.

Proposition 2.2 ([2])**.** *Let M be a regular frame and B a Wallman basis for M. Define $a \blacktriangleleft_B b$ in M by*

$$a \blacktriangleleft_B b \text{ iff there exists } c \in B \text{ such that } a \prec_B c \prec_B b.$$

Then $\blacktriangleleft_B$ is a strong inclusion on M.

From Proposition 2.2, the corresponding compactification associated with this Wallman basis B, denoted by $\gamma_B M$, is called the *Wallman compactification* of M. Here $\gamma_B M$ consists of all strongly regular ideals of M associated with $\blacktriangleleft_B$ and we have the join map $\bigvee : \gamma_B M \longrightarrow M$.

Baboolal [2] also showed how using the Wallman basis of a frame, one could obtain a Wallman basis for the corresponding Wallman compactification, using the join map.

Proposition 2.3 ([2]). *Let B be a Wallman basis of M, then $k(B)$ is a basis for $\gamma_B M$ where $k : M \longrightarrow \gamma_B M$ is the right adjoint of $\bigvee : \gamma_B M \longrightarrow M$.*

We now recall a result of Steiner [13], in spaces. Before generalising the result in frames, we also recall the statement of the Boolean Ultrafilter Theorem which is required in the next proof we present.

Proposition 2.4 ([13]). *If (X, d) is a compact metric space, then it has a base $\mathcal{B}$ of open regular sets which satisfies the following: $B_1, B_2 \in \mathcal{B}$ implies that $B_1 \cap B_2 \in \mathcal{B}$ and $B_1 \cup B_2 \in \mathcal{B}$. We say that $\mathcal{B}$ is a ring consisting of regular open sets.*

Definition 2.5. An element a of a frame M is called *regular* if $a = a^{**}$.

Remark 2.6. We note the following:

1. If X is a topological space, then an open set U is said to be regular open if $U = int(\overline{U})$.
2. It can be shown that an open set $U \in \mathcal{O}X$ is regular open if and only if $U = U^{**}$, where U^* refers to the pseudocomplement of U in the frame $\mathcal{O}X$.
   Thus an open set U is *regular open* if and only if $U \in \mathcal{O}X$ is a regular element.

Definition 2.7. Let M be a frame and $B \subseteq M$. B is called a *ring* in M, if $b_1, b_2 \in B$ implies that $b_1 \wedge b_2 \in B$ and $b_1 \vee b_2 \in B$.

Theorem 2.8 ([5], **(Boolean ultrafilter theorem)**). *Every non trivial Boolean algebra contains an ultrafilter (That is, a maximal proper filter).*

Lemma 2.9 ([5]). *The following are equivalent:*

1. *Every non trivial Boolean algebra contains an ultrafilter.*
2. *Every compact regular frame M is spatial.*

3. *$\Sigma M \neq \emptyset$, for every non-trivial, compact regular M.*

In the next proposition we provide a generalisation Steiner's result.

Proposition 2.10. *If (M,d) is a compact metric frame, then M has a base B of regular elements, and B is a ring.*

Proof. If (M,d) is a compact metric frame then (M,d) is compact regular, since every metric frame is regular. If we assume the Boolean ultrafilter theorem, then by Lemma 2.9, M is spatial. Thus

$$\eta : M \longrightarrow \mathcal{O}\Sigma M, \text{ given by } \eta(a) = \Sigma_a = \{\psi : M \longrightarrow \underline{2} \mid \psi(a) = 1\},$$

for $a \in M$, is an isomorphism. From [6], $(\Sigma M, \rho)$ is a metric space with metric given by

$$\rho(\xi, \eta) = \inf\{d(a) \mid \xi(a) = 1 = \eta(a)\}, \text{ for } \xi, \eta \in \Sigma M,$$

and τ_ρ (the topology on ΣM generated by ρ) is exactly $\mathcal{O}\Sigma M$. Furthermore, since M is compact, $\mathcal{O}\Sigma M$ is compact and therefore ΣM is compact. So $(\Sigma M, \rho)$ is a compact metric space and by Proposition 2.4, has a ring base $\mathcal{B}$ consisting of regular open sets of ΣM. Each $\Sigma_a \in \mathcal{B}$ is regular open in ΣM, so $\Sigma_a \in \mathcal{O}\Sigma M$ is a regular element of the frame $\mathcal{O}\Sigma M$. Since η is an isomorphism, $\eta^{-1}(\mathcal{B}) = B$ is a ring base for M consisting of regular elements. We can assume that $0_M, 1_M$ is also in B, without loss of generality, since $B \cup \{0_M, 1_M\}$ is still a ring base for M. □

The existence of a ring basis B of regular elements for a compact frame L, is now guaranteed by Proposition 2.10. Utilizing this, we can show that for any dense onto frame homomorphism $h : L \to M$ where L is compact, the image of B under h is a Wallman basis.

Proposition 2.11. *Let $h : L \longrightarrow M$ be a dense onto frame homomorphism. Suppose that L is compact and let B be a ring basis of regular elements of L. Then $h(B)$ is a Wallman basis of M.*

Proof. (1): Take any $h(b_1), h(b_2) \in h(B)$, for $b_1, b_2 \in B$. Then $h(b_1) \wedge h(b_2) = h(b_1 \wedge b_2)$, and since B is a ring, $h(b_1 \wedge b_2) \in h(B)$. Now $h(b_1) \vee h(b_2) = h(b_1 \vee b_2) \in h(B)$, since B is a ring. Also, $0_M = h(0_L) \in h(B)$ and $1_M = h(1_L) \in h(B)$.

(2): Take any $w \in M$. We will show that $w = \bigvee\{h(b) \mid b \in B, h(b) \prec_{h(B)} w\}$. Now $w = h(a)$, for some $a \in L$ since h is onto, and
$a = \bigvee\{b \mid b \in B, b \prec a\}$, since L is regular and B is a basis of L.
Claim 1: $b \prec a \Longleftrightarrow b \prec_B a$. (2.1)
For $b \prec a$, we have $b^* \vee a = 1_L$. Now $b^* = \bigvee\{c \mid c \in B, c \leq b^*\}$, so by the compactness of L, we have $c_1 \vee c_2 \vee ... \vee c_n \vee a = 1_L$, for suitable $c_i \leq b^*$ and $c_i \in B$ for $i = 1, ..., n$. Since B is closed under finite joins, then $c = c_1 \vee c_2 \vee ... \vee c_n \in B$, and so $c \vee a = 1_L$ with $c \in B$ and $c \leq b^*$. Hence $c \wedge b = 0_L$. Thus for $b \prec a$, we have shown that there exists $c \in B$ such that $b \wedge c = 0_L$ and $c \vee a = 1_L$. Hence $b \prec_B a$.
Now $b \prec_B a$ implies $b \prec a$ is immediate, hence $b \prec a$ if and only if $b \prec_B a$.

We also note that $b \prec_B a$ implies $h(b) \prec_{h(B)} h(a)$, since for $c \in B$ such that $b \wedge c = 0_L$ and $c \vee a = 1_L$, we have $h(b) \wedge h(c) = 0_M$, $h(c) \vee h(a) = 1_M$ and $h(c) \in h(B)$. Thus

$$\begin{aligned} w = h(a) &= h(\bigvee\{b \in B \mid b \prec a\}) \\ &= h(\bigvee\{b \in B \mid b \prec_B a\}) \\ &= \bigvee\{h(b) \mid b \in B, b \prec_B a\} \\ &\leq \bigvee\{h(b) \mid b \in B, h(b) \prec_{h(B)} h(a)\} \\ &= \bigvee\{h(b) \mid b \in B, h(b) \prec_{h(B)} w\} \\ &\leq w. \end{aligned}$$

So $w = \bigvee\{h(b) \mid b \in B, h(b) \prec_{h(B)} w\}$, as required.

(3): Take any $h(a), h(b) \in h(B)$ with $a, b \in B$, such that $h(a) \vee h(b) = 1_M$. Then $h(a \vee b) = 1_M$. We have to show that there exist $h(c), h(d) \in h(B)$ such that $h(c) \wedge h(d) = 0_M$ and $h(c) \vee h(a) = 1_M = h(d) \vee h(b)$. Now $a \vee b \in B$, so $a \vee b$ is regular.
Claim 2: If $x \in L$ is regular and $h(x) = 1_M$, then $x = 1_L$. (2.2)
Assume that $h(x) = 1_M$ where x is regular. Then,

$$\begin{aligned} & (h(x))^* = 0_M \\ \Longrightarrow\ & h(x^*) = 0_M \end{aligned}$$

$$\Longrightarrow x^* = 0_L \quad (\text{since } h \text{ is dense})$$
$$\Longrightarrow x^{**} = 1_L.$$

Since x is regular, $x = 1_L$, as claimed.

Hence $h(a \vee b) = 1_M$ implies $a \vee b = 1_L$. Now $a = \bigvee\{x \mid x \in B, x \prec_B a\}$, and $b = \bigvee\{y \mid y \in B,\ y \prec_B b\}$, therefore

$$\bigvee\{x \mid x \in B,\ x \prec_B a\} \vee \bigvee\{y \mid y \in B,\ y \prec_B b\} = 1_L.$$

Since M is compact, there exists $x \in B$ with $x \prec_B a$, and there exists $y \in B$ with $y \prec_B b$ such that $x \vee y = 1_L$. $x \prec_B a$ implies that there exists $c \in B$, such that $x \wedge c = 0_L$ and $c \vee a = 1_L$, and $y \prec_B b$ implies that there exists $d \in B$ such that $y \wedge d = 0_L$ and $d \vee b = 1_L$. Now, $c \wedge d = (c \wedge d) \wedge (x \vee y) = (c \wedge d \wedge x) \vee (c \wedge d \wedge y) = 0_L$. Hence $h(c) \wedge h(d) = h(c \wedge d) = 0_M$. Furthermore, $h(c) \vee h(a) = 1_M$, since $c \vee a = 1_L$ and $h(d) \vee h(b) = 1_M$, since $d \vee b = 1_L$. Hence condition (3) is satisfied.

We have shown that $h(B)$ is a Wallman basis of M. □

We briefly discuss an application of Proposition 2.11 to dense metric sublocales to guarantee the existence of a Wallman basis for all dense metric sublocales of compact frames. We recall the definition of a metric sublocale [9].

Definition 2.12 ([9])**.** Let (L, ρ) be a metric frame and $h : L \longrightarrow M$ be an onto frame homomorphism. For $a \in M$, let

$$d(a) = \inf\{\rho(x) \mid a \leq h(x), x \in L\},$$

then d is a compatible metric diameter on M, and (M, d) is called a *metric sublocale* of (L, ρ). Additionally, if h is a dense map, then we call (M, d) a *dense metric sublocale* of (L, ρ).

Corollary 2.13. *Let (M, d) be a dense metric sublocale of (L, ρ), with a dense onto homomorphism $h : L \longrightarrow M$. Suppose that L is compact and let B be a ring basis of regular elements of L. Then $h(B)$ is a Wallman basis of M.*

Proof. Follows immediately from Proposition 2.11. □

We now recall a result that follows directly from the work of Banaschewski in [4].

Theorem 2.14 ([4]). *Let M be a frame. Let (L, h) be a compactification of M associated with strong inclusion $\blacktriangleleft_1$, and let (N, f) be a compactification of M associated with strong inclusion $\blacktriangleleft_2$. If $\blacktriangleleft_1 = \blacktriangleleft_2$, then $L \cong N$.*

It is well-known in the literature that rather below relation, $\prec$, interpolates in a compact regular frame. We recall this fact below and then present an isomorphism theorem for the Wallman compactification of dense sublocales of a frame.

Proposition 2.15 ([5]). *Let L be a compact regular frame. Then for any $a, b \in L$, $a \prec b$ implies that there exists $c \in L$ such that $a \prec c \prec b$. We say that $\prec$ interpolates in a compact regular frame.*

Theorem 2.16. *With the conditions as in Proposition 2.13, the Wallman compactification $\gamma_{h(B)}M$ of M is isomorphic to L (as frames).*

Proof. By Proposition 2.2, $h(B)$ determines a strong inclusion on M given by: $x \blacktriangleleft y$ for $x, y \in M$ if and only if there exists $h(b)$ for $b \in B$, such that $x \prec_{h(B)} h(b) \prec_{h(B)} y$. Thus, $\gamma_{h(B)}M = \{J \mid J$ is a strongly regular ideal$\}$, where J is said to be strong regular if $x \in J$ implies there exists $y \in J$ such that $x \blacktriangleleft y$. $\gamma_{h(B)}M$ is a compact regular frame and the join map

$$\bigvee : \gamma_{h(B)}M \longrightarrow M$$
$$J \mapsto \bigvee J$$

makes $\gamma_{h(B)}M$ a compactification of M. We will show that $\gamma_{h(B)}M \cong L$. Let h_* be the right adjoint of h. We note that $h : L \longrightarrow M$ is a compactification of M (since L is a compact regular frame), and this induces a strong inclusion $\blacktriangleleft_1$ on M given by

$$x \blacktriangleleft_1 y \iff h_*(x) \prec h_*(y).$$

It suffices to show that $\blacktriangleleft = \blacktriangleleft_1$, for then by Theorem 2.14, $\gamma_{h(B)}M \cong L$. So suppose that $x \blacktriangleleft_1 y$, for $x, y \in M$. Then $h_*(x) \prec h_*(y)$ and therefore there exists $z \in L$ such that $h_*(x) \prec z \prec h_*(y)$, since $\prec$ interpolates in compact regular frames by Proposition 2.15. Now $h_*(x) \prec z$ implies $h_*(x)^* \vee z = 1_L$, and so $h_*(x)^* \vee \bigvee\{b \in B \mid b \leq z\} = 1_L$. Since L is compact and B is closed under finite joins, it follows that $h_*(x)^* \vee b = 1_L$, for some $b \in B$ with $b \leq z$. Now,

$$\begin{aligned} & h_*(x) \prec b \leq z \prec h_*(y) \\ \Longrightarrow\ & h_*(x) \prec b \prec h_*(y) \quad (b \in B) \\ \Longrightarrow\ & h_*(x) \prec_B b \prec_B h_*(y) \quad \text{(by equation (2.1))} \\ \Longrightarrow\ & hh_*(x) \prec_{h(B)} h(b) \prec_{h(B)} hh_*(y) \\ \Longrightarrow\ & x \prec_{h(B)} h(b) \prec_{h(B)} y \\ \Longrightarrow\ & x \blacktriangleleft y. \end{aligned}$$

Now suppose $x \blacktriangleleft y$, for $x, y \in M$. Then there exists $b_1 \in B$ such that

$$x \prec_{h(B)} h(b_1) \prec_{h(B)} y.$$

$x \prec_{h(B)} h(b_1)$ implies there exists $c_1 \in B$ such that $x \wedge h(c_1) = 0_M$ and $h(c_1) \vee h(b_1) = 1_M$. Now $h(h_*(x) \wedge c_1) = hh_*(x) \wedge h(c_1) = x \wedge h(c_1) = 0_M$. So, $h_*(x) \wedge c_1 = 0_L$, since h is a dense map. Furthermore, $c_1 \vee b_1 \in B$ and is therefore regular, so by equation (5.2), since $h(c_1 \vee b_1) = h(c_1) \vee h(b_1) = 1_M$, we must have $c_1 \vee b_1 = 1_L$. Hence we have shown that $h_*(x) \prec b_1$. Now, we observe that

$$\begin{aligned} & h(b_1) \leq y \\ \Longrightarrow\ & b_1 \leq h_*(y) \\ \Longrightarrow\ & h_*(x) \prec b_1 \prec h_*(y) \\ \Longrightarrow\ & h_*(x) \prec h_*(y) \\ \Longrightarrow\ & x \blacktriangleleft_1 y. \end{aligned}$$

Hence, we have shown that $\gamma_{h(B)}M \cong L$. □

3 S-metrizability and the Wallman basis

The purpose of this section is to provide one of the main results of this paper. We present a characterisation of S-metrizability in terms of the Wallman basis of a frame. S-metrizabilty of a frame is defined in terms of a connectedness property, called *Property S*, which is attributed to Sierpinski [12].

Definition 3.1. Let (L, d) be a metric frame. L is said to have *Property S* if, given any $\varepsilon > 0$, there exist $a_1, a_2, ..., a_n$ such that $\bigvee_{i=1}^{n} a_i = 1$, where a_i is connected and $d(a_i) < \varepsilon$ for each i.

Definition 3.2. Let (L, d) be a metric frame. Then (L, d) is *S-metrizable* if L admits a metric diameter that has Property S.

In what remains, we will let M be a locally connected frame. We briefly state required theory from [2].

Definition 3.3. An element $0 \neq c \in M$ is a *component* of an element $u \in M$ if:

1. c is connected and $c \leq u$,
2. c is maximally connected in u (that is, whenever $c \leq x \leq u$ and x is connected in M, then $c = x$).

Remark 3.4. We note that if c_α and c_β are components of $u \in M$, and $c_\alpha \neq c_\beta$, then $c_\alpha \wedge c_\beta = 0$

Definition 3.5. Let $B \subseteq M$ be a Wallman basis. Then B is *locally connected* if each component of each element of B is also in B.

Definition 3.6. A basis B of M is *uniformly connected* if whenever A is finite, $\bigvee A = 1$ and $A \subseteq B$, then there exists finite cover $C \subseteq B$, such that every $c \in C$ is connected and C is a refinement of A, denoted by $C \leq A$.

Definition 3.7. Let $\gamma_B M$ be the Wallman compactification associated with a Wallman basis B. An ideal $J \in \gamma_B M$ is said to be *insular* if whenever $x \in J$, there exists $y \in J$ having finitely many components, such that $y \in B$ and $x \blacktriangleleft y$.

In [2], Baboolal obtained the following characterisation for insular ideals of the Wallman compactification associated with a locally connect Wallman basis on a locally connected frame. This result plays an important role in the main result of this paper.

Theorem 3.8 ([2]). *Let B be a locally connected Wallman basis for the locally connected frame M. Then the following are equivalent*:

1. $\bigvee : \gamma_B M \longrightarrow M$ *is a perfect locally connected compactification of* M.
2. B *is uniformly connected.*
3. *Every ideal* J *in* $\gamma_B M$ *is insular.*

Although the following Lemma is known, it is difficult to find in the literature. We therefore, provide a proof for completeness.

Lemma 3.9. *Let M be a locally connected frame and c be a component of $v \in M$. Then $v \leq c \vee c^*$.*

Proof. By the local connectedness of M, $v = \bigvee_{\alpha \in I} c_\alpha$, where c_α are the components of v. Now $c = c_\alpha$, for some $\alpha \in I$. For $\beta \neq \alpha$, $c_\beta \wedge c_\alpha = 0_M$, so $c_\beta \leq c^*$. This implies that $\bigvee_{\beta \neq \alpha} c_\beta \leq c^*$, therefore $v = c \vee (\bigvee_{\beta \neq \alpha} c_\beta) \leq c \vee c^*$. □

Next we shall show that S-metrizability of a locally connected frame ensures the existence of a countable locally connected and uniformly connected Wallman basis. Before doing this, we need the following two propositions on *countability*.

Proposition 3.10. *Every compact metric frame has a countable base.*

Proof. Let (M, d) be a compact metric frame. For each $n \in \mathbb{N}$, $U^d_{\frac{1}{n}} = \{x \in M \mid d(x) < \frac{1}{n}\}$ is a cover of M. So by compactness of M, there exists a finite cover $F_n \subseteq U^d_{\frac{1}{n}}$, of M.

Let $B = \bigcup_{n=1}^{\infty} F_n$. Then B is countable. We shall show that B is a base for M. Take any $a \in M$. Then $a = \bigvee\{x \in M \mid x \lhd_d a\}$. Now for any $x \lhd_d a$, there exists $\varepsilon > 0$, such that $U^d_\varepsilon x \leq a$. Take $n \in \mathbb{N}$, such that $\frac{1}{n} < \varepsilon$. Then $U^d_{\frac{1}{n}} x \leq a$. Since F_n is a cover of M,

$$x = x \wedge \bigvee\{y \mid y \in F_n\} = \bigvee\{x \wedge y \mid y \in F_n, y \neq 0\}.$$

Now, $y \in F_n$ and $x \wedge y \neq 0$ imply that $y \leq a$ and therefore

$$x \leq \bigvee\{y \in F_n \mid x \wedge y \neq 0\} \leq a.$$

Since a is a join of the x's, it follows that a is a join of elements that come from B, since each $y \in F_n$ is in B. So B is a countable base. □

Proposition 3.11. *If (M, d) is a compact locally connected metric frame, then each $u \in M$ has only countably many components.*

Proof. Since M is locally connected, $u = \bigvee_{\alpha \in I} c_\alpha$, where c_α are the components of u. Let B be a countable base of M. The existence of a countable base follows from Proposition 3.10. Each c_α is a join of elements from B, so we can choose any $b_\alpha \in B$ such that $b_\alpha \leq c_\alpha$. Whenever $\alpha, \beta \in I$ and $\alpha \neq \beta$, then $c_\alpha \wedge c_\beta = 0$, therefore $b_\alpha \neq b_\beta$. Thus if I were uncountable, then $\{b_\alpha\}_{\alpha \in I}$ would be uncountable. But $\{b_\alpha\}_{\alpha \in I} \subseteq B$, and B is countable. Hence $\{b_\alpha\}_{\alpha \in I}$ is countable, which is a contradiction. Thus I is countable. □

Theorem 3.12 ([11])**.** *Let (M, d) be a connected, locally connected metric frame. Then (M, d) is S-metrizable if and only if (M, d) has a perfect locally connected metrizable compactification.*

We are now ready to present the main result of this section:

Proposition 3.13. *Let (M, d) be a connected metric frame. If M is S-metrizable then M has a countable, locally connected and uniformly connected Wallman basis.*

Proof. Assume that (M, d) is S-metrizable. Then by Theorem 3.12, (M, d) has a perfect locally connected metrizable compactification (just take the completion of (M, d)). Call it (L, ρ) and let $h : (L, \rho) \longrightarrow (M, d)$ be a dense surjection where $\rho(a) = d(h(a))$, for all $a \in L$. We know by Propositions 2.10 and 3.10, that whenever L is a compact metric frame, then L has a countable ring basis, call it B_0, consisting of regular elements. Let

$$C_0 = \{c \in L \mid c \text{ is a component of some } b \in B_0\},$$

and let $B_1 = \langle B_0 \cup C_0 \rangle$, where $\langle B_0 \cup C_0 \rangle$ denotes the ring generated by B_0 and C_0. We will now show that B_1 is the smallest ring containing B_0 and C_0. Since $B_1 = \langle B_0 \cup C_0 \rangle$, we have

$$B_1 = \{x \in L \mid x \text{ is a finite join of elements } y,\ y = \bigwedge_{i=1}^{n} t_i,\ t_i \in B_0 \cup C_0\}.$$

Take any $x, y \in B_1$. Then $x = \bigvee_{i=1}^{n} x_i$, where $x_i = s_1^i \wedge ... \wedge s_{k_i}^i$, for $s_j^i \in B_0 \cup C_0$, and $y = \bigvee_{i=1}^{m} y_i$, where $y_i = t_1^i \wedge ... \wedge t_{q_i}^i$, for $t_{q_i}^i \in B_0 \cup C_0$. Thus $x \vee y = \bigvee_{i=1}^{n} x_i \vee \bigvee_{i=1}^{m} y_i$, with x_i and y_i as described above, so $x \vee y \in B_1$. Now, $x \wedge y = \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} (x_i \wedge y_i)$, where $x_i \wedge y_i = s_1^i \wedge ... \wedge s_{k_i}^i \wedge t_1^i \wedge ... \wedge t_{q_i}^i$. So $x \wedge y \in B_1$. Hence B_1 is a ring containing B_0 and C_0, and B_1 is the smallest ring containing B_0 and C_0.

We now show that B_1 consists of regular elements. We first note that if x and y are regular then $x \wedge y$ is regular. For if $x = x^{**}$ and $y = y^{**}$, then $(x \wedge y)^{**} = x^{**} \wedge y^{**} = x \wedge y$ and so $x \wedge y$ is regular. If $c \in C_0$, then c is a component of some $b \in B_0$. Now $c \leq b$ implies that $c^{**} \leq b^{**} = b$, so $c \leq c^{**} \leq b$. Now, c is connected therefore c^{**} is connected. Since c is a component we must have $c = c^{**}$. Hence c is regular. Thus $B_0 \cup C_0$ consists of regular elements and finite meets of elements from $B_0 \cup C_0$ is regular. Let

$$H_1 = \{x \in L \mid x \text{ is a finite meet of elements from } B_0 \cup C_0\}.$$

Then H_1 consists of regular elements. For each $m > 1$, let

$$H_m = \{x \in L \mid x \text{ is a join of at most } m \text{ elements from } H_1\}.$$

We prove by induction that each H_m consists of regular elements. Let $m > 1$ and assume H_{m-1} consists of regular elements. Let $x \in H_m$. Then there exist $h_1, h_2, ..., h_m \in H_1$ such that $x = h_1 \vee h_2 \vee ... \vee h_m$. Take any h_k for $1 \leq k \leq m$. Now,

$$\begin{aligned} h_k &= b_1 \wedge ... \wedge b_t \wedge c_1 \wedge ... \wedge c_s \quad (\text{where } b_i \in B_0, c_j \in C_0) \\ &= b \wedge c_1 \wedge ... \wedge c_s, \end{aligned}$$

where $b = b_1 \wedge ... \wedge b_t \in B_0$, since B_0 is a ring. Each c_i is a component of some $v_i \in B_0$, so

$$h_k = b \wedge c_1 \wedge ... \wedge c_s$$

$$\leq b \wedge v_1 \wedge ... \wedge v_s = d_k \in B_0.$$

Claim: $d_k \leq h_k \vee h_k^*$.
$h_k \vee h_k^* = (b \wedge c_1 \wedge ... \wedge c_s) \vee (b \wedge c_1 \wedge ... \wedge c_s)^*$. Now $h_k = b \wedge c_1 \wedge ... \wedge c_s \leq c_i$, for $i = 1, ..., s$. So $c_i^* \leq h_k^*$, for each i, and thus $c_1^* \vee ... \vee c_s^* \leq h_k^*$. Hence,

$$\begin{aligned}
h_k \vee h_k^* &\geq (b \wedge c_1 \wedge ... \wedge c_s) \vee (c_1^* \vee ... \vee c_s^*) \\
&= (b \vee (c_1^* \vee ... \vee c_s^*)) \wedge (c_1 \vee (c_1^* \vee ... \vee c_s^*)) \wedge \ ... \ \wedge (c_s \vee (c_1^* \vee ... \vee c_s^*)) \\
&\geq b \wedge (c_1 \vee c_1^* \vee ... \vee c_s^*) \wedge (c_2 \vee c_1^* \vee ... \vee c_s^*) \wedge \ ... \ \wedge (c_s \vee c_1^* \vee ... \vee c_s^*) \\
&\geq b \wedge (c_1 \vee c_1^*) \wedge (c_2 \vee c_2^*) \wedge \ ... \ \wedge (c_s \vee c_s^*) \quad \text{(By Lemma 3.9)} \\
&\geq b \wedge v_1 \wedge v_2 \wedge \ ... \ \wedge v_s = d_k.
\end{aligned}$$

Thus proving the claim that $d_k \leq h_k \vee h_k^*$.
We now show that x is regular. Firstly, $x = h_1 \vee h_2 \vee ... \vee h_m \leq d_1 \vee d_2 \vee ... \vee d_m$. Hence $x^{**} \leq (d_1 \vee d_2 \vee ... \vee d_m)^{**} = d_1 \vee d_2 \vee ... \vee d_m$, since $d_i \in B_0$ and B_0 is a ring of regular elements. Fix any i, $1 \leq i \leq m$. Now $x = h_i \vee \bigvee_{j \neq i} h_j$, hence

$$\begin{aligned}
x \wedge h_i^* &\leq \bigvee_{j \neq i} h_j \\
\Longrightarrow \quad (x \wedge h_i^*)^{**} &\leq (\bigvee_{j \neq i} h_j)^{**} = \bigvee_{j \neq i} h_j \quad \text{(by the induction hypothesis)} \\
\Longrightarrow \quad x^{**} \wedge h_i^{***} &\leq \bigvee_{j \neq i} h_j \\
\Longrightarrow \quad x^{**} \wedge h_i^* &\leq \bigvee_{j \neq i} h_j
\end{aligned}$$

Hence for all i, we have $x^{**} \wedge h_i^* \leq \bigvee_{j \neq i} h_j$. Now,

$$\begin{aligned}
x^{**} &\leq d_1 \vee d_2 \vee ... \vee d_m \\
&\leq (h_1 \vee h_1^*) \vee (h_2 \vee h_2^*) \vee ... \vee (h_m \vee h_m^*) \\
&= (h_1 \vee ... \vee h_m) \vee (h_1^* \vee ... \vee h_m^*) \\
&= x \vee h_1^* \vee h_2^* ... \vee h_m^*.
\end{aligned}$$

Therefore,

$$x^{**} = x^{**} \wedge (x \vee h_1^* \vee h_2^* ... \vee h_m^*)$$

$$= (x^{**} \wedge x) \vee (x^{**} \wedge h_1^*) \vee (x^{**} \wedge h_2^*) \vee ... \vee (x^{**} \wedge h_m^*)$$
$$\leq x \vee \bigvee_{j\neq 1} h_j \vee \bigvee_{j\neq 2} h_j \vee ... \vee \bigvee_{j\neq m} h_j$$
$$\leq x.$$

Since $x \leq x^{**}$, we conclude that $x = x^{**}$, and so x is regular.
Thus by induction on m, H_m consists of regular elements for every $m > 1$. Thus $B_1 = \langle B_0 \cup C_0 \rangle$ consists of regular elements. Let $B_2 = \langle B_1 \cup C_1 \rangle$, where C_1 consists of components of elements from B_1. By a similar argument in which we showed that B_1 consists of regular elements, we can show that B_2 consists of regular elements. Thus $B = \bigcup_{n=0}^{\infty} B_n$, consists of regular elements. Also, B is a ring basis since $B_n \subseteq B_{n+1}$ and since each B_n is a ring basis. Hence by Proposition 2.13, $h(B)$ is a Wallman basis for (M, d).

Claim: $h(B)$ is countable.
B_0 is countable and by Proposition 3.11, since (L, ρ) is compact and locally connected, it follows that C_0 is countable. Thus the ring generated by B_0 and C_0 is countable. So B_1 is countable. It follows that all B_n's are countable. Hence $B = \bigcup_{n=0}^{\infty} B_n$ is countable. In addition, $h(B)$ would then be a countable base, as claimed.

We now show that $h(B)$ is a locally connected base. Take any $h(b) \in h(B)$, where $b \in B$. Let w be a component of $h(b)$. We will show that $w \in h(B)$. Now, $b \in B_n$ for some n. We know that $b = \bigvee_\alpha \{c_\alpha \mid c_\alpha \text{ is a component of } b\}$, therefore

$$h(b) = \bigvee_\alpha \{h(c_\alpha) \mid c_\alpha \text{ is a component of } b\}.$$

Since (L, ρ) is a perfect compactification, then each $h(c_\alpha)$ is connected in M. Now $w \leq h(b)$ implies $w \wedge h(c_\alpha) \neq 0_M$, for some component c_α of b. Therefore $w \leq w \vee h(c_\alpha) \leq h(b)$, with $w \vee h(c_\alpha)$ connected in M. Since w is a component of $h(b)$, $h(c_\alpha) \leq w$. Also,

$$w = w \wedge h(b) = (w \wedge h(c_\alpha)) \vee \bigvee_{\beta\neq\alpha} (w \wedge h(c_\beta)).$$

Furthermore,

$$(w \wedge h(c_\alpha)) \wedge \bigvee_{\beta\neq\alpha} (w \wedge h(c_\beta)) = w \wedge (h(c_\alpha) \wedge \bigvee_{\beta\neq\alpha} h(c_\beta)) = 0_M.$$

Whenever $\beta \neq \alpha$, then $h(c_\alpha) \wedge h(c_\beta) = h(c_\alpha \wedge c_\beta) = h(0_L) = 0_M$. So since w is connected and $w \wedge h(c_\alpha) \neq 0_M$, we must have $\bigvee_{\beta \neq \alpha}(w \wedge h(c_\beta)) = 0_M$. Hence $w = w \wedge h(c_\alpha) \leq h(c_\alpha)$, and therefore $w = h(c_\alpha)$. But c_α is a component of $b \in B_n$ for some n, so $c_\alpha \in B_{n+1} \subseteq B$. Thus $w = h(c_\alpha)$ with $c_\alpha \in B$, showing that $h(B)$ is a locally connected basis.
Lastly, we show that $h(B)$ is a uniformly connected base. We have $h : (L, \rho) \longrightarrow (M, d)$ is a perfect locally connected metrizable compactification of M, therefore by Proposition 2.16, the Wallman compactification $\gamma_{h(B)}M \cong L$, as frames. Thus $\gamma_{h(B)}M$ is a perfect locally connected compactification of M. By Theorem 3.8, $h(B)$ is uniformly connected. Thus $h(B)$ is a countable, locally connected and uniformly connected Wallman base for M. □

4 The Main Result

The following metrization theory from [9], is required for our main result:

Definition 4.1. A subset $X \subseteq M$ is said to be *locally finite* if there exists a cover W of M such that each $w \in W$ meets only finitely many elements from X.

Definition 4.2. A basis B of M is said to be $\sigma-$*locally finite* if $B = \bigcup_{n=1}^{\infty} B_n$ and each subset B_n is locally finite.

Theorem 4.3 ([9]). *Let M be a regular frame. M is metrizable if and only if M has a $\sigma-$locally finite basis.*

We now establish our main result in this section, which is a generalisation of a result of García-Máynez [7].

Theorem 4.4. *Let M be a connected and locally connected frame. The following are equivlent:*

1. *M is S-metrizable.*
2. *M has a countable locally connected and uniformly connected Wallman basis.*
3. *M has a countable locally connected Wallman basis B such that every ideal J of $\gamma_B M$ is insular.*

Proof. 1 $\Longrightarrow$ 2: Follows from Proposition 3.13.

2 $\Longleftrightarrow$ 3: Follows from Theorem 3.8.

2 $\Longrightarrow$ 1: Suppose then that M has a countable locally connected and uniformly connected Wallman basis B. By Theorem 3.8, $\bigvee : \gamma_B M \longrightarrow M$ is a perfect locally connected compactification of M. From Proposition 2.3, $k(B)$ is a basis for $\gamma_B M$, where $k : M \longrightarrow \gamma_B M$ is the right adjoint of $\bigvee : \gamma_B M \longrightarrow M$. Since B is countable, then $k(B)$ is countable. Thus $\gamma_B M$ has a countable basis and hence by Theorem 4.3 $\gamma_B M$ must be metrizable, since it is regular . So M has a perfect locally connected metrizable compactification and hence by Theorem 3.12 is S-metrizable. □

Remark 4.5. It should be noted that in [7], García-Máynez does not assume connectedness nor local connectedness. However, it is not expected that local connectedness could be relaxed in the point-free context.

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