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On the physical "massless" vector field in de Sitter space

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Abstract

The ”massless” vector field equation in de Sitter (dS) 4-dimensional space is gauge invariant under some special gauge transformations. It is also gauge invariant in ambient space notations in which the field equation is written in terms of the Casimir operators of dS group. In this paper the "massless" vector field equation has been solved in the physical case. It has been shown that the solution can be written as the multiplication of a generalized polarization vector and a ”massless” conformally coupled scalar field in the ambient space notations. The physical vector two-point function has been calculated using ambient space formalism and its zero curvature limit has been considered. It is shown that the physical vector two-point can be written in terms of the conformally coupled massless scalar two-point function in the ambient space notations. The two-point function is expressed in terms of dS intrinsic coordinates from its ambient space counterpart, which is dS-invariant and is free from any divergences.

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1. Introduction

The recent observational data are strongly in favor of a positive acceleration of the present universe. Therefore, in a first approximation, the space-time might be considered as a dS space-time and the quantization of the massless vector field (spin-1) on dS space, presents an excellent modality for further researches.

In dS space-time, a field is called "massless" if it propagates on the dS light-cone and corresponds to a massless Poincare' field at $H = 0$ and a field is "massive" if it propagates inside the light-cone and corresponds to a massive Poincare' field in the zero curvature limit.

In the previous studies, the "massless" vector field was considered in flat coordinates system covering only the one-half of the dS hyperboloid [1]. Allen and Jacobson [2] calculated the "massless" vector two-point function in terms of the geodesic distance. The two-point function contains a logarithmic divergent term [2]. "Massive" and "massless" free vector fields are considered in [3,4], respectively and "massive" and "massless" free tensor fields are investigated in [5,6] respectively. In this article, the physical sector (i.e. divergenceless) of the "massless" spin-1 field in dS ambient space has been considered. It is the only part, which appears in the interacting fields.

The organization of this paper is based on the following order. In Section 2, we recall the dS vector field equation. The field equation is written in terms of the Casimir operators of dS group and its gauge invariance is considered. The solution to the field equation, in ambient space notations is presented in Section 3. It is shown that the solution can be written as the multiplication of a generalized polarization vector and a "massless" conformally coupled scalar field in ambient space notations. Section 4 is devoted to the calculation of the physical "massless" vector two-point function $W_{\alpha\beta\alpha'\beta'}(x,x')$ in the ambient space notations, and its flat limit is considered. The two-point function is expressed in terms of the dS intrinsic coordinates from its ambient space counterpart. The two-point functions are free from the logarithmic divergence.

2. Notation

The dS space-time is made identical to the four dimensional one-sheeted hyperboloid:

$$X_H = \{x \in R^5;\ x^2 = \eta_{\alpha\beta}x^\alpha x^\beta = -H^{-2}\}, \qquad \alpha,\beta,\cdots = 0,1,2,3,4 \tag{1}$$

where $\eta_{\alpha\beta} = diag\,(1,-1,-1,-1,-1)$. The dS metric is:

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta\big|_{x^2=-H^{-2}} = g_{ab}^{dS}dX^a dX^b,\quad a,b,\cdots = 0,1,2,3, \tag{2}$$

where X^a's are the 4-space-time intrinsic coordinates in dS hyperboloid. Different coordinate systems can be chosen [7]. Any geometrical object in this space can be written in terms of the four local intrinsic coordinates X^a or in terms of the five global ambient space coordinates x^α. The metric (2) is a solution to Einstein's field equation with the cosmological constant $\Lambda = 3H^2$:

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$$R_{ab} - \frac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} = 0. \tag{3}$$

The Lagrangian density for free "massless" vector fields $A_\mu(X)$ propagating on dS space reads ($\hbar = 1$) [2]:

$$L = \frac{1}{4} F^{ab} F_{ab}, \tag{4}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ is the electromagnetic stress tensor. The variational principle applied to (4) yields the field equation:

$$\nabla_a F^{ab} = \nabla_a (\nabla^a A^b - \nabla^b A^a) = 0. \tag{5}$$

Since $[\nabla_a, \nabla_b] A_c = -H^2 (g_{ac} A_b - g_{bc} A_{ba})$, one obtains the wave equation:

$$(\nabla^2 + 3H^2) A_a(X) - \nabla_a \nabla \cdot A\,(X) = 0. \tag{6}$$

This field equation is identically satisfied by the gauge vector fields of the form $A_\alpha = \nabla_a \varphi$ because of the property $(\nabla^2 \nabla_a - \nabla_a \nabla^2)\varphi = -3H^2 \nabla_a \varphi$. Thus (6) is invariant under gauge transformation:

$$A_a \to A'_a = A_a + \nabla_a \varphi, \tag{7}$$

where φ is an arbitrary scalar field. The wave equation with gauge fixing parameter *c* reads:

$$(\nabla^2 + 3\mathrm{H}^2) A_a(X) - c \nabla_a \nabla \cdot A(X) = 0. \tag{8}$$

Our aim is now to write the field equation (6) in terms of the Casimir operator of the dS group $So_0(1,4)$. For this purpose we introduce a transverse vector field $K_a(x) (i \cdot e \cdot x \cdot K = 0)$ defined on the ambient space, which is related to $A_\mu(X)$ through:

$$A_a(X) = \frac{\partial x^\alpha}{\partial X^a} K_\alpha\big(x(X)\big). \tag{9}$$

The kinematical group of the dS space is the 10-parameter group $So_0(1,4)$ (connected component of the identity in $O(1,4)$), which is one of the two possible deformations of the Poincare' group. There are two Casimir operators:

$$Q_1^{(1)} = -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta}, \quad Q_1^{(2)} = -W_\alpha W^\alpha, \tag{10}$$

where

$W_a = -\frac{1}{8} \varepsilon_{\alpha\beta\gamma\delta\eta} L^{\beta\gamma} L^{\delta\eta}$, with 10 infinitesimal generators:

$$L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}. \tag{11}$$

The subscript 1 in $Q_1^{(1)}$ and $Q_1^{(2)}$ reminds that the carrier space is constituted by vectors. The orbital part $M_{\alpha\beta}$, and the action of the spinorial part $S_{\alpha\beta}$ on a vector field $K_\alpha(x)$ defined on the ambient space read respectively [5]:

$$M_{\alpha\beta} = -i\big(x_\alpha \partial_\beta - x_\beta \partial_\alpha\big),$$
$$S_{\alpha\beta} K_\gamma = -i\big(\eta_{\alpha\beta} K_\gamma - \eta_{\beta\gamma} K_\alpha\big). \tag{12}$$

The symbol $\varepsilon_{\alpha\beta\gamma\delta\eta}$ holds for the usual antisymmetrical tensor. The action of the Casimir operator $Q_1^{(1)}$ on $K_\alpha(x)$ can be written in the more explicit form:

$$Q_1^{(1)} K_\alpha(x) = \big(Q_1^{(1)} - 2\big) K_\alpha(x)$$
$$= 2x\bar{\partial} \cdot K(x) - 2\bar{\partial} x \cdot K_\alpha(x), \tag{13}$$

where, $Q_0^{(1)} = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} = -H^{-2} \big(\bar{\partial}\big)^2$, is the scalar Casimir operator and $\partial_\alpha = \bar{\theta}_{\alpha\beta} \partial^\beta$ and $\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$ is a transverse projector. It is easily shown that the metric g_{ab}^{dS} corresponds to the transverse projector $\theta_{\alpha\beta}$, that is:

$$g_{ab}^{dS}(X) = \frac{\partial x^\alpha}{\partial X^a} \frac{\partial x^\beta}{\partial X^b} \theta_{\alpha\beta}\big(x(X)\big). \tag{14}$$

We are now in position to express the wave equation (6) by using the Casimir operators. The d'Alembertian operator becomes

$$\nabla^c \nabla_c A_a = \nabla^2 A_a = \frac{\partial x^\alpha}{\partial X^a} \big(H^2 Q_0^{(1)} K_\alpha + H^2 K_\alpha$$
$$+ 2H^2 x_\alpha \bar{\partial} \cdot K\big), \tag{15}$$

and the Eq. (6) with this new notation reads

$$\big(Q_0^{(1)} - 2\big) K_\alpha(x) + 2x\bar{\partial} \cdot K(x) + D_1 \partial \cdot K(x) + D_1 \partial$$
$$\cdot K(x) = 0, \qquad D_1 = H^{-2} \bar{\partial}. \tag{16}$$

Finally using (13) one obtains the field equation formulated in terms of the Casimir operator $Q_1^{(1)}$ as:

$$Q_0^{(1)} K_\alpha(x) + D_1 \partial \cdot K(x) = 0, \tag{17}$$

which is invariant under the gauge transformation [6]:

$$K_\alpha(x) \to K'(x) = K(x) + \partial_\alpha \bar{\varphi}(x). \tag{18}$$

Because of the gauge freedom mentioned in Eq. (18), the field equation (17) can be written as

$$Q_0^{(1)} K_\alpha(x) + c D_1 \partial \cdot K(x) = 0, \tag{19}$$

The case $c = 1$ corresponds to the conformally invariant vector field equation, which has been discussed extensively in [8].
By physical state we mean the vector fields satisfying the divergenceless condition:

$$\partial \cdot K(x) = 0. \tag{20}$$

This condition is equivalente to the choice $c = 0$ for the gauge-fixing parameter. In this case the field equation (19) reduces to:

$$Q_1^{(1)} K_\alpha(x) = 0 = \left(Q_0^{(1)} - 2\right) K_\alpha(x). \tag{21}$$

In the following sections, the solution of this field equation is given and its physical two-point function is calculated, using ambient space formalism.

3. Solution to the physical vector field equation

The goal of this section is to give a solution for the physical field equation (21). Its general solution can be written as [8]:

$$K_\alpha(x) = \left[\bar{Z}_\alpha + a\frac{x \cdot Z}{x \cdot \xi}\bar{\xi}_\alpha\right]\phi(x). \qquad x \cdot \xi = 0, \tag{22}$$

Where Z_α is a constant 5-vector, $\bar{Z}_\alpha = \theta_{\alpha\beta} Z^\beta$, ξ is a light-like 5-vector and $\phi(x) = (Hx \cdot \xi)^\sigma$ is a scalar field, a is a constant coefficient to be determined. The condition $Z \cdot \xi = 0$ makes the degrees of freedom of the 5-vector Z to reduce from 5 to 4.
Implementation of $K_\alpha(x)$ in the wave equation (21) and using the divergenceless condition (20) we obtain:

$$\begin{cases} (a+1)(\sigma+4) = 0, & (I) \\ 2(a+1) + (\sigma+1)(\sigma = 2) = 0, & (II) \\ 2\sigma(a+1) + a(\sigma+1)(\sigma = 2) = 0. & (III) \end{cases} \tag{23}$$

Solution to the above system of equations can be written as:

$$\begin{cases} a = -1, \\ \sigma = -1, -2. \end{cases} \tag{24}$$

These mean that $\phi(x) = (Hx \cdot \xi)^\sigma$ is a massless conformally coupled scalar field in dS space and satisfies the following field equation [8, 9]:

$$\left(Q_0^{(1)} - 2\right)\phi = 0, \tag{25}$$

and the explicit form of the physical vector fields $K_\alpha(x)$ are:

$$K_\alpha(x) = \left[\bar{Z}_\alpha - \tfrac{x \cdot Z}{x \cdot \xi}\bar{\xi}_\alpha\right](Hx \cdot \xi)^\sigma. \qquad \sigma = -1, -2. \tag{26}$$

Eq. 26 shows that the solution can be written as the multiplication of a generalized polarization vector $\varepsilon_\alpha(x, Z, \xi)$ and a massless conformally coupled scalar field in ambient space notations. This result is consistent with that of [8]. An important difference with the Mincowskian case is that the polarization vectors $\varepsilon_\alpha(x, Z, \xi)$ are functions of the space-time variable x. Moreover, unlike the Mincowskian case these two solutions are not the complex conjugate of each other. The polarization vectors satisfy following properties

$$\varepsilon_\alpha(x, Z, \xi) = \varepsilon_\alpha(ax, Z, \xi), \\ \varepsilon_\alpha(x, Z, \xi) = \varepsilon_\alpha(x, Z, a\xi), \tag{27}$$

and the dS waves $K_\alpha(x)$ are homogeneous with degree σ. The solution presented by Eq. 26 is the same as that given in [8].

4. The physical two-point function

The vector two-point function $W_{\alpha\alpha'}(x, x')$ which is a solution to the wave equation (14) with respect to x or x', can be written in the following general form [5, 8]:

$$W_{\alpha\alpha'}(x, x') = \theta_\alpha \cdot \theta'_{\alpha'} F(Z) + H^2 (x \cdot \theta_{\alpha'})(x' \cdot \theta_\alpha) G(Z), \tag{28}$$

where $W_{\alpha\alpha'}(x, x')$ is a transverse bi-vector, $F(Z)$ and $G(Z)$ are two arbitrary functions of $Z = -H^2 x \cdot x'$. Comparing with the vector field (26), the two-point function (28) can be written as

$$W_{\alpha\alpha'}(x, x') = [\theta_\alpha \cdot \theta'_{\alpha'} f(Z) + H^2 (x \cdot \theta_{\alpha'})(x' \cdot \theta_\alpha) g(Z)] W_{cc}(x, x'), \tag{29}$$

where $W_{cc}(x, x')$ is the conformally coupled scalar two-point function [8-10] and:

$$W_{cc}(x, x') = -\frac{1}{8\pi}\left[\frac{1}{1-Z} - i\pi\varepsilon\left(x^0 - x'^0\right)\delta(1-Z)\right], \tag{30}$$

with:

$$\varepsilon\left(x^0 - x'^0\right) = \begin{cases} 1 & x^0 > x'^0, \\ 0 & x^0 = x'^0, \\ -1 & x^0 < x'^0, \end{cases}$$

The two-point function (30) satisfies the conformally coupled scalar field equation:

$$\left(Q_0^{(1)} - 2\right) W_{cc} = 0. \tag{31}$$

It is easy to show that (Appendix):

$$\frac{d}{dZ} W_{cc}(Z) = \frac{1}{1-Z} W_{cc}(Z). \tag{32}$$

By imposing the two-point function (29) to obey Eq. (21) with respect to *x* and using the divergenceless condition $\bar{\partial}_\alpha W^{\alpha\alpha'}(x,x') = 0$ it is easy to show that the new functions $f(Z)$ and $f(Z)$ satisfy the following system of differential equations

$$\begin{cases} 2f(Z) - 2Zg(Z) - 2(1-Z)f^{'}(Z) + \left(Z^2 - 1\right)f^{''}(Z) = 0, & (I) \\ \frac{-2}{1-Z}f(Z) - 2f^{'}(Z) + \frac{6-2Z}{1-Z}g(Z) - 2(1-3Z)g^{'}(Z) + \left(Z^2-1\right)g^{''}(Z) = 0, & (II) \\ \frac{4-3Z}{1-Z}f(Z) + Zf^{'}(Z) + (1-4Z)g(Z) - \left(Z^2-1\right)g^{'}(Z) = 0. & (III) \end{cases} \quad (33)$$

To obtain the above equations, the formulas given in [6] have been used. We suppose the following particular solutions to the above system of differential equations:

$$f(Z) = \frac{AZ+B}{1-Z}, \qquad g(Z) = \frac{CZ+D}{(1-Z)^2}, \quad (34)$$

with constant coefficients *A*, *B*, C and *D* which must be determined. Substituting these particular solutions in the above system of differential equations we leave with the following system of equations:

$$\begin{cases} (A+C)Z^2 + (D+B-A)Z + 2A + B = 0, & (I) \\ 2(A+C)Z + (2B+A+2D+3C) = 0, & (II) \\ -3(A+C)Z^2 + (5A-2B+3C-2D)Z + 4B + C + 3D = 0. & (I) \end{cases} \quad (35)$$

which results in

$$\begin{cases} C = -A, \\ B = -2A, \\ D = 3A. \end{cases} \quad (36)$$

Now using Eqs. 36 and 34 in Eq. 29 we obtain:

$$W_{aa'}(x,x') = A\left[\theta_\alpha \cdot \theta^{'}_{\alpha'}\frac{Z-2}{1-Z} + H^2\left(x\cdot\theta^{'}_{\alpha'}\right)(x' \cdot \theta_\alpha)\frac{3-Z}{(1-Z)^2}\right]W_{cc}(x,x'), \quad (37)$$

with an arbitrary constant coefficient *A*. It may be fixed through the flat limit and requirement that it must coincide on the Minkowsian vector two-point function. Through the zero curvature limit we have $A = -1$.

Now returning to Eq. 37, the explicit form of the physical vector two-point function is:

$$W_{aa'}(x,x') = \left[\theta_\alpha \cdot \theta^{'}_{\alpha'}\frac{2-Z}{1-Z} + H^2\left(x\cdot\theta^{'}_{\alpha'}\right)(x' \cdot \theta_\alpha)\frac{Z-3}{(1-Z)^2}\right]W_{cc}(x,x'), \quad (38)$$

which is ds-invariant and free of any logarithmic divergence.

The two-point function (38) is the physical sector of the massless vector two-point function in ambient space notations. It can be expressed in terms of the dS intrinsic coordinates [5, 6]:

$$Q_{aa'}(X,X') = \frac{1}{1-Z}\left[(2\cdot Z)g_{aa'} + (Z-5)n_a n_{a'}\right]W_{cc}(x,x'), \quad (39)$$

The two-point functions $W_{\alpha\alpha'}(x,x')$ and $Q_{\alpha\alpha'}(X,X')$ are related by:

$$Q_{\alpha\alpha'}(X,X') = \frac{\partial x^\alpha}{\partial X^\alpha}\frac{\partial x'^{\alpha'}}{\partial X'^{\alpha'}}W_{\alpha\alpha'}(x,x'). \quad (40)$$

5. Conclusion

The "massless" vector field equation in dS 4-dimensional space is gauge invariant under some special gauge transformations. It is also gauge invariant in ambient space notations in which the field equation is written in terms of the Casimir operators of dS group. In this article the physical dS "massless" vector wave field equation is solved using ambient space notations. It is shown that the solution can be written as the multiplication of a generalized polarization vector and a "massless" conformally coupled scalar field. Unlike the Minkowskian case, the generalized polarization vector is a function of space-time variables. The Physical "massless" vector two-point function is calculated in terms of the "massless" conformally coupled scalar two-point function. The explicit form of the two-point is given in terms of the basic bi-vectors of the flat 5-dimensional ambient space. The two-point function is expressed in terms of the dS intrinsic coordinates from its ambient space counterpart. The two-point functions are dS-invariant and free of any theoretical problem.

Appendix

Details of derivations of Eqs. 31 and 32

As mentioned in the text, the conformally coupled scalar two-point function is:

$$W_{cc}(x,x') = -\frac{1}{8\pi}\left[\frac{1}{1-Z} - i\pi\varepsilon\left(x^0 - x'^0\right)\delta(1-Z)\right]. \quad (A.1)$$

Taking its derivative we have:

$$\frac{d}{dZ}W_{cc}(x,x') = -\frac{1}{8\pi}\left[\frac{1}{(1-Z)^2} - i\pi\varepsilon\left(x^0 - x'^0\right)\frac{d}{dZ}\delta(1-Z)\right]. \quad (A.2)$$

Using the relation:

$$\int xf(x)\delta'(x)dx = \int [xf(x)]'\delta(x)dx, \quad (A.3)$$

we have $x\delta'(x) = -\delta(x)$ and returning to Eq. (A.2) we have

$$\frac{d}{dZ}W_{cc}(x,x') = -\frac{1}{8\pi}\left[\frac{1}{(1-Z)^2} - i\pi\varepsilon\left(x^0 - x'^0\right)\frac{\delta(1-Z)}{1-Z}\right]. \quad (A.4)$$

Therefore:

$$\frac{d}{dZ}W_{cc}(x,x') = \frac{1}{1-Z}W_{cc}(x,x'). \quad (A.5)$$

Eq. (A.5) is not anything other than Eq. 32.
Now Eq. (A.5) can be used to confirm the validity of Eq. 31. For this purpose one must note that, the scalar Casimir operator $Q_0^{(1)}$ can be written in the following form [6]:

$$Q_0^{(1)} = (1-Z^2)\frac{d^2}{dZ^2} - 4Z\frac{d}{dZ}. \quad (A.6)$$

Therefore:

$$Q_0^{(1)}W_{cc}(x,x') = \left(1-Z^2\right)\frac{d^2}{dZ^2}W_{cc}(x,x') - 4Z\frac{d}{dZ}W_{cc}(x,x'). \quad (A.7)$$

From (A.5) we have:

$$\frac{d^2}{dZ^2}W_{cc}(x,x') = \frac{2}{(1-Z)^2}W_{cc}(x,x'). \quad (A.8)$$

Substituting (A.5) and (A.8) in (A.7) we have:

$$Q_0^{(1)}W_{cc}(x,x') = \frac{2\left(1-Z^2\right)}{(1-Z)^2}W_{cc}(x,x') - \frac{4Z}{1-Z}W_{cc}(x,x') = \left[\frac{2\left(1-Z^2\right)}{(1-Z)^2} - \frac{4Z}{1-Z}\right]W_{cc}(x,x') = 2W_{cc}(x,x'). \quad (A.9)$$

This means that:

$$\left(Q_0^{(1)} - 2\right)W_{cc}(x,x') = 0. \quad (A.10)$$

Eq. (A.10) is not anything other than Eq. 31.

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