

# Analytical expressions for nucleon-nucleon phase shift at high energy using separable potential

A. Al-Jamel, M. Serhan, M. Abusini\*

Department of Physics, Faculty of science, Al al-Bayt University, Al-Mafraq, Jordan

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## Abstract

We obtain analytical expressions for the two-body  $T$ -matrix and phase shift at high energy ( $\sim 350$  MeV) scattering using nonlocal rank-two separable potential of Mongan type. The potential parameters are adjusted by fitting the available experimental nucleon-nucleon high energy scattering phase shift data for two single channels  $^1D_2$  and  $^1P_1$  with the corresponding derived analytical expressions. The results indicate that the analytical expressions obtained for the phase shifts are successful in reproducing the nucleon-nucleon scattering data.

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## 1. Introduction

Separable potentials are now widely used in electronic structure calculations on atoms, molecules and solids [1-4] and in the areas of nuclear and condensed matter [5-7]. In the particular case of the nucleon-nucleon interaction, separable potentials of two parameters were first introduced by Yamaguchi [8], which in case of the S-waves, were calculated in terms of the scattering length and effective range for each channel, so that they gave an adequate fit to the low-energy phase shifts. The phase shifts derived from these potentials, however, are positive for all energies, and thus badly represent the nucleon-nucleon interaction at high energies where the phase shifts change sign. As it is well known, this change of sign is due to a strong repulsion in the nucleon-nucleon interaction at short distances, so that in order to simulate it one needs at least a two-term separable potential with one term representing the attraction and the other the repulsion, such as the models proposed by Tabkin [9] and Mongan [10]. Two-term separable potentials, however, imply that one now has twice as many coupled amplitudes in the three-body equations, and consequently its numerical solution is more difficult. Ghirardi and Rimini [11] have examined general properties of separable potential. It can be shown that Schrödinger equation reduces to an algebraic equation in momentum space. The potential allows all quantities of interest, such as resonance energy, phase shift, etc., to be expressed in a completely analytical form. It is well known that separable ones of a certain rank [12] can accurately approximate almost all short-range potential

models. Moreover, a rank-two separable potential with Yamaguchi form factors leads to algebraic expressions for most quantities of interest. Such potentials have been studied as models for a variety of physical problems [13-7]

The main goal of the present work is to determine the nuclear potentials that, when using the scattering theory, reproduce nucleon-nucleon scattering data at high energy up to 350 MeV by adjusting the potential parameters until the calculations give a good fit to the experimental data. So we introduce a separable potential of rank-two by using Mongan-type for nucleon-nucleon scattering to obtain analytical expressions for  $\delta_l$  phase shift of  $^1D_2$  and  $^1P_1$  channels in order to adjust the potential parameters (attractive, repulsive) to reproduce the experimental phase shift data.

## 2. Separable potential

A separable potential model can be generally written as [18]:

$$\hat{V} = \sum_{i=1}^n |\phi_i\rangle \alpha_i \langle \phi_i|, \quad (1)$$

where  $n$  is the rank of the potential energy operator  $\hat{V}$ ,  $|\phi_i\rangle$  is state of the system, which is compatible with the domain of Hamiltonian, and  $\alpha_i$  is the potential energy strength, which is a real number in the unitary case.

If the interaction operator has the form:

$$\hat{V} = |\phi\rangle \alpha \langle \phi|, \quad (2)$$

which in configuration and momentum space written as:

\*Corresponding author: M. Abusini;  
E-mail: abusini@aabu.edu.jo  
Tel: (+962) 2629-7000  
Fax: (+962-2629-7034)

$$\langle r|V|r\rangle = \langle r|\phi\rangle\alpha\langle\phi|r\rangle' = \alpha\phi(r)\phi^*(r), \quad (3)$$

$$\langle q|V|q\rangle = \langle q|\phi\rangle\alpha\langle\phi|q\rangle' = \alpha\phi(q)\phi^*(q), \quad (4)$$

here  $\phi(q)$  and  $\phi(r)$  are the given functions, in this case the operator  $\hat{V}$  is called the first-rank separable operator, substituting equations (3) and (4) for  $T$  matrix we get:

$$T(E) = |\phi\rangle\alpha\langle\phi| + |\phi\rangle\alpha\langle\phi|G_0(E)T(E), \quad (5)$$

where  $G_0(E)$ , is the Greens operator, then the solution of equation (5) is given by:

$$T(E) = |\phi\rangle\zeta(E)\langle\phi|, \quad (6)$$

where  $\zeta(E)$ , is a function defined via the function  $\phi(q)$  or  $\phi(r)$ .

Substituting (6) into (5) we get:

$$|\phi\rangle\zeta(E)\langle\phi| = |\phi\rangle\alpha\langle\phi| + |\phi\rangle\alpha\langle\phi|G_0(E)|\phi\rangle\zeta(E)\langle\phi|, \quad (7)$$

and we obtain

$$\zeta(E) = [\alpha^{-1} - \langle\phi|G_0(E)|\phi\rangle]^{-1}, \quad (8)$$

where

$$\begin{aligned} \langle\phi|G_0(E)|\phi\rangle &= \int \frac{\langle\phi|q\rangle q^2 dq \langle q|\phi\rangle}{E + i\varepsilon - E(q)} \\ &= \int \frac{|\phi(q)|^2 q^2 dq}{E + i\varepsilon - E(q)}. \end{aligned} \quad (9)$$

For the total energy  $T$ -matrix we obtain:

$$\begin{aligned} \langle\phi|T(E)|\phi\rangle &= \phi(q)\zeta(E)\phi^*(q) \\ &= \frac{\phi(q)\phi^*(q)}{[\alpha^{-1} - \langle\phi|G_0(E)|\phi\rangle]} \end{aligned} \quad (10)$$

Thus, equations (10) and (9) give the solutions of the scattering problem for the separable potential (2).

For the partial scattering matrix we have the following expression:

$$S = 1 - 2\pi i \rho(E) T(E), \quad \text{where } \rho(E) = \left(\frac{\mu k}{\hbar^2}\right)^{-1} \quad (11)$$

which is related to the phase shift as:

$$S = e^{i2\delta} = 1 - 2\pi \frac{\hbar^2 k}{\mu} \zeta(E) |\phi(q)|^2. \quad (12)$$

The nonrelativistic Lippmann-Schwinger equation is written in the form:

$$T(k, k)' = V(k, k)' + \frac{2\mu}{\hbar^2} \int_0^\infty \frac{V(k, q) T(q, k)' q^2 dq}{k^2 + i\varepsilon - q^2}. \quad (13)$$

Using equations (12) and (13) the partial  $T$ -matrix can be written as:

$$T_l(E) = -\frac{1}{\pi l(E)} e^{i\delta_l(E)} \sin\delta_l(E). \quad (14)$$

Then equation (14) also can be written as:

$$\tan\delta_l(E) = \frac{ImT_l(E)}{ReT_l(E)}. \quad (15)$$

We see that  $\delta_l(E)$  can be evaluated obtaining  $T_l(E)$  from a direct integration of nonrelativistic Lippmann-Schwinger equation (14) and the phase shift is related to the potential via equations (12) and (13). So our task is to introduce a potential for nucleon-nucleon scattering to adjust its parameters to reproduce the experimentally observed phase shifts ( $\delta_l$ ). Here the potential chosen for the fits are the form rank-two parametrization separable potential of Mongan type [10], which has the following form:

$$V(k, k)' = g_l(k)g_l(k)' - h_l(k)h_l(k)', \quad (16)$$

where

$$g_l(k) = \frac{C_r k^l}{(k^2 + a_r^2)^{\frac{l+2}{2}}}; \quad h_l(k) = \frac{C_a k^l}{(k^2 + a_a^2)^{\frac{l+2}{2}}}, \quad (17)$$

where the  $N$ - $N$  parameters  $C_r, a_r, C_a, a_a$  correspond to repulsive and attractive channels, respectively.

With this potential using equation (14), the partial  $T$ -matrix has the form

$$T_l(k, k)' = \frac{A}{B}, \quad (18)$$

where

$$A = \left(g_l(k)g_l(k)' \left[1 + \frac{2\mu}{\hbar^2}\right] X - \left(h_l(k)h_l(k)' \left[1 - \frac{2\mu}{\hbar^2}\right] Y - \left(g_l(k)g_l(k)' \frac{2\mu}{\hbar^2}\right) Z\right), \quad (19)$$

$$B = \left[1 + \frac{2\mu}{\hbar^2}\right] X \cdot \left[1 - \frac{2\mu}{\hbar^2}\right] Y + \left[\frac{2\mu}{\hbar^2} Z\right]^2, \quad (20)$$

and

$$\begin{aligned} X &= \int_0^\infty \frac{h_l^2(q) q^2 dq}{k^2 + i\varepsilon - q^2}, \quad Y = \int_0^\infty \frac{g_l^2(q) q^2 dq}{k^2 + i\varepsilon - q^2}, \\ Z &= \int_0^\infty \frac{h_l(q)g_l(q) q^2 dq}{k^2 + i\varepsilon - q^2}. \end{aligned} \quad (21)$$

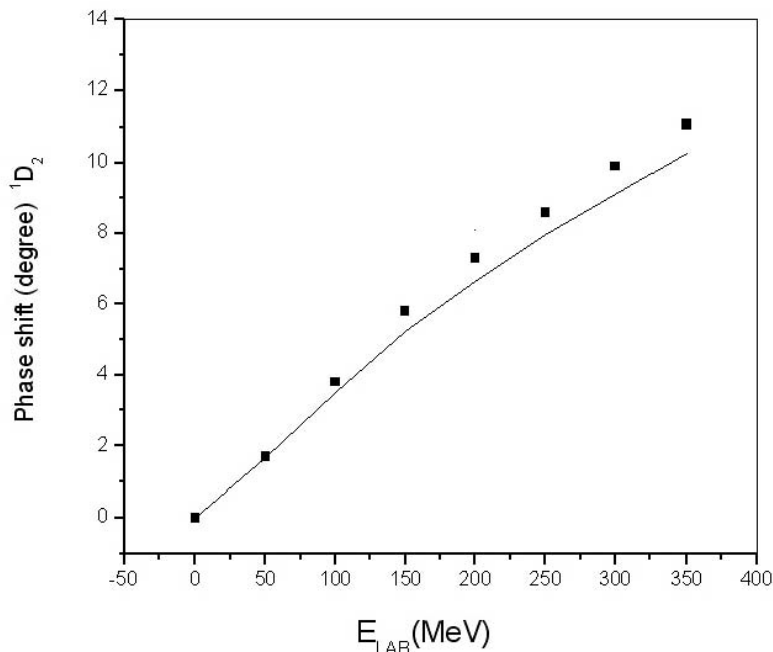


Fig. 1. The fit of experimental data of nucleon-nucleon scattering for the channel  $^1D_2$  up to energy  $E_{lab}= 350$  MeV, the experimental data (point) were taken from [18] .

The integrals in equation (21), and thus the phase shift  $\delta_l$  can be obtained analytically using Mathematica. So we will demonstrate a method to solve the integrals (21) for two channels of nucleon-nucleon scattering ( $^1D_2$  and  $^1P_1$ ).

For the channel  $^1D_2$  (purely attractive,  $g_l = 0$ ) we plug  $g_l(k)$  and  $h_l(k)$  from (17) on to equations (18-21) then we get:

$$A = -h_l(k)h_l('k)'; \quad B = \left[1 + \frac{2\mu}{\hbar^2}\right]X. \tag{22}$$

Using principal value theorem, we can write the integral  $X$  in equation (21) as:

$$\int_0^\infty \frac{h_l^2(q)q^2 dq}{k^2 + i\varepsilon - q^2} = P\mathcal{F} - i\pi\mathcal{X}, \tag{23}$$

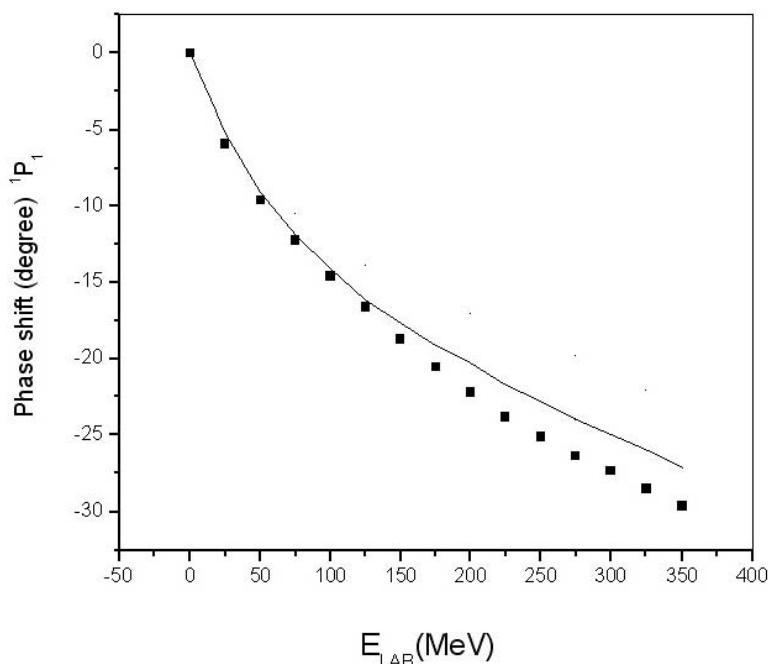


Fig. 2. The fit of experimental data of nucleon-nucleon scattering for the channel  $^1P_1$  up to energy  $E_{lab}= 350$  MeV, where the experimental data (points) were taken from [19] .

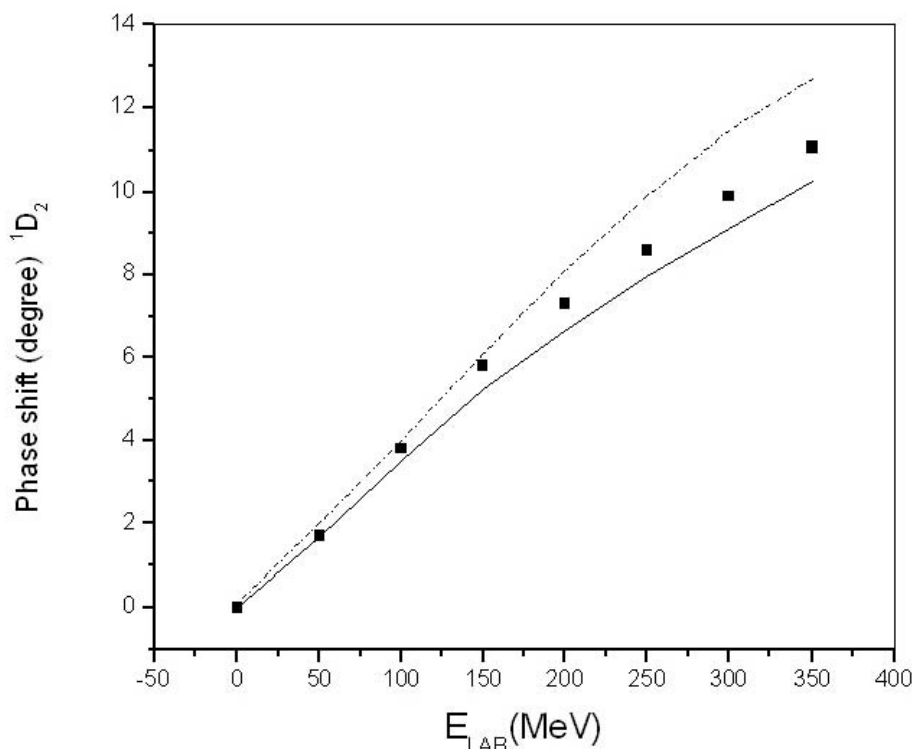


Fig. 3. Comparison of fits up to energy  $E_{lab}= 350$  MeV for the channel  $^1D_2$  where the solid line correspond to fitting by our analytical expression, the dashed line correspond to fitting by Mongan parameters, the experimental data (points) were taken from [18] .

where

$$\mathcal{F} = \int_0^\infty \frac{h_l^2(q)q^2 dq}{k^2 - q^2}, \tag{24}$$

$$\kappa = \int_0^\infty \delta(k^2 - q^2)h_l^2(q)q^2 dq. \tag{25}$$

On substituting the values of  $g_l(k)$  and  $h_l(k)$  from equation (22) in to equation (25) we get:

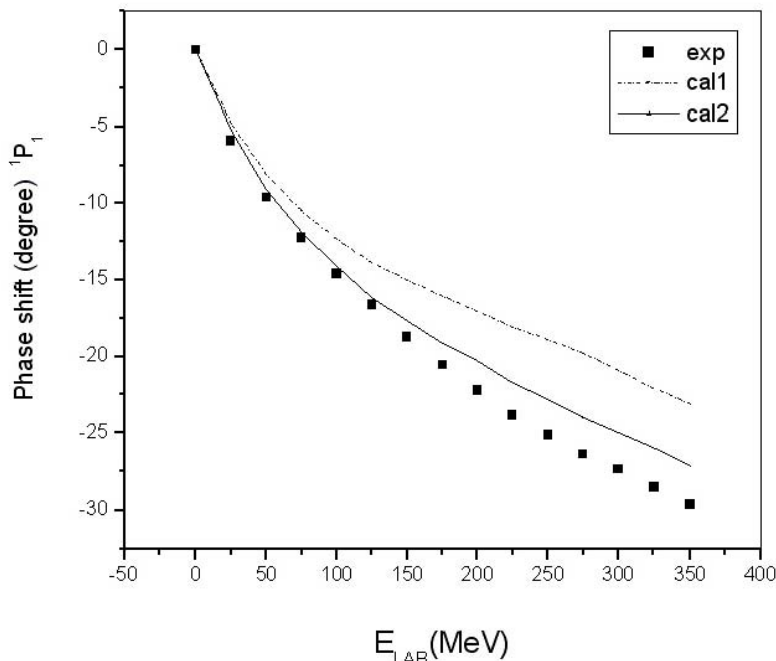


Fig 4. Comparison of fits up to energy  $E_{lab}= 350$  MeV for the channel  $^1P_1$  . Where the solid line correspond to fitting by our analytical expression, the dashed line correspond to fitting by Mongan parameters the experimental data (points)were taken from [19] .

$$\kappa = \frac{C_a^2 k^5}{(k^2 + a_a^2)^4}. \quad (26)$$

By breaking the integral in equation (24) and substituting equation (26) in (23) then in (18-22) we get analytical expression for the partial  $T$ -matrix nucleon-nucleon scattering for the attractive channel  $^1D_2$  (singlet,  $l = j = 2$ )

Finally using the equations (15-26) we get analytical expression for phase shift  $\delta_l$  for the attractive channel  $^1D_2$  of neutron-proton scattering in the form of:

$$\delta_{^1D_2}(E) = \tan^{-1} \left[ 16\pi k^5 \mu \frac{a_a^2 C_a^2}{(\hbar^2 + \pi\mu)(2k^8 - k^6 a_a^2 + 17k^4 a_a^4 + 5k^2 a_a^6 + a_a^8)} \right]. \quad (27)$$

The procedure was adopted in finding analytical expression for the phase shift of the channel  $^1P_1$ , we obtain the following expression for the channel  $^1P_1$  of neutron-proton scattering (singlet,  $l = j = 1$ ):

$$\delta_{^1P_1}(E) = \tan^{-1} \left[ -4\pi k^3 \mu \frac{a_r^2 C_r^2}{(k^2 + a_r^2)(8k^4 \hbar^2 + 3\pi\mu k^4 + 18\pi\mu k^2 a_r^2 + 7\pi\mu a_r^4)} \right]. \quad (28)$$

In our work we determined the parameters  $C_a, a_a C_r, a_r$  (attractive or repulsive) so that each of these expressions (27, 28) reproduce the nucleon-nucleon experimental data of the corresponding phase shifts  $\delta_l(E)$  for a given range of energy. The set of values  $C_a, a_a C_r, a_r$  that make the  $\delta_l(E)$  a good fit to the experimental data of phase shift uniquely define the scattering separable potential (16,17). It should be noted that Mongan used an approach similar to ours to obtain the parameters  $C_a, a_a C_r, a_r$ , except he does not obtain analytical expression for  $\delta_l(E)$ . Mongan in fact relies on numerical evaluations on equations (19, 20 and 22) to extract the potential parameters.

### 3. Results and Discussions

The results were obtained for the data taken from [18, 19] for both channels  $^1D_2$  and  $^1P_1$  up to lab. energy 350 MeV. Fig. 1 shows the phase shift as a function of energy of nucleon-nucleon scattering for the attractive channel  $^1D_2$  up to lab. energy 350 MeV. This is our fit to the experimental data taken from Ref. [18] and is based on Mongan separable potential which consists of two parts: an attractive channel  $^1D_2$  and a repulsive one  $^1P_1$ . The solid line represents the fit obtained using our analytical expression (27) for  $^1D_2$  channel

and the dotted line represents the experimental phase shift data. The discrepancy is small specially at low energies. That is as the energy increases the discrepancy increases. Fig. 2 shows the phase shift as a function of energy of nucleon-nucleon scattering for the repulsive channel  $^1P_1$  up to lab. energy 350 MeV. This is our fit for the experimental data taken from Ref. [19]. The solid line represents the fit obtained using our analytical expression (28) and the dotted line represents the experimental phase shift data. The discrepancy is also here small and in particular at low energies.

In order to determine the accuracy of our results, we compare our results with those using the parameters obtained using Mongan parameters. It should be noted that Mongan used an approach similar to ours to obtain the parameters  $C_a, a_a C_r, a_r$ , except that he does not obtain analytical expression for  $\delta_l(E)$ . Mongan in fact relies on numerical evaluations of equations (19, 20 and 22) to extract the potential parameters. The comparison shows that the fits obtained using analytical expressions are much better than the fits obtained using Mongan's parameters (Figs. 3 and 4 for both channels  $^1D_2$  and  $^1P_1$ ). This result is more obvious in the repulsive channel  $^1P_1$ .

In Table 1 we list the parameters obtained by our calculations and those obtained by Mongan for both channels.

Our results show that the analytical expressions obtained for phase shift are successful in reproducing  $N-N$  scattering data, and we also see that our approach is simple and the interesting result of our work, is that the separable potentials, which were used earlier for lower energy [20], can also be successfully applied to higher energies.

### 4. Conclusion

In this work we have used rank-two separable potential of Mongan type for the nucleon-nucleon scattering phase shift results. Namely we have adjusted the parameters of the potential to reproduce the experimentally observed phase shifts for two single channels  $^1D_2$  and  $^1P_1$ . Our results are in good agreement with the experimental results and better than those found using Mongan potential.

As a future elaboration of this work we could use this procedure to analyze the phase shift data from coupled channels. The analysis given here could also be repeated for other separable potentials such as Reid-soft-core potential and super-soft-core potentials.

**Table 1. Our fitting parameters and the Mongan parameters for both channels.**

Channels	Mongan parameters	Fitting parameters
$^1P_1$	$C_r = 72.65 \text{ (MeV. fm)}^{1/2}$ $a_r = 2.23 \text{ (fm)}^{-1}$	$C_r = 40.88 \text{ (MeV. fm)}^{1/2}$
$^1D_2$	$C_r = 34.21 \text{ (MeV. fm)}^{1/2}$ $a_r = 2.61 \text{ (fm)}^{-1}$	$C_r = 21.09 \text{ (MeV. fm)}^{1/2}$

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