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The investigation and application of two approximate analytical methods for the solution of nonlinear differential equation of beam elastic deformation

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Abstract

In this Paper, we apply two approximate analytical methods of Perturbation Method (PM) and Homotopy Perturbation Method (HPM) to solve the equation of beam deformation with two fixed end and under uniform distributed load. The presented results in this paper reveal that these two methods are very effective and can be easily extended to other nonlinear systems and can therefore be found widely applicable in engineering and other sciences.

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1. Introduction

Nonlinear systems have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion and deformation. The study of nonlinear systems is of interest to many researchers and various methods of solution have been proposed. Surveys of the literature with numerous references, and useful bibliographies, have been given by Nayfeh [1], Mickens [2], Jordan and Smith [3] and more recently by He [4].

The solving of governing equations due to limitation of existing exact solutions have been one of the most time-consuming and difficult affairs among researchers of nonlinear problems.

With the rapid development of nonlinear science, there appears an ever-increasing interest of scientists in the analytical asymptotic techniques for nonlinear Problems and several analytical approximate methods have been developed to solve linear and nonlinear ordinary and partial differential equations.

Some of these techniques include Perturbation Method (PM) [4-7], Variational Iteration Method (VIM) [8-12], Homotopy Perturbation Method (HPM) [13-21], Energy Balance Method (EBM) [22-26], Variational Approach Method (VAM) [27-30], Parameter-Expansion Method (PEM) [31-37], Amplitude-Frequency Formulation (AFF) [38-43], Iteration Perturbation Method (IPM) [44, 45] and etc.

Among these methods, the Perturbation and Homotopy Perturbation Methods are considered to be two of powerful methods capable of handling strongly nonlinear behaviors and can converge to an accurate solution for smooth nonlinear systems.

One of the responsibilities of the structural design engineer is to devise arrangements and proportions of members that can withstand, economically and efficiently, the conditions anticipated during the lifetime of a structure. A central aspect of this function is the calculation of the beam deformation, which has very wide applications in structural engineering.

The main objective of this paper is to approximately solve nonlinear differential equation of beam elastic deformation with two fixed end and under uniform distributed load (Fig. 1), by applying the Perturbation Method (PM) and Homotopy Perturbation Method (HPM) and to compare the approximate results with formula in mechanics of materials for beams with two fixed end and under uniform distributed load.

The results presented in this paper reveal that the methods are very effective for solution of nonlinear differential equations of beam elastic deformation and can be easily extended to other nonlinear systems and can therefore be found widely applicable in engineering and other sciences.

The equation of beam elastic deformation with two fixed end and under uniform distributed load is in the following form [6]:

$$\left(\frac{d^2}{dx^2}y(x)\right)-\left(\frac{M(x)}{EI}\right)\cdot\left(1+\left(\frac{d}{dx}y(x)\right)^2\right)^{\frac{3}{2}}=0. \qquad (1)$$

In Eq. (1):

$$M(x)=\left(\frac{W}{12}\cdot\left(6Lx-L^2-6x^2\right)\right).$$

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In this equation, *M* is bending moment, *E* is the elastic modulus and *I* is the second moment of area. *I* must be calculated with respect to axis perpendicular to the applied load.
With the boundary conditions:

$$y(0) = y(L) = 0, \quad y'(0) = y'(L) = 0 \tag{2}$$

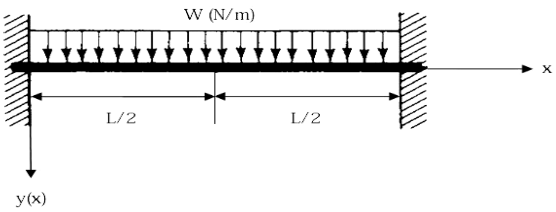

Fig. 1. Beam with two fixed end and under uniform distributed load

2. Computational method

1.2. The basic idea of perturbation method

Perturbation method is based on assuming that a parameter in the system is small. The approximate solution obtained by the perturbation methods, in most cases are valid only for small value of the small parameter.

Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we cannot rely fully on approximations, because there is no criterion on which the small parameter should exist. Thus, it is essential to check the validity of approximations numerically or experimentally [4-7].

For a very small ε << 1, let us assume a regular perturbation expansion and calculate the first three terms, thus we assume [4]:

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2. \tag{3}$$

With substituting Eq. (3) in the nonlinear differential equation and after expansion and rearranging based on coefficient of ε-term we have:

Coefficient of ε^0: Differential Equation in $\theta_0(\tau) = f(u)$. (4)

Coefficient of ε^1: Differential Equation in $\theta_1(\tau)$ and $\theta_0(\tau) = 0$. (5)

Coefficient of ε^2: Differential Equation in $\theta_1(\tau)$, $\theta_2(\tau)$ and $\theta_0(\tau) = 0$. (6)

and finally with three-term expansion:

$$\theta(\tau) = \varepsilon^0\theta_0(\tau) + \varepsilon^1\theta_1(\tau) + \varepsilon^2\theta_2(\tau). \tag{7}$$

2.2. The basic idea of homotopy perturbation method

Until recently, the application of the homotopy perturbation method in nonlinear problems has been devoted by scientists and engineers, because this method is to continuously deform a simple problem easy to solve into the difficult problem under study. To illustrate the basic ideas of this method, we consider the following equation [13]:

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{8}$$

with the boundary condition of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Omega \tag{9}$$

where *A* is a general differential operator, *B* a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.
A can be divided into two parts which are *L* and *N*, where *L* is linear and *N* is nonlinear. Therefore, it can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{10}$$

Homotopy perturbation structure is shown as follows:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{11}$$

where,

$$v(r,p): \Omega \times [0,1] \to R \tag{12}$$

In Eq. (11), $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (11) can be written as a power series in *P*, as:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots, \tag{13}$$

and the best approximation for solution is:

$$v = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots \tag{14}$$

2.3. The application of perturbation method

To solve Eq. (1) by means of perturbation method, first we change Eq. (1) to following form:

$$\left(\frac{d^2}{dx^2}y(x)\right)^2 - \left(\frac{W}{144(EI)^2}\cdot(6Lx - L^2 - 6x^2)^2\right) \cdot \left(1 + \varepsilon\cdot\left(\frac{d}{dx}y(x)\right)^2\right)^3 = 0. \tag{15}$$

For very small ε, let us assume a regular perturbation expansion and calculate the first three terms, thus we assume:

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x). \quad (16)$$

By substituting Eq. (16) in the Eq. (15) and after expansion and rearranging based on coefficient of ε-term we have:

$$\varepsilon^0: \left(\frac{d^2}{dx^2} y_0(x)\right)^2 - \left(\frac{1}{144}\frac{W^2(6Lx-L^2-6x^2)^2}{(EI)^2}\right) = 0. \quad (17)$$

$$\varepsilon^1: 2\left(\frac{d^2}{dx^2} y_0(x)\right)\cdot\left(\frac{d^2}{dx^2} y_1(x)\right) - \left(\frac{1}{48}\frac{W^2(6Lx-L^2-6x^2)^2}{(EI)^2}\right)\cdot\left(\frac{d}{dx} y_0(x)\right)^2 = 0. \quad (18)$$

$$\varepsilon^2: 2\left(\frac{d^2}{dx^2} y_0(x)\right)\cdot\left(\frac{d^2}{dx^2} y_2(x)\right) + \left(\frac{d^2}{dx^2} y_1(x)\right)^2 - \left(\frac{1}{144}\cdot\frac{W^2(6Lx-L^2-6x^2)^2}{(EI)^2}\right)\cdot\left(6\left(\frac{d}{dx} y_0(x)\right)\left(\frac{d}{dx} y_1(x)\right) + 3\left(\frac{d}{dx} y_0(x)\right)^4\right) = 0. \quad (19)$$

By solving the Eq. (17), Eq. (18) and Eq. (19), we have:

$$y_0(x) = \left(\frac{W}{12EI}\right)\cdot\left(-Lx^3 + \frac{1}{2}L^2x^2 + \frac{1}{2}x^4\right). \quad (20)$$

$$y_1(x) = -\left(\frac{W^3}{1152(EI)^3}\right)\cdot\left(\frac{4}{15}x^{10} - \frac{4}{3}Lx^9 + \frac{11}{4}L^2x^8 - 3L^3x^7 + \frac{11}{6}L^4x^6 - \frac{3}{5}L^5x^5 + \frac{1}{12}L^6x^4\right). \quad (21)$$

$$y_2(x) = -\left(\frac{5W^5}{663552(EI)^5}\right)\cdot\left(\frac{2}{5}x^{16} - \frac{16}{5}Lx^{15} + \frac{80}{7}L^2x^{14} - 24L^3x^{13} + \frac{197}{6}L^4x^{12} - \frac{153}{5}L^5x^{11} + \frac{197}{10}L^6x^{10} - \frac{26}{3}L^7x^9 + \frac{5}{2}L^8x^8 - \frac{3}{7}L^9x^7 + \frac{1}{30}L^{10}x^6\right). \quad (22)$$

According to the perturbation method, we can conclude that:

$$\lim_{\varepsilon\to 1} y(x) = y_0(x) + y_1(x) + y_2(x). \quad (23)$$

Therefore, substituting the values of $y_0(x)$, $y_1(x)$ and $y_2(x)$ from Eq. (20), Eq. (21) and Eq. (22) into Eq. (23) yields:

$$y(x) = \left(\frac{W}{12EI}\right)\cdot\left(-Lx^3 + \frac{1}{2}L^2x^2 + \frac{1}{2}x^4\right) + \left(\frac{W^3}{1152(EI)^3}\right)\left(\frac{4}{15}x^{10} - \frac{4}{3}Lx^9 + \frac{11}{4}L^2x^8 - 3L^3x^7 - \frac{3}{5}L^5x^5 + \frac{1}{12}L^6x^4\right) + \left(\frac{5W^5}{663552(EI)^5}\right)\cdot\left(\frac{2}{5}x^{16} - \frac{16}{5}Lx^{15} + \frac{80}{7}L^2x^{14} - 24L^3x^{13} + \frac{197}{6}L^4x^{12} - \frac{153}{5}L^5x^{11} + \frac{197}{10}L^6x^{10} - \frac{26}{3}L^7x^9 + \frac{5}{2}L^8x^8 - \frac{3}{7}L^9x^7 + \frac{1}{30}L^{10}x^6\right). \quad (24)$$

2.4. The application of homotopy perturbation method

To solve Eq. (1) by means of homotopy perturbation method, first we change Eq. (1) to following form:

$$\left(\frac{d^2}{dx^2} y(x)\right) - \left(\frac{W}{11EI}\cdot(6Lx - L^2 - 6x^2)\right)\cdot\left(1 + \frac{3}{2}\cdot\left(\frac{d}{dx} y(x)\right)^2\right) = 0. \quad (25)$$

To solve Eq. (25) by means of Homotopy Perturbation Method, we consider the following process after separating the linear and nonlinear parts of the equation. A Homotopy can be constructed as follows:

$$H(y,p) = (1-p)\cdot\left(\frac{d^2}{dx^2} y(x) - \frac{W}{12EI}\cdot(6Lx - L^2 - 6x^2)\right) + p\cdot\left(\frac{d^2}{dx^2} y(x) - \frac{W}{12EI}\cdot(6Lx - L^2 - 6x^2)\cdot\left(1 + \frac{3}{2}\left(\frac{d}{dx} y(x)\right)^2\right)\right) = 0. \quad (26)$$

We can assume that the solution of Eq. (26) can be written as a power series in *p*, as:

$$y(x) = y_0(x) + p y_1(x) + p^2 y_2(x). \quad (27)$$

By substituting Eq. (27) into Eq. (26) and after expansion and rearranging based on coefficient of *p*-term we have:

$$p^0: \left(\frac{d^2}{dx^2} y_0(x)\right) + \left(\frac{Wx^2}{2EI} - \frac{WLx}{2EI} + \frac{WL^2}{12EI}\right) = 0. \quad (28)$$

$$p^1: \left(\frac{d^2}{dx^2} y_1(x)\right) + \left(\frac{3Wx^2}{4EI} - \frac{3WLx}{4EI} + \frac{WL^2}{8EI}\right)\left(\frac{d}{dx} y_0(x)\right)^2 = 0. \quad (29)$$

$$p^2: \left(\frac{d^2}{dx^2} y_2(x)\right) + \left(\frac{3Wx^2}{2EI} - \frac{3WLx}{2EI} + \frac{WL^2}{4EI}\right)\cdot\left(\frac{d}{dx} y_0(x)\right)\cdot\left(\frac{d}{dx} y_1(x)\right) = 0. \quad (30)$$

By solving Eq. (28), Eq. (29) and Eq. (30), we have:

$$y_0(x) = \left(\frac{W}{12EI}\right)\cdot\left(-Lx^3 + \frac{1}{2}L^2x^2 + \frac{1}{2}x^4\right). \quad (31)$$

$$y_2(x) = \left(\frac{W^2}{1152(EI)^3}\right)\left(\frac{4}{15}x^{10} - \frac{4}{3}Lx^9 + \frac{11}{4}L^2x^8 - 3L^3x^7 + \frac{11}{6}L^4x^6 - \frac{3}{5}L^5x^5 + \frac{1}{12}L^6x^4\right). \quad (32)$$

$$y_2(x) = \left(\frac{5W^5}{663552(EI)^5}\right) \cdot \left(\frac{2}{5}x^{16} - \frac{16}{5}Lx^{15} + \frac{80}{7}L^2x^{14} - 24L^3x^{13} + \frac{197}{6}L^4x^{12} - \frac{153}{5}L^5x^{11} + \frac{197}{10}L^6x^{10} - \frac{26}{3}L^7x^9 + \frac{5}{2}L^8x^8 - \frac{3}{7}L^9x^7 + \frac{1}{30}L^{10}x^6\right). \quad (33)$$

According to the Homotopy Perturbation Method, we can conclude that:

$$\lim_{p\to 1} y(x) = y_0(x) + y_1(x) + y_2(x). \quad (34)$$

Therefore, substituting the values of $y_0(x)$, $y_1(x)$, and $y_2(x)$ from Eq. (31), Eq. (32) and Eq. (33) into Eq. (34) yields:

$$y(x) = \left(\frac{W}{12EI}\right) \cdot \left(-Lx^3 + \frac{1}{2}L^2x^2 + \frac{1}{2}x^4\right) + \left(\frac{W^3}{1152(EI)^3}\right)\left(\frac{4}{15}x^{10} - \frac{4}{3}Lx^9 + \frac{11}{4}L^2x^8 - 3L^3x^7 + \frac{11}{6}L^4x^6 - \frac{3}{5}L^5x^5 + \frac{1}{12}L^6x^4\right) + \left(\frac{5W^5}{663552(EI)^5}\right)\left(\frac{2}{5}x^{16} - \frac{16}{5}Lx^{15} + \frac{80}{7}L^2x^{14} - 24L^3x^{13} + \frac{197}{6}L^4x^{12} - \frac{153}{5}L^5x^{11} + \frac{197}{10}L^6x^{10} - \frac{26}{3}L^7x^9 + \frac{5}{2}L^8x^8 - \frac{3}{7}L^9x^7 + \frac{1}{30}L^{10}x^6\right). \quad (35)$$

3. Results

In this section, we compare the results of Perturbation method (PM) and Homotopy Perturbation Method (HPM) with formula in mechanics of materials for beams with two fixed end and under uniform distributed load.

In mechanics of materials for beams with two fixed end and under uniform distributed load the deformation is computed by following formula [6]:

$$y(x) = \frac{1}{24} \cdot \frac{Wx^2(L-x)^2}{EI}. \quad (36)$$

The approximate analytical results are in good agreement with the results obtained by the formula in mechanics of materials for beams with two fixed end and under uniform distributed load. The results of comparison between Perturbation Method (PM) with formula in mechanics of materials for beams with two fixed end and under uniform distributed load are given in Tables (1-5).

Table 1. Comparison of perturbation method with formula in mechanics of materials for beams with two fixed end and under uniform distributed load when ($EI = 500\frac{N}{m^2}, L = 1m, W = 100\ N$)

X (displacement from left support)	Perturbation Method (PM)	Formula in Mechanics of Material
0.10	0.000067	0.000067
0.20	0.000213	0.000213
0.30	0.000367	0.000367
0.40	0.000480	0.000480
0.50	0.000521	0.000521

Table 2. Comparison of perturbation method with formula in mechanics of materials for beams with two fixed end and under uniform distributed load when ($EI = 1000\frac{N}{m^2}, L = 1m, W = 100\ N$)

X (displacement from left support)	Perturbation Method (PM)	Formula in Mechanics of Material
0.10	0.000034	0.000034
0.20	0.000107	0.000107
0.30	0.000184	0.000184
0.40	0.000240	0.000240
0.50	0.000260	0.000260

Table 3. Comparison of perturbation method with formula in mechanics of materials for beams with two fixed end and under uniform distributed load when ($EI = 1500\frac{N}{m^2}, L = 1m, W = 100\ N$)

X (displacement from left support)	Perturbation Method (PM)	Formula in Mechanics of Material
0.10	0.000022	0.000022
0.20	0.000071	0.000071
0.30	0.000122	0.000122
0.40	0.000160	0.000160
0.50	0.000174	0.000174

Table 4. Comparison of perturbation method with formula in mechanics of materials for beams with two fixed end and under uniform distributed load when ($EI = 500\frac{N}{m^2}, L = 2m, W = 100\ N$)

X (displacement from left support)	Perturbation Method (PM)	Formula in Mechanics of Material
0.10	0.0003	0.0003
0.20	0.0011	0.0011
0.30	0.0022	0.0022
0.40	0.0034	0.0034
0.50	0.0047	0.0047
0.60	0.0059	0.0059
0.70	0.0069	0.0069
0.80	0.0077	0.0077
0.90	0.0082	0.0082
1.00	0.0083	0.0083

Table 5. Comparison of perturbation method with formula in mechanics of materials for beams with two fixed end and under uniform distributed load when ($EI = 500\frac{N}{m^2}, L = 3m, W = 100\ N$)

X (displacement from left support)	Perturbation Method (PM)	Formula in Mechanics of Material
0.10	0.0007	0.0007
0.20	0.0026	0.0026
0.30	0.0055	0.0055
0.40	0.0090	0.0090
0.50	0.0130	0.0130
0.60	0.0173	0.0173
0.70	0.0216	0.0216
0.80	0.0258	0.0258
0.90	0.0298	0.0298
1.00	0.0334	0.0334
1.10	0.0364	0.0364
1.20	0.0390	0.0390
1.30	0.0407	0.0407
1.40	0.0418	0.0418
1.50	0.0422	0.0422

The results of comparison between Homotopy Perturbation Method (HPM) with formula in mechanics of materials for beams with two fixed end and under uniform distributed load are given in Tables (6-10).

Table 6. Comparison of homotopy perturbation method with formula in mechanics of materials for beams with two fixed end and under uniform distributed load when ($EI = 500\frac{N}{m^2}, L = 1m, W = 100\ N$)

X (displacement from left support)	Homotopy Perturbation Method (HPM)	Formula in Mechanics of Material
0.10	0.000067	0.000067
0.20	0.000213	0.000213
0.30	0.000367	0.000367
0.40	0.000480	0.000480
0.50	0.000521	0.000521

Table 7. Comparison of homotopy perturbation method with formula in mechanics of materials for beams with two fixed end and under uniform distributed load when ($EI = 1000\frac{N}{m^2}, L = 1m, W = 100\ N$)

X (displacement from left support)	Homotopy Perturbation Method (HPM)	Formula in Mechanics of Material
0.10	0.000034	0.000034
0.20	0.000107	0.000107
0.30	0.000184	0.000184
0.40	0.000240	0.000240
0.50	0.000260	0.000260

Table 8. Comparison of homotopy perturbation method with formula in mechanics of materials for beams with two fixed end and under uniform distributed load when ($EI = 1500\frac{N}{m^2}, L = 1m, W = 100\ N$)

X (displacement from left support)	Homotopy Perturbation Method (HPM)	Formula in Mechanics of Material
0.10	0.000022	0.000022
0.20	0.000071	0.000071
0.30	0.000122	0.000122
0.40	0.000160	0.000160
0.50	0.000174	0.000174

Table 9. Comparison of homotopy perturbation method with formula in mechanics of materials for beams with two fixed end and under uniform distributed load when ($EI = 500\frac{N}{m^2}, L = 2m, W = 100\ N$)

X (displacement from left support)	Homotopy Perturbation Method (HPM)	Formula in Mechanics of Material
0.10	0.0003	0.0003
0.20	0.0011	0.0011
0.30	0.0022	0.0022
0.40	0.0034	0.0034
0.50	0.0047	0.0047
0.60	0.0059	0.0059
0.70	0.0069	0.0069
0.80	0.0077	0.0077
0.90	0.0082	0.0082
1.00	0.0083	0.0083

Table 10. Comparison of homotopy perturbation method with formula in mechanics of materials for beams with two fixed end and under uniform distributed load when ($EI = 5000\frac{N}{m^2}, L = 3m, W = 100\ N$)

X(displacement from left support)	Homotopy Perturbation Method (HPM)	Formula in Mechanics of Material
0.10	0.0007	0.0007
0.20	0.0026	0.0026
0.30	0.0055	0.0055
0.40	0.0090	0.0090
0.50	0.0130	0.0130
0.60	0.0173	0.0173
0.70	0.0216	0.0216
0.80	0.0258	0.0258
0.90	0.0298	0.0298
1.00	0.0334	0.0334
1.10	0.0364	0.0364
1.20	0.0390	0.0390
1.30	0.0407	0.0407
1.40	0.0418	0.0418
1.50	0.0422	0.0422

4. Conclusion

In this paper, the Homotopy Perturbation Method and Perturbation method have been successfully applied to the nonlinear differential equation of beam deformation with two fixed end and under uniform distributed load. These methods enable to convert a difficult problem into a simple problem which can easily be solved. Comparisons of the results obtained here provide more realistic solutions, reinforcing the conclusions pointed out by many researchers about the efficiency of these two methods. Therefore the Homotopy Perturbation Method and Perturbation Method are powerful mathematical tools that can be widely applied to structural engineering such as beam problems.

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