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Calculation of exact eigenspectra of the two dimensional noncommutative superoscillators, using supersymmetry

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Abstract

The purpose of this paper is to present a formulation of supersymmetry quantum mechanics on noncommutative space. In particular we construct Supersymmetric Quantum Mechanics (SQM) in terms of N = 2 real supercharges on noncommutative space in arbitrary dimensions. The exact eigenspectra of the two dimensional noncommutative superoscillators is obtained.

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1. Introduction

The algebraic technique of supersymmetry in quantum mechanics (SQM) was first introduced by witten [1,2]. One of the main problems which are involved in many physical is the difference of energy state between ground state and first excited state for potential wells. This is generally solved by using the approximation methods. Recently we calculated these different values for a five fold and seven fold potential wells using supersymmetry in quantum mechanics. We finally generalized it to find a relation for (2n+1)–fold wells [3,4]. We started with a free theory of a bosonic and a fermonic oscillator of unequal frequencies witch is not supersymmetry and showed that in the presence of interactions this theory can become supersymmetry [5]. In this paper we will start supersymmetry in non relativistic quantum mechanics (QM) and calculate the exact eigenspectra of the two dimensional noncommutative superoscillators, using the supersymmetry [6]. At this critical surface, the energy-spectrum of the bosonic sector is infinitely degenerate, while the degeneracy in the spectrum of the fermionic sector gets enhanced by a factor of two for each pair of reduced canonical coordinates.

In this paper we will make use of the momentum space Green's function and Clifford algebra [7]. Nonlinear supersymmetry is a natural generalization of the usual linear supersymmetry. It is realized variously in such different systems as the parabosonic and parafermionic [8].

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2. General Formulation

Consider the noncommutative algebra:

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij},$$
$$[\hat{p}_i, \hat{p}_j] = iB_{ij},$$
$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} + i(1-\delta_{ij})C_{ij},$$
$$i, j = 1,2, \ldots., N \quad (1)$$

where θ_{ij} and B_{ij} are real, antisymmetric matrices are independent of the hermitian operetors $\hat{x}_i, \hat{p}_i$. The elements of the matrix C_{ij} are taken to be zero and its off diagonal elements do not depend on the space and momentum coordinates. We introduce 2N elements ε_i satisfying the following real Clifford algebra:

$$\{\varepsilon_i, \varepsilon_j\} = 2g_{ij} \quad , \quad g_{ij} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \quad (2)$$

where I is an $N \times N$ identity matrix. The signature of the matrix g_{ij} is such that the square of the ε_i's is equal to 1 or -1 depending on whether $i > N$ or $i \leq N$, respectively. In particular:

$$\varepsilon_{N+1}^2 = -\varepsilon_i^2 = 1 \quad , \quad i = 1,2, \ldots\ldots, N \quad (3)$$

We use a particular matrix representation of the Clifford algebra so that the following additional relations are also satisfied:

$$\varepsilon_i^+ = -\varepsilon_i \quad , \quad \varepsilon_{N+1}^+ = \varepsilon_{N+1} \quad (4)$$

where X^+ denotes the hermitian adjoint of X. Such a matrix representation of the Clifford algebra is required to construct hermitian Hamiltonian within our approach. We further introduce the hermitian operator γ [9]:

$$\gamma = \varepsilon_1\varepsilon_2 \dots \varepsilon_{2N-1}\varepsilon_{2N}, \tag{5}$$

That anticommutes with all the ε_i's and $\gamma^2 = 1$. The elements of the Clifford algebra $\varepsilon_i, \varepsilon_{N+1}$ commute with the noncommutative bosonic variables $\hat{x}_i$ and $\hat{p}_i$:

$$[\hat{x}_i, \varepsilon_j] = [\hat{p}_i, \varepsilon_j] = 0,\ [\hat{x}_i, \varepsilon_{N+j}] = [\hat{p}_i, \varepsilon_{N+j}] = 0, \qquad \forall i,j \tag{6}$$

It naturally follows that γ also commutes with all the bosonic coordinates.

We now introduce the supercharges Q_1 and Q_2:

$$Q_1 = \frac{1}{\sqrt{2}}\sum_{i=1}^{N}\left[-i\varepsilon_i\hat{p}_i + \varepsilon_{N+1}\widehat{W}_i\right] \quad , \quad Q_2 = -i\gamma Q_1. \tag{7}$$

The superpotential $\widehat{W}_i$ are real functions of the noncommutative coordinate $\hat{x}_i$ and in general $[\widehat{W}_i, \widehat{W}_j] \neq 0$. Note that both Q_1 and Q_2 are constructed to be hermitian operators for real $\widehat{W}_i$.

The supercharges Q_1 and Q_2 satisfy the following standard superalgebra:

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha,\beta}H \quad , \quad [H, Q_\alpha] = 0 \quad , \quad \alpha,\beta = 1,2: \tag{8}$$

where the Hamiltonian H is given by:

$$H = \frac{1}{2}\sum_{i=1}^{N}\left(\hat{p}_i^2 + \widehat{W}_i^2\right) - \frac{i}{4}\sum_{i,j=1}^{N}\left(B_{ij}\varepsilon_i\varepsilon_j + 2\varepsilon_i\varepsilon_{N+1}[\hat{p}_i, \widehat{W}_j] + i\varepsilon_{N+i}\varepsilon_{N+j}[\widehat{W}_i, \widehat{W}_j]\right). \tag{9}$$

The Hamiltonian H is hermitian, since it is given by the square of the hermitian operator $Q_1(Q_2)$. The hermiticity of H can also be checked explicitly using Eq. (4). The term containing B_{ij} arises due to the noncommutativity among momentum operators. Similary, the last term in H that is proportional to $[\widehat{W}_i, \widehat{W}_j]$ arises due to the noncommutativity among space coordinates. Such a term is absent for supersymmetry quantum mechanics on commutative space.

A few comments are in order at this point:

(i) if we allow θ_{ij}, B_{ij} and C_{ij} to be functions of the noncommutative coordinates $\hat{x}_i, \hat{p}_i$ instead of c-number marrices, the whole analysis up to the construction of the superhamiltonian (Eq. (9)) remains valid. It is worth mentioning here that the Jacobi identities severely restrict the choice of the operators $\theta_{ij}, B_{ij}, C_{ij}$ for such more general theories.

(ii) In the standard construction of supersymmetric quantum mechanics on the commutative space, one usually introduces fermionic variables ψ_i and its conjugate ψ_i^+:

$$\psi_i = \frac{i}{2}(\varepsilon_i - \varepsilon_{N+i}) \quad ; \quad \psi_i^+ = \frac{i}{2}(\varepsilon_i + \varepsilon_{N+i}), \tag{10}$$

So that the eigenstates can be labeled in terms of the total fermion number $N_F = \sum_i \psi_i^+\psi_i$. However, one can check that H contains terms of the form $\psi_i\psi_j$, $\psi_i^+\psi_j^+$ for $B_{ij}, \theta_{ij} \neq 0$, implying N_f is not a conservsd quantity.

(iii) let us define another set of supercharges q_1 and q_2:

$$q_1 = -\frac{1}{\sqrt{2}}\sum_{i=1}^{N}\left(\varepsilon_{N+i}\hat{p}_i + i\varepsilon_i\widehat{W}_i\right), \quad q_2 = -i\gamma q_1 \tag{11}$$

These two supercharges satisfy the standard superalgebra (Eq. (8)) with the Hamiltonian $h \equiv q_1^2 = q_2^2$:

$$h = \frac{1}{2}\sum_{i=1}^{N}\left(\hat{p}_i^2 + \widehat{W}_i^2\right) + \frac{i}{4}\sum_{i,j=1}^{N}\left(B_{ij}\varepsilon_{N+i}\varepsilon_{N+j} - 2\varepsilon_i\varepsilon_{N+j}[\hat{p}_j, \widehat{W}_i] + i\varepsilon_i\varepsilon_j[\widehat{W}_i, \widehat{W}_j]\right). \tag{12}$$

In the commutative limit $B_{ij} = \theta_{ij} = C_{ij} = 0$, the two Hamiltonian H and h are identical. Moreover, in the same limit, the pair of charges (Q_1, q_1) and the pair (Q_2, q_2) satisfy the superalgebra (Eq. (8)) separately. Although, (Q_1, Q_2, q_1, q_2) do not close under an enlarged N = 4 superalgebra, we have the freedom of choosing any pair of the supercharges $\{(Q_1, Q_2), (q_1, q_2), (Q_1, q_1), (Q_2, q_2)\}$ for a given N = 2 super-Hamiltonian H = h on the commutative space. However, if we take any of the parameters $\theta_{ij}, B_{ij}, C_{ij}$ non-zero, such a freedom is completely lost. We are compelled to choose either the pair (Q_1, Q_2) or (q_1, q_2) and of course, in general, the super-Hamiltonian H and h are not identical.

3. Superoscillators

We will be working with Q_1, Q_2 and H in the rest of the paper. We also consider $\theta_{ij}, B_{ij}, C_{ij}$ as constant matrices from now on. For the case of superoscillator (SO), we choose $\widehat{W}_i = \omega\hat{x}_i$.

The Hamiltonian now reads:

$$H = H_b - H_f$$
$$H_b \equiv \frac{1}{2}\sum_{i=1}^{N}(\hat{p}_i^2 + \omega^2\hat{x}_i^2)$$
$$H_f \equiv \frac{\omega}{2}\left(\sum_{i=1}^{N}\varepsilon_i\varepsilon_{N+i} - \sum_{i\neq j}\varepsilon_{N+i}\varepsilon_j C_{ij}\right) + \frac{i}{4}\sum_{i,j=1}^{N}\left(B_{ij}\varepsilon_i\varepsilon_j - \omega^2\theta_{ij}\varepsilon_{N+i}\varepsilon_{N+j}\right). \tag{13}$$

In the limit of $B_{ij}, \theta_{ij}, C_{ij} \to 0$, the Hamiltonian of the superoscillator on commutative space is recovered. Note that H_b is a function of the noncommutative coordinates and the momenta only, whereas H_f is solely expressed in terms of the elements of the Clifford algebra. This implies that H_b and H_f can be diagonalized separately.

A comment is in order at this point. The N dimensional superoscillator is described in terms of 2N elements of the Clifford algebra (Eq. 2). So, the Hamiltonian H_f can be expressed in terms of the linear combination of the N (2N-1) generators $\sum_{ij}^{1,2,3}$:

$$\sum\nolimits_{ij}^{1} = \frac{i}{4}[\varepsilon_i, \varepsilon_j],$$

$$\sum\nolimits_{ij}^{2} = \frac{i}{4}\left[\varepsilon_{N+i}, \varepsilon_{N+j}\right],$$
$$\sum\nolimits_{ij}^{3} = \frac{i}{4}\left[\varepsilon_{i}, \varepsilon_{N+j}\right], \tag{14}$$

of the group SO (N,N) of rank N [10]. Thus, in general, the eigenvalues of H_f can be expressed in terms of N quantum numbers. Further, in the matrix representation of the Clifford algebra (Eq. 2), both the generators $\sum_{ij}^{1,2,3}$, and the Hamiltonian H_f can be expressed in terms of $2^N \times 2^N$ dimensional matrices. Thus, each of the N quantum numbers can take only two values.

4. Two dimensional superoscillator

We now specialize in this section to the noncommutative plane for which the antisymmetric matrices B_{ij} and θ_{ij} can be parametrized in terms of signal parameters B and θ, respectively. In particular:

$$C_{12} \equiv \varphi_1 \;,\; C_{21} \equiv -\varphi_2 \;,\; B_{ij} \equiv \epsilon_{ij} B \;,\; \theta_{ij} \equiv \epsilon_{ij}\theta \;,\quad i,j = 1,2. \tag{15}$$

With this choise for B_{ij} and θ_{ij} there are many physically equivalent representations [5,6] of the algebra (Eq. (1)) in terms of commutative canonically conjugate variables x_i and p_i satifying:

$$\left[x_i, x_j\right] = 0,\; \left[p_i, p_j\right] = 0,\; \left[x_i, p_j\right] = i\delta_{ij}. \tag{16}$$

As shown in Ref. [11,12], the Hamiltonian H_b with N = 2 can be equivalently written as a two-dimensional anisotropic oscillator. In particular:

$$H_b = \tfrac{1}{2}\left[\Omega_+(p_1^2 + x_1^2) + \Omega_-(p_2^2 + x_2^2)\right]$$
$$2\Omega_\pm = \sqrt{(\omega^2\theta - B)^2 + 4\omega^2 + \omega^2(\varphi_1 + \varphi_2)^2} \pm \sqrt{(\omega^2\theta + B)^2 + \omega^2(\varphi_1 - \varphi_2)^2}\,,\quad k > 0$$
$$2\Omega_\pm = \pm\sqrt{(\omega^2\theta - B)^2 + 4\omega^2 + \omega^2(\varphi_1 + \varphi_2)^2} + \sqrt{(\omega^2\theta + B)^2 + \omega^2(\varphi_1 - \varphi_2)^2}\,,\quad k < 0$$
$$\Omega_+ = \sqrt{(\omega^2\theta + B)^2 + \omega^2(\varphi_1 - \varphi_2)^2},$$
$$\Omega_- = 0\,,\quad k = 0, \tag{17}$$

where the positive and the negative values of the parameter $k \equiv \omega^2(1 - B\theta + \varphi_1\varphi_2)$ correspond to two different phases of the noncommutative oscillators, the critical value being k = 0. The energy eigenvalues E_b of H_b are:

$$E_b = \left(n_+^b + \tfrac{1}{2}\right)\Omega_+ + \left(n_-^b + \tfrac{1}{2}\right)\Omega_-, \tag{18}$$

where the quantum numbers $n_\pm^b$ can take any nonnegative integer values.

In order to diagonalize H_f, we use the following matrix representation of the Clifford algebra:

$$\varepsilon_1 = i\sigma_1 \otimes \sigma_2 \quad ; \quad \varepsilon_2 = i\sigma_2 \otimes \sigma_2,$$
$$\varepsilon_3 = -\sigma_3 \otimes \sigma_2 \quad ; \quad \varepsilon_4 = I \otimes \sigma_3, \tag{19}$$

where $\sigma_{1,2,3}$ are the three Pauli matrices and I is a 2×2 identity matrix. The Hamiltonian H_f is a 4×4 matrix:

$$H_f = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix},$$
$$\alpha \equiv \begin{pmatrix} \frac{B}{2} & \frac{\omega}{2}(i + \varphi_1) \\ \frac{\omega}{2}(-i + \varphi_1) & -\frac{B}{2} \end{pmatrix},$$
$$\beta \equiv \begin{pmatrix} -\frac{\omega^2\theta}{2} & \frac{\omega}{2}(i + \varphi_2) \\ \frac{\omega}{2}(-i + \varphi_2) & \frac{\omega^2\theta}{2} \end{pmatrix}. \tag{20}$$

The eigenvalues of H_f are,

$$\left\{-\tfrac{1}{2}(\Omega_+ - \Omega_-), \tfrac{1}{2}(\Omega_+ - \Omega_-), -\tfrac{1}{2}(\Omega_+ + \Omega_-), \tfrac{1}{2}(\Omega_+ + \Omega_-)\right\}. \tag{21}$$

For $\Omega_- = 0$, there are only two independent eigenvalues $\pm\frac{1}{2}\Omega_+$, each of them being doubly degenerate. The frequency Ω_- vanishes only in the critical phase k = 0.

5. Conclusions

We have constructed supersymmetric quantum mechanics on the noncommutative space in terms of N = 2 real supercharges. This construction is valid in any arbitrary dimensions for arbitrary superpotential.

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