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Conformal graviton two-point function in de Sitter space

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Abstract

The conformally invariant linearized de Sitter gravitational wave equation has recently received a lot of attention. In this article, using the ambient space notations, we solve this field equation in five various cases. In each case, it has been shown that the solution can be written as the product of a generalized symmetric polarization tensor of rank 2 and a massless minimally or conformally coupled scalar field in de Sitter space-time. The conformally covariant graviton two-point functions have been calculated in terms of the massless minimally or conformally coupled scalar two-point functions, using ambient space notations. In the case of massless minimally coupled scalar field, the Krien space quantization has been used to avoid the violation of de Sitter invariance. The two-point functions are expressed in terms of de Sitter intrinsic coordinates from their ambient space counterparts, which are free of any theoretical problem.

Keywords: Conformal gravity, Conformal two-point function, De Sitter space

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Introduction

The recent observational data are strongly in favor of a positive acceleration of the present universe. Therefore, in a first approximation, the background space-time might be considered as a de Sitter (dS) space-time, and the quantization of the massless tensor field (spin-2) on dS space (dS linear quantum gravity) presents an excellent modality for further researches.

Gravitational fields are long ranged and seem to travel with the speed of light. In the first approximation, their equations are expected to be conformally invariant. Einstein's theory of gravitation, in the background field method ($g_{ab} = g_{ab}^{(BG)} + h_{ab}$) and linear approximation, can be considered as a massless symmetric tensor field of rank 2, h_{ab} on a fixed background $g_{ab}^{(BG)}$, such as dS space. It is well known that the massless fields propagate on the light cone and are invariant under the conformal group $SO(2,4)$.

Einstein's classical theory of gravitation, as well as the equation of h_{ab}, is not conformally invariant, thus could not be considered as a comprehensive universal theory of gravitational fields. Many believe that conformal invariance may be the key that will solve the problems of quantum gravity.

In dS space, mass is not an invariant parameter for the set of observable transformations under the dS group $SO(2,4)$. The concept of light cone propagation, however, does exist and leads to conformal invariance. 'Massless' is used in reference to propagation on the dS light cone (conformal invariance). The term 'massive' refers to fields that in their Minkowskian limit (zero curvature) reduce to the massive Minkowskian fields [1].

The organization of this paper is as follows: The 'Notation' section is devoted to a brief review of the ambient space notations. The conformal space, Dirac's six-cone formalism, and conformally invariant gravitational field equations are introduced in the 'dS conformal field equation' section. In 'Solution to the conformal field equation' section, the conformally invariant gravitational field equation is solved in five different cases using ambient space formalism. It is shown that the solution can be written as the multiplication of a generalized symmetric polarization tensor of rank 2 and a massless minimally or conformally coupled scalar field in dS space. The

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conformally invariant graviton two-point functions have been calculated in 'The conformal two-point functions' section using ambient space notations. It is shown that the conformally invariant graviton two-point functions can be written in terms of the two-point functions of the massless minimally or conformally coupled scalar two-point functions in dS ambient space. In the case of massless minimally coupled scalar field, in order to avoid violation of dS invariance, the Krein space quantization has been used [2]. The two-point functions are written in terms of dS intrinsic coordinates from their ambient space counterparts. Finally, a brief discussion and conclusion has been given in the 'Conclusions' section.

Notation

The dS space-time is made identical to the four dimensional one-sheeted hyperboloid

$$X_H = \left\{x \in R^5; x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2}\right\},\quad \alpha, \beta, \ldots = 0, 1, 2, 3, 4, \tag{1}$$

where $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1)$. The dS metric is

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta|_{x^2 = -H^{-2}} = g_{ab}^{dS} dX^a dX^b,\quad a, b, \ldots = 0, 1, 2, 3, \tag{2}$$

where X^a's are the four space-time intrinsic coordinates in dS hyperboloid. Different coordinate systems can be chosen [3]. Any geometrical object in this space can be written in terms of the four local intrinsic coordinates X^a or in terms of the five global ambient space coordinates x^α.

The metric (2) is a solution to Einstein's equation with the cosmological constant $\Lambda = 3H^2$

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0 \tag{3}$$

The linearized field equation is obtained by setting $g_{ab} = g_{ab}^{(\mathrm{dS})} + h_{ab}$ where $g_{ab}^{(\mathrm{dS})}$ is the background metric (2) and h_{ab} is its fluctuation. The wave equation which is obtained through the above linear approximation is [4-6]

$$\frac{1}{2}\left(\nabla_a \nabla^c h_{bc} + \nabla_b \nabla^c h_{ac} - \nabla^2 h_{ab} - \nabla_a \nabla_b h' + 2H^2 h_{ab}\right) + \frac{1}{2} g_{ab}^{(dS)}\left(\nabla^2 h' - \nabla_c \nabla_d h^{cd} + H^2 h'\right) = 0 \tag{4}$$

where $\nabla^2 \equiv \nabla_a \nabla^a$ is the Laplace-Beltrami operator on dS space, and $h' \equiv h_a^a$ is the trace of h_{ab} with respect to the background metric. Here, ∇^b is the background covariant derivative; indices are raised and lowered by the background metric. The field equation (4) is invariant under the gauge transformation

$$h_{ab} \to h_{ab}^{(gt)} + \nabla_a \zeta_b + \nabla_b \zeta_a \tag{5}$$

where ζ_a is an arbitrary vector.

Now, we adopt the tensor field $K_{\alpha\beta}(x)$ in ambient space notations. In these notations, the solutions to the field equations are easily written out in terms of scalar fields. The symmetric tensor field $K_{\alpha\beta}(x)$ is defined on dS space-time and satisfies the transversality condition [7,8]

$$x.K_{\alpha\beta}(x) = 0, \quad i.e., \quad x^\alpha K_{\alpha\beta}(x) = 0 = x^\beta K_{\alpha\beta}(x) \tag{6}$$

The covariant derivative in the ambient space notations is

$$D_\beta T_{\alpha 1..\alpha i..\alpha n} = \bar{\partial}_\beta T_{\alpha 1..\alpha i..\alpha n} - H^2 \sum_{i=1}^{n} x_{\alpha i} T_{\alpha 1..\beta..\alpha n} \tag{7}$$

where $\bar{\partial}_\alpha$ is a tangential (or transverse) derivative in dS space

$$\bar{\partial}_\alpha = \theta_{\alpha\beta} \partial^\beta = \partial_\alpha + H^2 x_\alpha x.\partial, \quad x.\bar{\partial} = 0 \tag{8}$$

$\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$ is the transverse projector.

In order to express Equation 4 in terms of the ambient space notations, we use the fact that the 'intrinsic' field $h_{ab}(X)$ is locally determined by the transverse tensor field $K_{\alpha\beta}(x)$ through

$$K_{\alpha\beta}(x) = \frac{\partial X^a}{\partial x^\alpha} \frac{\partial X^b}{\partial x^\beta} h_{ab}(X) \tag{9}$$

It is easily shown that the metric $g_{ab}^{(\mathrm{dS})}$ corresponds to the transverse projector $\theta_{\alpha\beta}$.

The linearized field equation in ambient space notations is

$$\left(Q_2^{(1)} + 6\right) K_{\alpha\beta} + D_2(\partial_2.K) = 0 \tag{10}$$

where $\partial_2.K = \partial.K - H^2 x K' - \frac{1}{2}\bar{\partial}K'$, $D_2(\partial_2.K) = H^{-2}S(\bar{\partial} - H^2 x)(\partial_2.K)$, and $Q_2^{(1)}$ is one of the two Casimir operators of dS group and

$$Q_2^{(1)}K = \left(Q_0^{(1)} - 6\right)K + 2\eta K' + 2Sx(\partial.K) - 2S\partial(x.K) \tag{11}$$

$$Q_1^{(1)}\Lambda(x) = \left(Q_0^{(1)} - 2\right)\Lambda(x) + 2x\bar{\partial}.\Lambda(x) - 2\partial x.\Lambda(x) \tag{12}$$

for any arbitrary vector Λ. The symmetrizer S is defined for the two vectors ξ_α and ω_β by $S(\xi_\alpha\omega_\beta) = \xi_\alpha\omega_\beta + \omega_\alpha\xi_\beta$. K' is the trace of K, and $Q_0^{(1)} = -H^{-2}\bar{\partial}^2$.

The field equation (10) is invariant under the following gauge transformation [8]

$$K_{\alpha\beta} \rightarrow K_{\alpha\beta}^{(gt)} = K_{\alpha\beta} + D_2\Lambda_g \tag{13}$$

where Λ_g is an arbitrary vector. Because of this gauge freedom, the field equation (10) can be written in the following form

$$\left(Q_2^{(1)} + 6\right)K_{\alpha\beta} + cD_2(\partial_2.K) = 0 \tag{14}$$

where c is a gauge-fixing parameter. The field equation (14) has been considered in [9,10]. In the physical case $\bar{\partial}.K = 0 = K'$ or equivalently $c = 0$, the field equation (10) or (14) reduce to the following eigenvalue equation

$$\left(Q_2^{(1)} + \left\langle Q_2^{(1)}\right\rangle\right)K_{\alpha\beta} = 0 \text{ where } \left\langle Q_2^{(1)}\right\rangle = -6 \tag{15}$$

which has been considered in our previous work [11].

dS conformal field equation

Barut and Bohm [1] have shown that for the unitary irreducible representation of the conformal group, the value of the conformal Casimir operator is 9. However, according to the calculation of Binegar et al. [12] for the tensor field of rank 2 and conformal degree 0, this value becomes 8 and the tensor field of rank 2 does not correspond to any unitary irreducible representation of the conformal group. In other words, the tensor field that carries physical representations of the conformal group must be a tensor field of higher rank. The conformally invariant wave equations for scalar and vector fields have been obtained in [13] using Dirac's six-cone formalism, which transformed according to the unitary irreducible representation of dS group. The conformal space and six-cone formalism was first used by Dirac to obtain the conformally invariant equations [7]. This formalism was developed by Mack and Salam [14] and many others [15-17]. This approach to conformal symmetry leads to the best path to exploit the physical symmetry in contrast to approaches based on group theoretical treatment of state vector spaces. The conformal invariance, as well as the light cone propagation, constitutes the basis for constructing 'massless' field in dS space. The conformally invariant field equations for a rank-2 tensor field has been obtained in [11],

$$\left(Q_2^{(1)} + 4\right)\left(Q_2^{(1)} + 6\right)K_{\alpha\beta} = 0, \quad \partial.K = 0 = K' \tag{16}$$

based on Dirac's six-cone formalism, which corresponds to the two representations of dS group, namely $\Pi_{2,1}^{\pm}$ and $\Pi_{2,2}^{\pm}$. However, this equation does not transform according to the dS and conformal groups simultaneously.

Using a mixed symmetric field of rank 3 in Dirac's six-cone formalism, it has been shown that if we want $K_{\alpha\beta}$ to be a physical state of the dS and conformal groups simultaneously, it must satisfy a field equation of order 6 [18],

$$\left(Q_2^{(1)} + 4\right)^2\left(Q_2^{(1)} + 6\right)K_{\alpha\beta} = 0, \quad \partial.K = 0 = K' \tag{17}$$

In the next section, we solve this field equation in five various cases. In each case, it is shown that the solution can be written as the product of a generalized symmetric polarization tensor of rank 2 and a massless minimally or conformally coupled scalar field in dS space-time.

Solution to the conformal field equation

A general solution of Equation 17 can be constructed from the combination of a scalar field and two vector fields. Let us first introduce a traceless and transverse tensor field $K_{\alpha\beta}$ in terms of a five-dimensional constant vector $Z_1(Z_{1\alpha})$ and a scalar field ϕ_1 and two vector fields, K_1 and K_g, by putting [8,11]

$$K = \theta\phi_1 + S\bar{Z}_1K_1 + D_2K_g \tag{18}$$

$$K' = 2\phi_1 + 2Z_1.K_1 + 2H^2(x.Z_1)(x.K_1) + 2H^{-2}\bar{\partial}.K_g - 2x.K_g = 0 \tag{19}$$

where $\bar{Z}_1\alpha = \theta_{\alpha\beta} Z_1^{\beta}$. In solving Equation 17, the following various cases are distinguishable.

The physical case

A simple solution to the conformally covariant field equation (17) is the solution of the following equation

$$\left(Q_2^{(1)} + 6\right) K_{\alpha\beta} = 0 \tag{20}$$

This is an eigenvalue equation with the eigenvalue $\langle Q_2^{(1)} \rangle = -6$. From the group theoretical point of view, this corresponds to unitary irreducible representations of dS group in the sense of $\Pi_{2,2}{}^{\pm}$ discrete series which reduce to the physical representations of the Poincare group in the zero curvature limit. This is why we call it the physical case.

Using ansatz (18) to the above field equation, we have

$$\begin{cases} \left(Q_0^{(1)} + 6\right)\phi_1 = -4 Z_1.K_1, & (I) \\ \left(Q_1^{(1)} + 2\right)K_1 = 0, \;\; x.K_1 = 0 = \partial.K_1, & (\mathrm{II}) \\ \left(Q_1^{(1)} + 6\right)K_g = 2H^2 x.Z_1 K_1 & (\mathrm{III}) \end{cases} \tag{21}$$

The solution to this system of differential equations is [11]

$$\begin{cases} \phi_1 = -\frac{2}{3} Z_1.K_1, \\ K_1 = \bar{Z}_2 \phi_2 - \frac{1}{2} D_1 \left[Z_2.\bar{\partial}\phi_2 + 2H^2 (x.Z_2)\phi_2\right], \\ K_g = \frac{1}{3}\left[H^2 (x.Z_1) K_1 + \frac{1}{9} D_1 (Z_1.K_1)\right], \end{cases} \tag{22}$$

where Z_1 and Z_2 are two constant 5 vectors, and $x.K_g = 0$, $\bar{\partial}.K_g = \frac{1}{3} H^2 Z_1.K_1$. ϕ_2 is a massless minimally coupled scalar field in dS space

$$Q_0^{(1)}\phi_2 = 0, \;\; \phi_2 = (Hx.\xi)^{\sigma}, \;\; \sigma = 0, \; -3, \;\; \xi^2 = 0 \tag{23}$$

It is easily shown that

$$K_{\alpha\beta} = \varepsilon_{\alpha\beta}(x, Z_1, Z_2, \xi)\phi_2 \tag{24}$$

where $\varepsilon_{\alpha\beta}$ is a generalized symmetric polarization tensor

$$\begin{aligned} \varepsilon_{\alpha\beta} &= S\left(A\bar{Z}_{1\alpha}\bar{Z}_{2\beta} + B\bar{Z}_{1\alpha}\bar{\xi}_{\beta} + C\bar{Z}_{2\alpha}\bar{\xi}_{\beta} + D\bar{\xi}_{\alpha}\bar{\xi}_{\beta} + E\theta_{\alpha\beta}\right) \\ A &= -\frac{\sigma}{27}(\sigma + 21), \\ B &= -\frac{\sigma}{27}\left(\sigma^2 + 21\sigma + 36\right)\frac{x.Z_2}{x.\xi}, \\ C &= -\frac{\sigma}{27}\left(\sigma^2 + 12\sigma + 9\right)\frac{x.Z_1}{x.\xi}, \\ D &= -\frac{\sigma(\sigma-1)}{54(x.\xi)^2}\left[\sigma H^{-2} Z_1.Z_2 + \left(\sigma^2 + 12\sigma + 18\right)(x.Z_1)(x.Z_2)\right], \\ E &= -\frac{\sigma}{54}\left[(\sigma - 9) Z_1.Z_2 + (\sigma+3)(\sigma+2) H^2 (x.Z_1)(x.Z_2)\right], \end{aligned} \tag{25}$$

where the conditions $Z_1.\,\xi = 0 = Z_2.\,\xi$ are used for simplicity. These conditions reduce the number of degrees of freedom of constant 5 vectors Z_1 and Z_2 from 5 to 4. It is obvious that

$$\begin{aligned} K' = -\frac{\sigma(\sigma+3)}{27}&\left[(\sigma+2) Z_1.Z_2 + \left(\sigma^2 + 16\sigma + 46\right)\right. \\ &\left.\times H^2 (x.Z_1)(x.Z_2)\right] \; \phi_2 = 0 \;\text{ for }\; \sigma = 0, -3. \end{aligned} \tag{26}$$

The two solutions that correspond to two different values of σ are

$$K_{\alpha\beta}^{(1)}(x) = 0 \quad \text{for} \quad \sigma = 0$$

and

$$\begin{aligned} K_{\alpha\beta}^{(2)}(x) = 2S\Big[&\bar{Z}_{1\alpha}\bar{Z}_{2\beta} - \frac{x.Z_2}{x.\xi}\bar{Z}_{1\alpha}\bar{\xi}_{\beta} - \frac{x.Z_1}{x.\xi}\bar{Z}_{2\alpha}\bar{\xi}_{\beta} \\ &-\frac{1}{3} Z_1.Z_2\theta_{\alpha\beta} + \frac{1}{3}\left(H^{-2}(Z_1.Z_2)\right. \\ &\left.+3(x.Z_1)(x.Z_2)\right)\frac{\bar{\xi}_{\alpha}\bar{\xi}_{\beta}}{(x.\xi)^2}\Big](Hx.\xi)^{-3} \;\text{ for }\; \sigma = -3. \end{aligned}$$

The semi-physical case

Another simple solution to the conformally covariant field equation (17) is the solution of the equation

$$\left(Q_2^{(1)} + 4\right) K_{\alpha\beta} = 0 \tag{27}$$

This is also an eigenvalue equation with the eigenvalue $\langle Q_2^{(1)} \rangle = -4$. From the group theoretical point of view, this corresponds to unitary irreducible representations of dS group in the sense of $\Pi_{2,1}^{\pm}$ discrete series. It has no Poincare correspondence in the zero curvature limit. This is why we call it the semi-physical case.

Using ansatz (18) to the above field equation, we have

$$\begin{cases} \left(Q_1^{(1)} + 4\right)\phi_1 = -4Z_1.K_1, & (I) \\ Q_1^{(1)}K_1 = 0, \quad x.K_1 = 0 = \partial.K_1, & \text{(II)} \\ \left(Q_1^{(1)} + 4\right)K_g = 2H^2 x.Z_1 K_1 & \text{(III)} \end{cases} \tag{28}$$

The solution to this system of differential equations is [11]

$$\begin{cases} \phi_1 = -\frac{2}{3}Z_1.K_1, \\ K_1 = \left(\bar{Z}_2 - \frac{x.Z_2}{x.\xi}\bar{\xi}\right)\phi, \\ K_g = \frac{1}{2}H^2\left[3(x.Z_1)K_1 - x(Z_1.K_1) + H^{-2}(Z_1.\bar{\partial})K_1 - \frac{1}{3}D_1(Z_1.K_1)\right], \end{cases} \tag{29}$$

where ϕ is a massless conformally coupled scalar field in dS space,

$$\left(Q_0^{(1)} - 2\right)\phi = 0, \quad \phi = (Hx.\xi)^\sigma, \quad \sigma = -1, \ -2, \ \xi^2 = 0 \tag{30}$$

It is easily shown that

$$K_{\alpha\beta} = \varepsilon'_{\alpha\beta}(x, \ Z_1, \ Z_2, \ \xi)\phi \tag{31}$$

where ε' is a generalized symmetric polarization tensor

$$\begin{aligned} \varepsilon'_{\alpha\beta} &= S\left(A'\bar{Z}_{1\alpha}\bar{Z}_{2\beta} + B'\bar{Z}_{1\alpha}\bar{\xi}_\beta + C'\bar{Z}_{2\alpha}\bar{\xi}_\beta + D'\bar{\xi}_\alpha\bar{\xi}_\beta + E'\theta_{\alpha\beta}\right) \\ A' &= \frac{1}{2}(\sigma + 5), \\ B' &= -\frac{1}{2}(\sigma + 5)\frac{x.Z_2}{x.\xi}, \\ C' &= -\frac{1}{2}(\sigma - 1)(\sigma + 3)\frac{x.Z_1}{x.\xi}, \\ D' &= -\frac{(\sigma + 3)(\sigma - 1)}{6(x.\xi)^2}\left[H^{-2}Z_1.Z_2 + 3(x.Z_1)(x.Z_2)\right], \\ E' &= -\frac{1}{6}(\sigma + 5) \end{aligned} \tag{32}$$

where the conditions $Z_1.\xi = 0 = Z_2.\xi$ are used for simplicity. These conditions reduce the number of degrees of freedom of constant 5 vectors Z_1 and Z_2 from 5 to 4. It is obvious that

$$\begin{aligned} K' &= -\frac{(\sigma + 1)(\sigma + 2)}{2}(Z_1.Z_2)\phi \\ &= 0 \text{ for } \sigma = -1, \quad -2 \end{aligned} \tag{33}$$

The two solutions that correspond to two different values of σ are

$$\begin{aligned} K^{(1)}_{\alpha\beta}(x) = 2S\Big[&\bar{Z}_{1\alpha}\bar{Z}_{2\beta} - \frac{x.Z_2}{x.\xi}\bar{Z}_{1\alpha}\bar{\xi}_\beta - \frac{x.Z_1}{x.\xi}\bar{Z}_{2\alpha}\bar{\xi}_\beta \\ &- \frac{1}{3}Z_1.Z_2\theta_{\alpha\beta} + \frac{1}{3}\big(H^{-2}(Z_1.Z_2) \\ &+ 3(x.Z_1)(x.Z_2)\big)\frac{\bar{\xi}_\alpha\bar{\xi}_\beta}{(x.\xi)^2}\Big](Hx.\xi)^{-1} \text{ for } \sigma = -1 \end{aligned}$$

and

$$\begin{aligned} K^{(2)}_{\alpha\beta}(x) = \frac{3}{2}S\Big[&\bar{Z}_{1\alpha}\bar{Z}_{2\beta} - \frac{x.Z_2}{x.\xi}\bar{Z}_{1\alpha}\bar{\xi}_\beta - \frac{x.Z_1}{x.\xi}\bar{Z}_{2\alpha}\bar{\xi}_\beta \\ &- \frac{1}{3}Z_1.Z_2\theta_{\alpha\beta} + \frac{1}{3}\big(H^{-2}(Z_1.Z_2) \\ &+ 3(x.Z_1)(x.Z_2)\big)\frac{\bar{\xi}_\alpha\bar{\xi}_\beta}{(x.\xi)^2}\Big](Hx.\xi)^{-2} \text{ for } \sigma = -2. \end{aligned}$$

Special case I

A special solution to the conformal field equation (17) may be considered as the solution of the equation

$$\left(Q_2^{(1)} + 4\right)^2 K_{\alpha\beta} = 0. \tag{34}$$

Using the ansatz (18) to the above field equation, we have

$$\begin{cases} \left(Q_0^{(1)} + 4\right)^2\phi_1 + 4\left(Q_0^{(1)} + 4\right)Z_1.K + 4Q_1^{(1)}Z_1.K_1, & (I) \\ \left[Q_1^{(1)}\right]^2 K_1 = 0, \quad x.K_1 = 0 = \partial.K_1, & \text{(II)} \\ \left(Q_1^{(1)} + 4\right)^2 K_g = 2H^2\left[x.Z_1\left(Q_1^{(1)} + 4\right)K_1 + Q_1^{(1)}x.Z_1K_1\right] & \text{(III)} \end{cases} \tag{35}$$

The simplest solution to this system of differential equations is [11]

$$\begin{cases} \phi_1 = -\frac{2}{3}Z_1.K_1, \\ K_1 = \left(\bar{Z}_2 - \frac{x.Z_2}{x.\xi}\bar{\xi}\right)\phi, \quad \phi : \text{conformally coupled} \\ K_g = \frac{1}{2}H^2\left[3(x.Z_1)K_1 - x(Z_1.K_1) + H^{-2}(Z_1.\bar{\partial})K_1 - \frac{1}{3}D_1(Z_1.K_1)\right] \end{cases} \tag{36}$$

The solutions are the same as that given in the semi-physical case.

Special case II

A simple solution to the conformally covariant field equation may be the solution of the equation

$$\left(Q_2^{(1)} + 4\right)\left(Q_2^{(1)} + 6\right)K_{\alpha\beta} = 0, \tag{37}$$

which is considered as a special case.

Using ansatz (18) to the above field equation, the two following cases are distinguishable:

First case

$$\begin{cases} \left(Q_0^{(1)} + 4\right)\left(Q_0^{(1)} + 6\right)\phi_1 + 8Q_0^{(1)}Z_1.K_1 + 16Z_1.K_1, & (I) \\ \left(Q_1^{(1)} + 2\right)K_1 = 0, \quad x.K_1 = 0 = \partial.K_1, & (\mathrm{II}) \\ \left(Q_1^{(1)} + 4\right)\left(Q_1^{(1)} + 6\right)K_g = 4H^2\left[\left(Q_1^{(1)} + 5\right)x.Z_1K_1 + H^{-2}(Z_1.\bar{\partial})K_1 - xZ_1.K_1\right]. & (\mathrm{III}) \end{cases} \quad (38)$$

In this case, we have

$$\begin{cases} \phi_1 = -\dfrac{2}{3}Z_1.K_1, \\ K_1 = \bar{Z}_2\phi_2 - \dfrac{1}{2}D_1\left(Z_2.\bar{\partial}\phi_2 + 2H^2x.Z_2\phi_2\right), \\ K_g = \dfrac{1}{9}\left[H^2xZ_1.K_1 + Z_1.\bar{\partial}K_1 + \dfrac{2}{3}H^2D_1Z_1.K_1\right]. \end{cases} \quad (39)$$

So, the solutions are the same as that given in the physical case.

Second case

$$\begin{cases} \left(Q_0^{(1)} + 4\right)\left(Q_0^{(1)} + 6\right)\phi_1 + 8Q_0^{(1)}Z_1.K_1 + 16Z_1.K_1, & (I) \\ Q_1^{(1)}K_1 = 0, \quad x.K_1 = 0 = \partial.K_1, & (\mathrm{II}) \\ \left(Q_1^{(1)} + 4\right)\left(Q_1^{(1)} + 6\right)K_g = 4H^2\left[\left(Q_1^{(1)} + 5\right)x.Z_1K_1 + H^{-2}(Z_1.\bar{\partial})K_1 - xZ_1.K_1\right]. & (\mathrm{III}) \end{cases} \quad (40)$$

with the solutions

$$\begin{cases} \phi_1 = -\dfrac{2}{3}Z_1.K_1, \\ K_1 = \left(\bar{Z}_2 - \dfrac{x.Z_2}{x.\xi}\bar{\xi}\right)\phi, \quad \phi: \text{conformally coupled} \\ K_g = \dfrac{1}{2}H^2\left[3(x.Z_1)K_1 - x(Z_1.K_1) + H^{-2}(Z_1.\bar{\partial})K_1 - \dfrac{1}{3}D_1(Z_1.K_1)\right]. \end{cases} \quad (41)$$

The solutions are the same as that given in the semi-physical case.

The general case

In this subsection, I solve the conformally invariant linearized gravitational field equation in its general form, that is

$$\left(Q_2^{(1)} + 4\right)^2\left(Q_2^{(1)} + 6\right)K_{\alpha\beta} = 0. \quad (42)$$

Using ansatz (18) to this general form of the conformally covariant field equation, the two following cases are distinguishable:

First case

$$\begin{cases} \left(Q_0^{(1)}+4\right)^2\left(Q_0^{(1)}+6\right)\phi_1 + 8\left(Q_0^{(1)}+4\right)\left(Q_0^{(1)}+2\right)Z_1.K_1 + 4Q_1^{(1)}\left(Q_1^{(1)}+2\right)Z_1.K_1 = 0, & (I) \\ \left(Q_1^{(1)}+2\right)K_1 = 0, \quad x.K_1 = 0 = \partial.K_1, & (\mathrm{II}) \\ \left(Q_1^{(1)}+4\right)^2\left(Q_1^{(1)}+6\right)K_g = 2H^2\left[x.Z_1Q_1^{(1)}\left(Q_1^{(1)}+2\right)K_1 + Q_1^{(1)}\left(Z_1.K_1\left(Q_1^{(1)}+6\right)\right)K_1 + Q_1^{(1)}\left(Q_1^{(1)}+4\right)(x.Z_1K_1) + 4x.Z_1\left(Q_1^{(1)}+6\right)K_1\right] & (\mathrm{III}) \end{cases} \quad (43)$$

In this case, we have

$$\begin{cases} \phi_1 = -\dfrac{2}{3}Z_1.K_1, \\ K_1 = \bar{Z}_2\phi_2 - \dfrac{1}{2}D_1\left[Z_2.\bar{\partial}\phi_2 + 2H^2(x.Z_2)\phi_2\right], \\ K_g = \dfrac{1}{3}H^2\left[(x.Z_1)K_1 + \dfrac{1}{9}D_1(Z_1.K_1)\right]. \end{cases} \quad (44)$$

So, the solutions are the same as that given in the physical case.

Second case

$$\begin{cases} \left(Q_0^{(1)}+4\right)^2\left(Q_0^{(1)}+6\right)\phi_1+8\left(Q_0^{(1)}+4\right)\left(Q_0^{(1)}+2\right)Z_1.K_1+4Q_1^{(1)}\left(Q_1^{(1)}+2\right)Z_1.K_1=0, & (I)\\ Q_1^{(1)}K_1=0, \quad x.K_1=0=\partial.K_1, & (\mathrm{II})\\ \left(Q_1^{(1)}+4\right)^2\left(Q_1^{(1)}+6\right)K_g=2H^2\left[x.Z_1Q_1^{(1)}\left(Q_1^{(1)}+2\right)K_1+Q_1^{(1)}\left(Z_1.K_1\left(Q_1^{(1)}+6\right)\right)K_1+Q_1^{(1)}\left(Q_1^{(1)}+4\right)(x.Z_1K_1)+4x.Z_1\left(Q_1^{(1)}+6\right)K_1\right] & (\mathrm{III}) \end{cases} \tag{45}$$

with the solutions

$$\begin{cases} \phi_1=-\frac{2}{3}Z_1.K_1,\\ K_1=\left(\bar{Z}_2-\frac{x.Z_2}{x.\xi}\bar{\xi}\right)\phi, \quad \phi: \text{ conformally coupled}\\ K_g=\frac{1}{2}H^2\left[3(x.Z_1)K_1-x(Z_1.K_1)+H^{-2}(Z_1.\bar{\partial})K_1-\frac{1}{3}D_1(Z_1.K_1)\right]. \end{cases} \tag{46}$$

The solutions are the same as that given in the semi-physical case.

In summary, so far, we have shown that the general solution to the conformal field equation can be written in terms of the physical and semi-physical solutions of the wave equations. In the next section, we will calculate the physical and semi-physical two-point functions.

The conformal two-point functions

The physical two-point function

The two-point function $W_{\alpha\beta\alpha'\beta'}(x,x')$, which is a solution of the wave equation with respect to x or x', can be found simply in terms of the scalar two-point function. Very similar to the recurrence formula (18), let us try the following possibility [11]:

$$W(x,x')=\theta\theta' W_0(x,x')+SS'\theta.\theta' W_1(x,x') + D_2D_2' W_g(x,x'), \tag{47}$$

where W, W_1, and W_g are transverse bi-vectors, W_0 is bi-scalar, and $D_2D'_2=D'_2D_2$. By imposing this two-point function to obey Equation 20, with respect to x, it is easy to show that

$$\begin{cases} \left(Q_0^{(1)}+6\right)\theta' W_0=-4S'\theta'.W_1, & (I)\\ \left(Q_1^{(1)}+2\right)W_1=0, \quad \partial.W_1=0, & (\mathrm{II})\\ \left(Q_1^{(1)}+6\right)W_g=2H^2S'(x.\theta')W_1 & (\mathrm{III}) \end{cases} \tag{48}$$

The solutions to Equation 48 are as follows [11]:

$$\theta' W_0(x,x')=-\frac{2}{3}S'\theta'.W_1(x,x'), \tag{49}$$

$$D_2' W_g=\frac{1}{3}H^2S'\left[\frac{1}{9}D_1(\theta'.W_1)+x(\theta'.W_1)\right], \tag{50}$$

$$W_1(x,x')=\left(\theta.\theta'-\frac{1}{2}D_1\left[\theta'.\partial+2H^2(x.\theta')\right]\right)W_{mc}(x,x'), \tag{51}$$

where $W_{mc}(x,x')$ is the two-point function for the massless minimally coupled scalar field. In the 'Gupta-Bleuler vacuum' state [19],

$$W_{mc}(x,x')=\frac{iH^2}{8\pi^2}\varepsilon\left(x^0-x'^0\right) \times\left[\delta(1-Z(x,x'))+\theta(Z(x,x')-1)\right], \tag{52}$$

with

$$\varepsilon\left(x^0-x'^0\right)=\begin{cases} 1 & x^0>x'^0,\\ 0 & x^0=x'^0,\\ -1 & x^0<x'^0. \end{cases} \qquad Z=-H^2x.x' \tag{53}$$

The two-point function (47) also satisfies the field equation (18) with respect to x'; in this case, one can obtain

$$\begin{cases} \left(Q_0'^{(1)}+6\right)\theta W_0=-4S\theta.W_1, & (I)\\ \left(Q_1'^{(1)}+2\right)W_1=0, \quad \partial'.W_1=0, & (\mathrm{II})\\ \left(Q_1'^{(1)}+6\right)D_2W_g=2H^2S(x'.\theta)W_1 & (\mathrm{III}) \end{cases} \tag{54}$$

with the solutions [12]

$$\theta W_0(x,x')=-\frac{2}{3}S\theta.W_1(x,x'), \tag{55}$$

$$D_2W_g=\frac{1}{3}H^2S\left[\frac{1}{9}D_1'(\theta.W_1)+x'(\theta.W_1)\right], \tag{56}$$

$$W_1(x,x')=\left(\theta'.\theta-\frac{1}{2}D_1'\left[\theta.\partial+2H^2x'.\theta\right]\right)W_{mc}(x,x'), \tag{57}$$

where the primed operators act on the primed coordinates only.

Using Equations 49 to 52 in Equation 47, we have

$$W_{\alpha\beta\alpha'\beta'}(x,x') = \frac{-2Z}{27(1-Z^2)^2}SS'\Big[(1-Z^2)(3Z^2-2)\theta_{\alpha\beta}\theta'_{\alpha'\beta'} + 3H^2(1+Z^2)\theta'_{\alpha'\beta'}(x'.\theta_\alpha)(x'.\theta_\beta) + 3H^2(1+Z^2)\theta_{\alpha\beta}(x.\theta'_{\alpha'})(x.\theta'_{\beta'}) + \frac{3H^4}{1-Z^2}(21-2Z^2-3Z^4)(x.\theta'_{\alpha'})(x.\theta'_{\beta'}) \times (x'.\theta_\alpha)(x'.\theta_\beta) - \frac{2}{Z}(20+Z^2-9Z^4) \times H^2(\theta_\alpha.\theta'_{\alpha'})(x'.\theta_\beta)(x.\theta'_{\beta'}) + (1-Z^2) \times (11-9Z^2)(\theta_\alpha.\theta'_{\alpha'})(\theta_\beta.\theta'_{\beta'})\Big]\frac{d}{dZ}W_{mc}(Z), \quad (58)$$

where the relation $Q_0^{(1)}W_{mc} = 0$ has been used and

$$\frac{d}{dZ}W_{mc}(Z) = \frac{iH^2}{8\pi^2}\frac{Z-2}{Z-1}\varepsilon(x^0 - x'^0)\delta(Z-1). \quad (59)$$

It is evident that if one uses Equations 55 to 57 instead of Equations 49 to 51 in Equation 47, the final result is none other than Equation 58.

Equation 58 is the explicit form of the physical (traceless and divergenceless) two-point function in ambient space notations. It can be expressed in terms of the dS intrinsic coordinates [8,11]

$$Q_{aba'b'}(X,X') = \frac{-2}{27(1-Z^2)}SS'\big[Z(3Z^2-2)g_{ab}g'_{a'b'} + 3Z(1+Z^2)(g'_{a'b'}n_an_b + g_{ab}n_{a'}n_{b'}) + (40+32Z-20Z^2-6Z^3+9Z^4-9Z^5) \times n_an_bn_{a'}n_{b'} + (40+9Z^2+9Z^4)g_{aa'}n_bn_{b'} + Z(11-9Z^2)g_{aa'}g_{bb'}\big]\frac{d}{dZ}W_{mc}(x,x'), \quad (60)$$

which is dS-invariant and free of any divergences.

The two-point functions $W_{\alpha\beta\alpha'\beta'}(x,x')$ and $Q_{aba'b'}(X,X')$ are related through

$$Q_{aba'b'}(X,X') = \frac{\partial x^\alpha}{\partial X^a}\frac{\partial x^\beta}{\partial X^b}\frac{\partial x'^{\alpha'}}{\partial X'^{a'}}\frac{\partial x'^{\beta'}}{\partial X'^{b'}}W_{\alpha\beta\alpha'\beta'}(x,x').$$

The semi-physical two-point function

Substituting the two-point function (47) in Equation 27 with respect to x, we have

$$\begin{cases} \left(Q_0^{(1)}+4\right)\theta' W_0 = -4S'\theta'.W_1, & (I) \\ Q_1^{(1)}W_1 = 0, \qquad \bar\partial.W_1 = 0, & (II) \\ \left(Q_1^{(1)}+4\right)D'_2W_g = 2H^2S'[x.\theta' W_1] & (III) \end{cases} \quad (61)$$

The solution to part II of Equation 61 is [20]

$$W_1(x,x') = \left[\theta.\theta'\frac{Z-2}{1-Z} + H^2(x.\theta')(x'.\theta)\frac{3-Z}{(1-Z)^2}\right]W_{cc}(x,x'), \quad (62)$$

in which W_{cc} is the conformally coupled scalar two-point function and

$$\left(Q_0^{(1)}-2\right)W_{cc} = 0, \quad (63)$$

$$W_{cc}(x,x') = -\frac{1}{8\pi}\left[\frac{1}{1-Z} + i\pi\varepsilon(x^0-x'^0)\theta(Z-1)\right], \quad (64)$$

$$\frac{d}{dZ}W_{cc}(x,x') = \frac{1}{1-Z}W_{cc}(x,x'). \quad (65)$$

In summary, the solution to the above system of equations is [21]

$$\theta' W_0(x,x') = -\frac{2}{3}S'\theta'.W_1(x,x') \quad (66)$$

$$D'_2W_g(x,x') = \frac{1}{3}H^2S'\left[\frac{1}{9}D_1\theta'. + x\theta'.\right]W_1(x,x'), \quad (67)$$

$$W_1(x,x') = \left[\theta.\theta'\frac{Z-2}{1-Z} + H^2(x.\theta')(x'.\theta)\frac{3-Z}{(1-Z)^2}\right]W_{cc}(x,x'). \quad (68)$$

The two-point function (47) also satisfies the field equation (27) with respect to x'; in this case, one can obtain

$$\begin{cases} \left(Q_0'^{(1)}+4\right)\theta W_0 = -4S\theta.W_1, & (I) \\ Q_1'^{(1)}W_1 = 0, \qquad \partial'.W_1 = 0, & (II) \\ \left(Q_1'^{(1)}+4\right)D_2W_g = 2H^2S(x'.\theta)W_1 & (III) \end{cases} \quad (69)$$

with the solutions

$$\theta W_0(x,x') = -\frac{2}{3}S\theta.W_1(x,x'), \tag{70}$$

$$D_2 W_g(x,x') = \frac{1}{2}H^2 S\big[3(x'.\theta)W_1 + H^{-2}\theta.\bar{\partial}W_1 + x(\theta.W_1) - \frac{1}{3}D_1'(\theta.W_1)\big], \tag{71}$$

and

$$W_1(x,x') = \left[\theta.\theta'\frac{Z-2}{1-Z} + H^2(x.\theta')(x'.\theta)\frac{3-Z}{(1-Z)^2}\right]W_{cc}(x,x'). \tag{72}$$

Using Equations 66 to 68 in Equation 47, after some calculation, we have

$$W_{\alpha\beta\alpha'\beta'}(x,x') = \frac{2}{3(1-Z)^3}SS'\big[(1-Z)(Z^2-3Z+1)\theta_{\alpha\beta}\theta'_{\alpha'\beta'} + H^2(Z-4)\theta'_{\alpha'\beta'}(x'.\theta_\alpha)(x'.\theta_\beta) + H^2(Z-4) \times \theta_{\alpha\beta}(x.\theta'_{\alpha'})(x.\theta'_{\beta'}) + \frac{3H^4}{1-Z}(-8+5Z-Z^2) \times (x.\theta')(x.\theta')(x'.\theta)(x'.\theta) + 6(Z^2-4Z+5) \times H^2(\theta.\theta')(x'.\theta)(x.\theta') + (1-Z) \times (-8+9Z-3Z^2)(\theta.\theta')(\theta.\theta')\big]W_{c.c.}(x,x'). \tag{73}$$

It is evident that if one uses Equations 70 to 72 instead of Equations 66 to 68 in Equation 47, the final result is none other than Equation 73. Equation 73 is the explicit form of the traceless and divergenceless semi-physical two-point function in the ambient space notations. It can be expressed in terms of the dS intrinsic coordinates [8,11]

$$Q_{aba'b'}(X,X') = \frac{2}{3(1-Z)^2}SS'\big[(Z^2-3Z+1)g_{ab}g'_{a'b'} + (Z-4)(1+Z)(g'_{a'b'}n_a n_b + g_{ab}n_{a'}n_{b'}) + 2(-31+5Z+11Z^2-3Z^3-6Z^4) \times n_a n_b n_{a'} n_{b'} + 2(1-Z)(23-21Z+6Z^2) \times g_{aa'}n_b n_{b'} + (-8+9Z-3Z^2)g_{aa'}g_{bb'}\big] \times W_{c.c.}(x,x'), \tag{74}$$

which is free of any theoretical divergences.

Conclusions

The conformally invariant linearized gravitational field equation has been solved in five different cases using ambient space formalism. It has been shown that the solution can be written as the product of a generalized symmetric polarization tensor of rank 2 and a massless minimally or conformally coupled scalar field in dS space. The conformally invariant graviton two-point functions have been calculated using ambient space notations. It has been shown that the conformally invariant graviton two-point functions can be written in terms of the two-point functions of the massless minimally or conformally coupled scalar two-point functions in dS space. In the case of massless minimally coupled scalar field, the Krein space quantization has been used to avoid violation of dS invariance. The two-point functions are written in terms of dS intrinsic coordinates from their ambient space counterparts. The results of this paper may open the road to the quantization of gravitational fields without any theoretical problem.

Competing interests

The author declares that he has no competing interests.

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