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Semi-analytic algorithms for the electrohydrodynamic flow equation

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Abstract

In this paper, we consider the nonlinear boundary value problem for the electrohydrodynamic (EHD) flow of a fluid in an ion-drag configuration in a circular cylindrical conduit. This phenomenon is governed by a nonlinear second-order differential equation. The degree of nonlinearity is determined by a nondimensional parameter α. We present two semi-analytic algorithms to solve the EHD flow equation for various values of relevant parameters based on optimal homotopy asymptotic method (OHAM) and optimal homotopy analysis method. In 1999, Paullet has shown that for large α, the solutions are qualitatively different from those calculated by Mckee in 1997. Both of our solutions obtained by OHAM and optimal homotopy analysis method are qualitatively similar with Paullet's solutions.

Keywords: Optimal homotopy asymptotic method (OHAM), Optimal homotopy analysis method, Electrohydrodynamic flow, Square residual error, Gauss quadrature

MSC: 34B15, 34B16, 76E30

Background

The electrohydrodynamic flow of a fluid in an ion-drag configuration in a circular cylindrical conduit is governed by a nonlinear second-order ordinary differential equation. Perturbation solutions of fluid velocities for different orders of nonlinearities were given by McKee et al. [1]. In their study, a description of the problem was presented in which the governing equations were reduced to the following nonlinear boundary value problem (BVP):

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + H^2\left(1 - \frac{u}{1-\alpha u}\right) = 0, \quad 0 < r < 1, \quad (1)$$

subject to boundary conditions

$$u'(0) = 0, u(1) = 0, \quad (2)$$

where $u(r)$ is the fluid velocity, r is the radial distance from the centre of the cylindrical conduit, H is the Hartman electric number and the parameter α is a measure of the strength of the nonlinearity. In [1], the authors used a regular perturbation technique to obtain two perturbation solutions given by Equations 4 and 6 depending on the value of the nonlinearity control parameter α.

For $\alpha << 1$ and assuming a solution of the form

$$u(r) = \sum_{n=0}^{\infty} \alpha^n u_n(r;\alpha). \quad (3)$$

Mckee et al. [1] obtained the $O(\alpha^3)$ perturbation solution as

$$u(r;\alpha) = 1 - \frac{I_0(Hr)}{I_0(H)} + \alpha[(u_1(Hr) + C_1)I_0(Hr) + v_1(Hr)K_0(Hr)] + \alpha^2[(u_2(Hr) + C_2)I_0(Hr) + v_2(Hr)K_0(Hr)]. \quad (4)$$

Similarly, for $\alpha >> 1$, the authors [1] proposed that the solution to the BVP could be expanded in the series of the form

$$u(r) = \sum_{n=0}^{\infty} \alpha^{-n} u_n(r;\alpha) \quad (5)$$

with an $O(1)$ leading-order term and obtained the perturbation solution as

$$u(r;\alpha) = \frac{H^2}{4}\left(1 + \frac{1}{\alpha}\right)(1 - r^2) + \frac{1}{\alpha^2}\left(2\int_0^r \frac{\log(1-s^2)}{s}ds + \frac{\pi^2}{6}\right). \quad (6)$$

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Paullet [2] proved the existence and uniqueness of the solution to the BVP (1) and (2) in the following theorem:

Theorem 1. *For any $\alpha > 0$ and any $H^2 \neq 0$, there exists a solution to the BVP (1) and (2). Furthermore, this solution is monotonically decreasing and satisfies $0 < u(r) < 1 / (\alpha + 1)$ for all $r \in (0,1)$.*

Remark 1. By a solution of Equations 1 and 2, we mean a function $u(r) \in C[0,1] \cap C^2(0,1)$ that satisfies Equation 1 for $0 < r < 1$ along with Equation 2. In order for such a function to be a solution, we must necessarily have $u(r) < 1/\alpha$ on (0,1); if $u(r)$ ever equals $1/\alpha$, it is no longer C^2, owing to the term $u(r) / (1 - \alpha u(r))$ in Equation 1 [2].

Paullet [2] claimed an error in the perturbation and numerical solutions given in [1] for large values of α. This stems from the fact that for large α, the solutions are $O(1/\alpha)$, not $O(1)$ as proposed in the perturbation expansion used in [1]. For $\alpha << 1$, our solutions obtained by the two semi-analytic algorithms (proposed in the 'Application of OHAM to EHD flow problem' and 'Application of optimal homotopy analysis method to EHD flow problem' subsections) are in complete agreement with those of [1] and [2], but for $\alpha >> 1$, the proposed solution profiles are similar to those of [2]. Thus, based on our work in this paper, we support Paullet's solution profiles for $\alpha >> 1$.

Recently, Mastroberardino [3] proposed an analytical method based on the homotopy analysis method (HAM) to find the solutions of Equations 1 and 2 for $\alpha \in (0,1]$ and H^2 up to 4. The author [3] has shown that the homotopy perturbation method (HPM) yields a divergent solution for all of the cases considered. The HAM solutions are quite accurate for lower values of the parameters α and H^2, but the accuracy decreases rather fast for higher values of these parameters even though fairly higher order (20 to be precise) solutions were considered, as shown in Table one of [3]. Further, from Figure two of [3], we observe that even a slight deviation from the optimal value of $\hbar$ causes a huge square residual error for $\alpha = 0.5$, 1 and $H^2 = 4$. This, along with the qualitative difference between the solution profiles of Mckee et al. [1] and Paullet [2] for $\alpha >> 1$, motivated us to look for algorithms giving accurate solutions for higher values of the parameters as well.

The aim of the present work is to propose two algorithms for the solutions of the above BVP (1) and (2) for all values of relevant parameters using optimal homotopy asymptotic method (OHAM) and optimal homotopy analysis method. We show that even the third- and fourth-order solutions obtained from OHAM and optimal homotopy analysis method, respectively, are highly accurate for $\alpha >> 1$. From Figures 1 and 2, we see that the square residual errors E_3/E_4 are stable even for larger deviations from the optimal value of C_1 (in the case of OHAM) or c_1 (in the case of optimal homotopy analysis method) as compared to the deviations in $\hbar$ (in the case of HAM). A comparison is made between OHAM, optimal homotopy analysis method and HAM via exact square residual errors. It is shown that for higher values of α and H^2, the respective third- and fourth-order OHAM and optimal homotopy analysis method solutions are more accurate than the 20th-order HAM solutions. Further, the central processing unit (CPU) time is also calculated and compared for these methods, establishing the superiority of OHAM and optimal homotopy analysis method over the HAM solution. Also, the solution profiles shown for $\alpha = 4$, 10, $H^2 = 1$ and $\alpha = 4$, 10, $H^2 = 10$ by Figures 3 and 4 respectively match Paullet's solution profiles shown in Figures one and two of [2] for the corresponding values of the parameters.

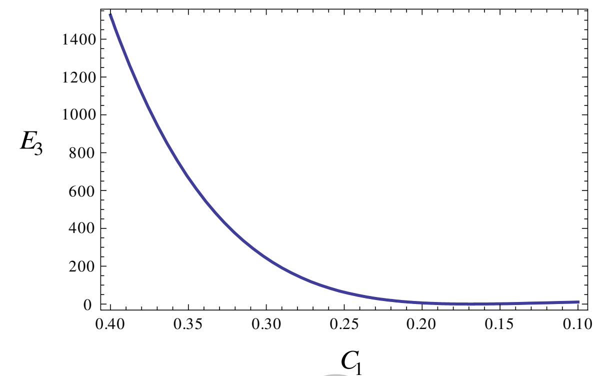

Figure 1 **Square residual error E_3 for the third-order OHAM ($C_2 = -0.1809055$, $C_3 = 0.275378$).**

Analysis of the method

Optimal homotopy asymptotic method

Since the last two decades, homotopy perturbation method [4] and homotopy analysis method [5] based on the topological concept of homotopy have become very popular in solving nonlinear ordinary/partial differential

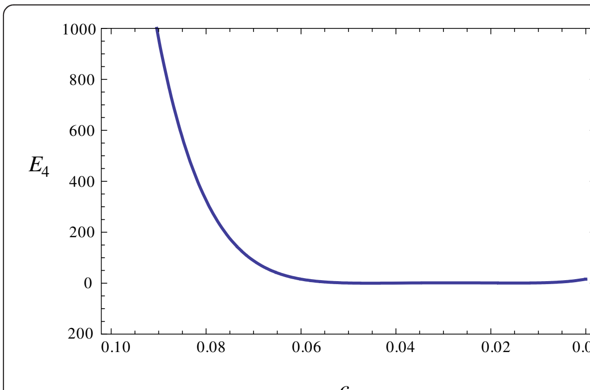

Figure 2 **Square residual error E_4 of the fourth-order optimal HAM ($c_2 = 0.0242099$, $c_3 = -6.19495$).**

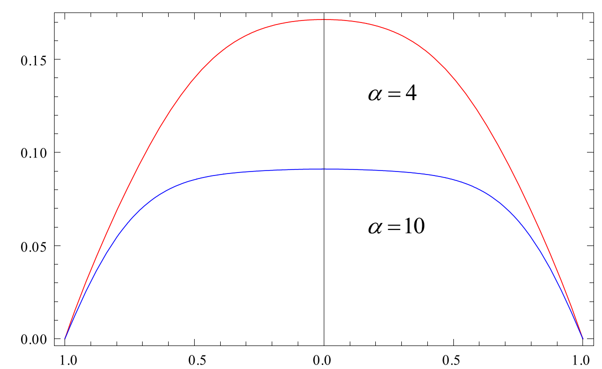

Figure 3 OHAM solution *u*(*r*) for H^2 = 1 and *α* = 4 (red), *α* = 10 (blue).

equations [6,7]. Later, in 2008, Marinca et al. [8-11] introduced a new analytical method known as OHAM to solve a variety of nonlinear problems. This method is straightforward and reliable, and it does not need to look for $\hbar$ curves like HAM. This method provides us a convenient way to control the convergence of the series solution and allows the adjustment of the convergence region wherever it is needed via unspecified number of convergence control parameters. These parameters are determined in such a way that the optimal values are yielded unlike the $\hbar$ curve method used in HAM. The OHAM solution generally agrees with the exact solution at larger domains as compared to HPM and HAM solutions. OHAM is based on a generalized zeroth-order deformation equation (8) and does not consider the *m*th-order deformation equation like HAM.

We apply OHAM to the following nonlinear differential equation:

$$A(u(r)) + f(r) = 0,\ B(u) = 0, \Leftrightarrow L(u(r)) + f(r) + N(u(r)) = 0,\ \ B(u) = 0, \tag{7}$$

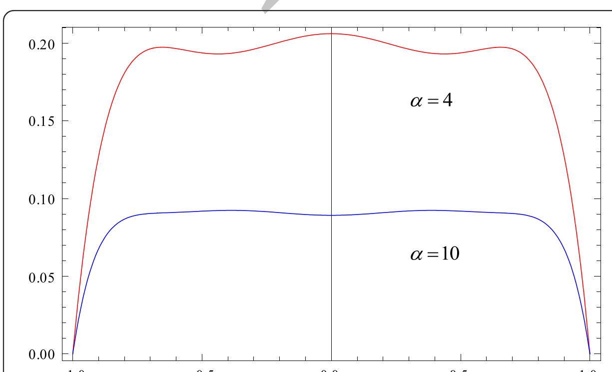

Figure 4 OHAM solution *u*(*r*) for H^2 = 10 and *α* = 4 (red), *α* = 10 (blue).

where, $A = L + N$, L is a linear operator, N is a nonlinear operator, r denotes the independent variable, $u(r)$ is an unknown function, $f(r)$ is a known function and B is a boundary operator.

A homotopy $h(\varphi(r,q),q)$: $R \times [0,1] \rightarrow R$ is constructed satisfying

$$(1-q)[L(\varphi(r,q)) + f(r)] = H(q)[L(\varphi(r,q)) + f(r) + N(\varphi(r,q))],\ \ B(\varphi(r,q)) = 0, \tag{8}$$

where, $q \in [0,1]$ is an embedding parameter, $H(q)$ is a nonzero auxiliary function for $q \neq 0$ and $H(0) = 0$. As the embedding parameter q increases from 0 to 1, the $\varphi(r,q)$ varies from the initial approximation $u_0(r)$ to the solution $u(r)$.

The auxiliary function $H(q)$ is chosen as

$$H(q) = qC_1 + q^2C_2 + q^3C_3 + \cdots, \tag{9}$$

where C_1, C_2, $C_3, \cdots$ are constants to be determined. It is very important to choose these constants properly since the convergence of the solution depends on them.

Expanding $\varphi(r,q)$ in a power series with respect to the parameter q, we get

$$\phi(r,q,C_1,C_2,\cdots) = u_0(r) + u_1(r,C_1)q + u_2(r,C_1,C_2)q^2 + \cdots. \tag{10}$$

Substituting Equation 10 into Equation 8 and equating the coefficients of like powers of q, we obtain the following equations:

$$L(u_0(r)) + f(r) = 0,\ \ B(u_0) = 0, \tag{11}$$

$$L(u_1(r)) = L(u_0(r)) + f(r) + C_1[L(u_0(r)) + N_0(u_0(r)) + f(r)],\ \ B(u_1) = 0, \tag{12}$$

$$L(u_2(r)) = L(u_1(r)) + C_1[L(u_1(r)) + N_1(u_0(r),u_1(r))] + C_2[L(u_0(r)) + N_0(u_0(r)) + f(r)], B(u_2) = 0, \tag{13}$$

$$L(u_3(r)) = L(u_2(r)) + C_1[L(u_2(r)) + N_2(u_0(r),u_1(r),u_2(r))] + C_2[L(u_1(r)) + N_1(u_0(r),u_1(r))] + C_3[L(u_0(r)) + N_0(u_0(r)) + f(r)],\ \ B(u_3) = 0, \tag{14}$$

$$\vdots$$
$$\begin{aligned} L(u_m(r)) =& L(u_{m-1}(r)) + C_1[L(u_{m-1}) \\ &+ N_{m-1}(u_0, u_1, \cdots, u_{m-1})] \\ &+ C_2[L(u_{m-2}) + N_{m-2}(u_0, u_1, \cdots, u_{m-2})] \\ &+ \cdots + C_{m-1}[L(u_1(r)) + N_1(u_0(r), u_1(r))] \\ &+ C_m[L(u_0(r)) + N_0(u_0(r)) + f(r)], \\ B(u_m) =& 0, \end{aligned} \tag{15}$$

where $N_m(u_0, u_1, \cdots, u_m)$ is the coefficient of q^m in the expansion of $N(\phi(r,q))$ about the embedding parameter q.

The above equations are called the zeroth-, first-, second- and mth-order problems, respectively.

As $q \to 1$, in Equation 10,

$$u(r, C_i) = u_0(r) + \sum_{k=1}^{\infty} u(r, C_i). \tag{16}$$

Truncating Equation 16 at level $k = m$, the mth-order solution is given by

$$\begin{aligned} \widetilde{u}_m(r, C_1, C_2, \cdots, C_m) =& u_0(r) \\ &+ \sum_{i=1}^{m} u_i(r, C_1, C_2, \cdots, C_i). \end{aligned} \tag{17}$$

Substituting Equation 17 into Equation 7, one gets the following residual:

$$\begin{aligned} R_m(r, C_1, C_2, \cdots, C_m) =& L(\widetilde{u}_m(r, C_1, C_2, \cdots, C_m)) \\ &+ f(r) \\ &+ N(\widetilde{u}_m(r, C_1, C_2, \cdots, C_m)). \end{aligned} \tag{18}$$

If $R_m = 0$, then $\tilde{u}_m$ will be the exact solution, which does not happen in practice, especially in nonlinear problems. In order to find the optimal values of C_i, $i = 1, 2, 3, \cdots$, we first construct the functional (called the square residual error)

$$E_m(C_1, C_2, \cdots, C_m) = \int_a^b R^2(r, C_1, C_2, \cdots, C_m)dr, \tag{19}$$

([a,b] being the domain of the problem), and then minimizing it, we get

$$\frac{\partial E_m}{\partial C_1} = \frac{\partial E_m}{\partial C_2} = \cdots = \frac{\partial E_m}{\partial C_m} = 0. \tag{20}$$

Substituting the optimal values of C_i's obtained from Equation 20 into Equation 17, the mth-order approximate solution $\tilde{u}_m$ is obtained.

As discussed in [12], computing $E_n(C_1, C_2, C_3, \cdots, C_n)$ directly with a symbolic computational software is impractical. Thus, we approximate Equation 19 using a Gaussian quadrature with eight nodes followed by minimizing Equation 19 using the Mathematica function *Minimize*; the optimal values of these convergence control parameters are obtained.

OHAM faces the practical problem of computing higher order iterates since as many number of parameters C_m are to be computed as the order of iterates. The method is well suited for the electrohydrodynamic (EHD) problem as shown by the various solution profiles and tables.

Application of OHAM to EHD flow problem

Choosing $L = \frac{d^2}{dr^2}$ and $f = H^2$ and using Equation 11, the zeroth-order problem for Equation 1 with boundary conditions (2)

$$\begin{aligned} \frac{d^2u_0(r)}{dr^2} + H^2 &= 0, {u_0}'(1) = 0, u_0(0) \\ &= 0 \text{ gives } u_0(r) \\ &= \frac{1}{2}\left(H^2 - H^2r^2\right). \end{aligned} \tag{21}$$

As the fourth-order approximate solution gives a very accurate solution even for higher values of the nonlinearity parameter α and the Hartmann electric number H. These iterates are obtained from Equations 12 to 14, and the first three iterates are listed below:

- First-order problem:

$$\begin{aligned} \frac{d^2u_1(r)}{dr^2} =& \frac{d^2u_0(r)}{dr^2} + H^2 + C_1\left[\frac{d^2u_0(r)}{dr^2} + \frac{1}{r}\frac{du_0(r)}{dr} \right. \\ & -\alpha u_0(r)\frac{d^2u_0(r)}{dr^2} - \frac{\alpha}{r}u_0(r)\frac{du_0(r)}{dr} \\ & \left. + H^2(1 - (1+\alpha)u_0(r))\right], \\ & {u_1}'(1) = 0, u_1(0) = 0. \end{aligned} \tag{22}$$

- Second-order problem:

$$\frac{d^2u_2(r)}{dr^2} = \frac{d^2u_1(r)}{dr^2} + C_1\Bigg[\frac{d^2u_1(r)}{dr^2} + \frac{1}{r}\frac{du_1(r)}{dr} - \alpha u_0(r)\frac{d^2u_1(r)}{dr^2} - \alpha u_1(r)\frac{d^2u_0(r)}{dr^2} - \frac{\alpha}{r}u_0(r)\frac{du_1(r)}{dr} - \frac{\alpha}{r}u_1(r)\frac{du_0(r)}{dr} - H^2(1+\alpha)u_0(r)\Bigg] + C_2\left[Au_0 + H^2\right], u_2'(1) = 0, u_2(0) = 0. \tag{23}$$

- Third-order problem:

$$\frac{d^2u_3(r)}{dr^2} = \frac{d^2u_2(r)}{dr^2} + C_1\Bigg[\frac{d^2u_2(r)}{dr^2} + \frac{1}{r}\frac{du_2(r)}{dr} - \alpha u_0(r)\frac{d^2u_2(r)}{dr^2} - \alpha u_1(r)\frac{d^2u_1(r)}{dr^2} - \alpha u_2(r)\frac{d^2u_0(r)}{dr^2} - \frac{\alpha}{r}u_0(r)\frac{du_2(r)}{dr} - \frac{\alpha}{r}u_1(r)\frac{du_1(r)}{dr} - \frac{\alpha}{r}u_2(r)\frac{du_0(r)}{dr} - H^2(1+\alpha)u_2(r)\Bigg] + C_2\Bigg[\frac{d^2u_1(r)}{dr^2} + \frac{1}{r}\frac{du_1(r)}{dr} - \alpha u_0(r)\frac{d^2u_1(r)}{dr^2} - \alpha u_1(r)\frac{d^2u_0(r)}{dr^2} - \frac{\alpha}{r}u_0(r)\frac{du_1(r)}{dr} - \frac{\alpha}{r}u_1(r)\frac{du_0(r)}{dr} - H^2(1+\alpha)u_0(r)\Bigg] + C_3\left[Au_0 + H^2\right],\ u_3'(1) = 0, u_3(0) = 0. \tag{24}$$

Solving Equations 22 to 24, we get the first three iterates as follows:

$$u_1(r) = \frac{1}{24}H^2C_1\left(-1+r^2\right)\left[12 + H^2\left(-5+r^2\right)(-1+\alpha)\right], \tag{25}$$

$$u_2(r) = \frac{1}{720}H^2\left(-1+r^2\right)\Bigg[30(C_1+C_2) \times \left(12 + H^2\left(-5+r^2\right)(-1+\alpha)\right) + {C_1}^2\Big\{720 + 10H^2(47 - 77\alpha + r^2(-7+13\alpha) + H^4\Big(61 - 260 + 199\alpha^2 - 14r^2\left(1 - 5\alpha + 4\alpha^2\right) + r^4\left(1 - 10\alpha + 9\alpha^2\right)\Big)\Big\}\Bigg], \tag{26}$$

$$u_3(r) = -\frac{1}{604,800}H^2\left(-1+r^2\right)[25,200(C_2+C_3) \times \left(12 + H^2\left(-5+r^2\right)(-1+\alpha)\right) + 1,680{C_1}^2\{720 + 10H^2\left(47 - 77\alpha + r^2(-7+13\alpha)\right) + H^4(61 - 260 + 199\alpha^2 - 14r^2\left(1-5\alpha+4\alpha^2\right) + r^4\left(1 - 10\alpha + 9\alpha^2\right))\} + {C_1}^3\{1,209,600 + 5,600H(193 - 424\alpha + r^2(-23 + 62\alpha)) + 56H^4\left(4,853 - 26,310\alpha + 25,597\alpha^2 + r^2\left(-922 + 5,940\alpha - 6,278\alpha^2\right) + r^4(53 - 710\alpha + 897\alpha^2\right)) + H^6\left(20,775 - 203,959\alpha + 456,665\alpha^2 - 273,481\alpha^3 r^4\left(405 - 9,989\alpha + 31,835\alpha^2 - 22,251\alpha^3\right) + 15r^6\left(-1 + 49\alpha - 299\alpha^2 + 191\alpha^3\right) + r^2(4,845 + 56,861\alpha - 139,315\alpha^2 + 87,299\alpha^3)\right))\} + 1,680C_1\{180(1+4C_2) + H^4C_2\left(61 - 260\alpha + 199\alpha^2 - 14r^2\left(1 - 5\alpha + 4\alpha^2\right) + r^4\left(1 - 10\alpha + 9\alpha^2\right)\right) + 5H^2(3\left(-5+r^2\right)(-1+\alpha) + 2C_2(47 - 77\alpha + r^2(-7+13\alpha))\}]. \tag{27}$$

Substituting the above iterations in Equation 17, the mth-order approximate solution is obtained as

$$\widetilde{u}_m(r, C_1, C_2, C_3, \cdots, C_m) = u_0(r) + u_1(r, C_1) + u_2(r, C_1, C_2) + \cdots u_m(r, C_1, C_2, C_3, \cdots, C_m). \tag{28}$$

From Equation 18, the mth-order residual is

$$R_m(r, C_1, C_2, C_3, \cdots, C_m) = \frac{d^2\widetilde{u}_m}{dr^2} + \frac{1}{r}\frac{d\widetilde{u}_m}{dr} + H^2\left(1 - \frac{\widetilde{u}_m}{1 - \alpha\widetilde{u}_m}\right) \quad m = 1, 2, 3\cdots. \tag{29}$$

Substituting Equation 29 in Equation 19 and computing the square residual error E_m numerically using the Gauss quadrature formulae with eight node points followed by minimizing E_m, the optimal values of the convergence control parameters C_1, C_2, C_3,$\cdots$, C_m are obtained.

Optimal homotopy analysis method

The optimal homotopy analysis method was first proposed by Liao [12] containing exactly three convergence control parameters at any level of approximation in contrast to OHAM. The optimal homotopy analysis method is based on a generalized zeroth-order deformation equation (31). Liao [12] used special deformation functions which are determined completely by only one characteristic parameter $|c_2| < 1$ and $|c_3| < 1$, respectively. To illustrate the procedure, consider the nonlinear equation

Table 1 Comparison of exact square residual errors at α = 0.5 and different values of H^2

Auxiliary parameters C_i		$\alpha = 0.5$, $H^2 = 0.5$	$\alpha = 0.5$, $H^2 = 1$	$\alpha = 0.5$, $H^2 = 2$	$\alpha = 0.5$, $H^2 = 4$
OHAM	C_1	−0.108084	−0.128187	−0.064627	−0.0236008
	C_2	−0.478377	−0.295243	−0.564313	−0.78968
	C_3	0.527761	0.270073	0.724121	1.19611
	E_3	7.527668×10^{-12}	1.51604×10^{-8}	5.41811×10^{-6}	7.36637×10^{-4}
Optimal homotopy analysis method	c_1	−0.566753	−0.374439	−0.0316471	−0.276117
	c_2	0.0199658	−0.321575	−0.357213	0.389223
	c_3	−0.0812378	−0.391183	−8.72346	−0.37244
	E_4	2.40146×10^{-12}	8.90742×10^{-8}	4.68025×10^{-6}	2.05172×10^{-4}
HAM	$\hbar$	−0.375	−0.276	−0.275	−0.205
	E_{20}	7.772×10^{-12}	1.230×10^{-9}	5.319×10^{-8}	4.568×10^{-5}

$$N[u(r)] = 0. \tag{30}$$

Marinca and Herisanu [9] constructed a general form of the zeroth-order deformation equation:

$$(1 - B(q))L[\phi(r;q) - u_0(r)] = c_1 A(q) N[\phi(r;q)], \;\; r \in \Omega, q \in [0,1], \tag{31}$$

where N is a nonlinear operator, L is a linear operator and $A(q)$ and $B(q)$ are called deformation functions satisfying

$$A(0) = B(0) = 0, A(1) = B(1) = 1, \tag{32}$$

the Taylor series of which:

$$A(q) = \sum_{m=1}^{\infty} \mu_m q^m, B(q) = \sum_{m=1}^{\infty} \sigma_m q^m, \tag{33}$$

exist and are convergent $|q| \leq 1$.

There are infinite number of deformation functions satisfying the properties (32) and (33). For the sake of computational efficiency, we use the following one-parameter deformation functions:

$$A_1(q;c_2) = \sum_{m=1}^{\infty} \mu_m(c_2) q^m, \quad B_1(q;c_3) = \sum_{m=1}^{\infty} \sigma_m(c_3) q^m, \tag{34}$$

where $|c_2| < 1$ and $|c_3| < 1$ are constants called the convergence control parameters, and

$$\mu_1(c_2) = 1 - c_2, \quad \mu_m(c_2) = (1 - c_2) c_2^{\,m-1}, \quad m > 1, \tag{35}$$

$$\sigma_1(c_3) = 1 - c_3, \quad \sigma_m(c_3) = (1 - c_3) c_3^{\,m-1}, \quad m > 1. \tag{36}$$

Using these deformation functions, the zeroth-order deformation equation takes the form

$$(1 - B_1(q;c_3))L[\varphi(r;q) - u_0(r)] = c_1 A(q;c_2) N[\varphi(r;q)], \quad r \in \Omega, q \in [0,1], \tag{37}$$

where $c_1 \neq 0$ is an auxiliary parameter called the convergence control parameter. Thus, we have at most three

Table 2 Comparison of exact square residual errors at α = 1 and different values of H2

Auxiliary parameters C_i		$\alpha = 1$, $H^2 = 0.5$	$\alpha = 1$, $H^2 = 1$	$\alpha = 1$, $H^2 = 2$	$\alpha = 1$, $H^2 = 4$
OHAM	C_1	−0.377725	−0.35981	−0.30792	−0.168637
	C_2	0.0420993	0.0623337	0.06334952	−0.1809055
	C_3	−0.01487	−0.0164235	−0.00274481	0.275378
	E_3	2.591989×10^{-11}	1.55047×10^{-9}	1.05203×10^{-7}	1.17518×10^{-4}
Optimal homotopy analysis method	c_1	−0.835839	−0.950907	−0.022845	−0.0455211
	c_2	0.587823	−0.411825	−0.3021	0.0242099
	c_3	0.546788	−0.037081	−12.3406	−6.19495
	E_4	1.74228×10^{-11}	-1.0616×10^{-9}	1.5504×10^{-4}	4.27912×10^{-4}
HAM	$\hbar$	−0.303	−0.292	−0.254	−0.198
	E_{20}	4.634×10^{-11}	4.996×10^{-9}	2.363×10^{-6}	3.461×10^{-4}

Table 3 Comparison of exact square residual error at α = 1.5 and different values of H^2

Auxiliary parameters C_i		$\alpha = 1.5, H^2 = 0.5$	$\alpha = 1.5, H^2 = 1$	$\alpha = 1.5, H^2 = 2$	$\alpha=1.5, H^2 = 4$
OHAM	c_1	−0.429884	−0.443299	−0.6898	−0.499435
	c_2	0.0742146	0.139540	0.0298953	0.07564
	c_3	−0.0128337	−0.02021099	0.5407644	1.06037
	E_3	2.0763995×10^{-9}	3.8790×10^{-7}	2.6342×10^{-6}	1.5866×10^{-4}
Optimal homotopy analysis method	c_1	−0.675245	−0.800339	−0.341231	−0.0427887
	c_2	0.2038278	0.741063	−0.301007	0.111194
	c_3	−0.030015	−1.36725	−0.623547	−7.00114
	E_4	1.3619×10^{-8}	2.563×10^{-5}	6.744×10^{-5}	1.6×10^{-3}
HAM	$\hbar$	−0.393372	−0.0164281	−0.6237173	−0.32296
	E_9	8.195638×10^{-8}	8.1234×10^{-4}	0.177825	0.551513

convergence control parameters: c_1, c_2 and c_3. As the embedding parameter q increases from 0 to 1, $\varphi(r;q)$ deforms continuously from the initial guess $u_0(r)$ to the exact solution $u(r)$ since $\varphi(r;0) = u_0(r)$ and $\varphi(r;1) = u(r)$.

Note that $\varphi(r;q)$ is determined by the auxiliary operator L, the initial guess $u_0(r)$ and the convergence control parameters c_1, c_2 and c_3, and we have great freedom to choose them. Assuming that all of them are so properly chosen that $\varphi(r;q)$ has the Taylor series representation as $\varphi(r;q) = u_0(r) + \sum_{m=1}^{\infty} u_m(r) q^m$, which converges at $q = 1$, thus the solution by optimal homotopy analysis method will be given as

$$u(r) = u_0(r) + \sum_{m=1}^{\infty} u_m(r), \tag{38}$$

where $u_m(r) = \frac{1}{m!} \frac{\partial^m \varphi(r;q)}{\partial q^m} \big|_{q=0}$ is the mth-order homotopy derivative [9].

Taking the mth-order homotopy derivative on both sides of Equation 37, we get the mth-order deformation equation:

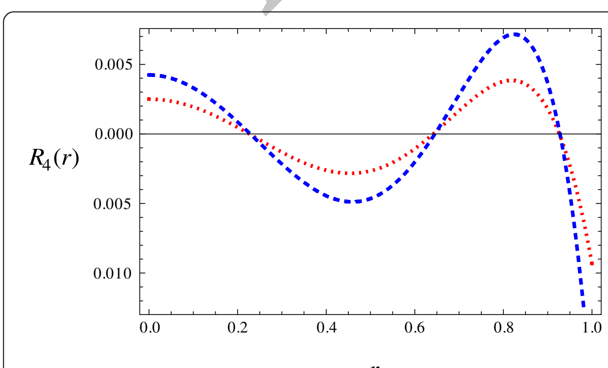

Figure 5 Residual $R_4(r)$ for H^2 = 4, α = 0.5 (red - OHAM, blue - optimal HAM).

$$L\left[u_m(r) - \chi_m \sum_{k=1}^{m-1} \sigma_{m-k}(c_3) u_k(r)\right] = c_1 \sum_{k=0}^{m-1} \mu_{m-k}(c_2) N[u_k(r)], \tag{39}$$

subject to the given boundary conditions.

Application of optimal homotopy analysis method to EHD flow problem

Now, we apply the optimal homotopy analysis method as developed above to the EHD flow (Equations 1 and 2). Assuming the initial guess $u_0(r) = 0$ and the linear operator $L = \frac{d^2}{dr^2}$, Equation 39 becomes

$$\frac{d^2}{dr^2}\left[u_m(r) - \chi_m \sum_{k=1}^{m-1} \sigma_{m-k}(c_3) u_k(r)\right] = c_1 \sum_{k=0}^{m-1} \mu_{m-k}(c_2) N[u_k(r)], \tag{40}$$

where

$$\begin{aligned} N[u_k(r)] = {} & u_k''(r) + \frac{1}{r} u_k'(r) \\ & + H^2\left(1 - \chi_k - (1+\alpha) u_{k-1}(r)\right) \\ & - \alpha \sum_{i=1}^{k} \{u_i(r) u_{k-i}''(r)\} \\ & - \frac{\alpha}{r} \sum_{i=0}^{k} \{u_i(r) u_{k-i}'(r)\}, \end{aligned} \tag{41}$$

and $\sigma_m(c_3)$ and $\mu_m(c_2)$ are given in Equations 35 and 36.

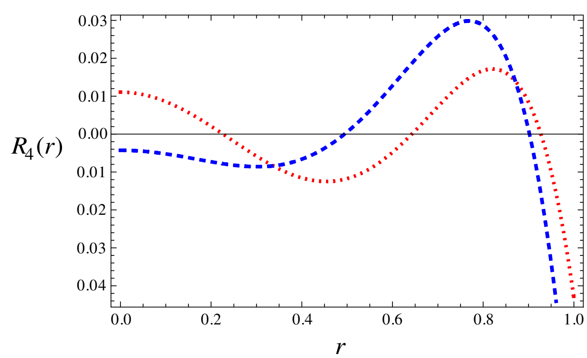

Figure 6 Residual $R_4(r)$ for $H^2 = 4$, $\alpha = 1$ (red - OHAM, blue - optimal HAM).

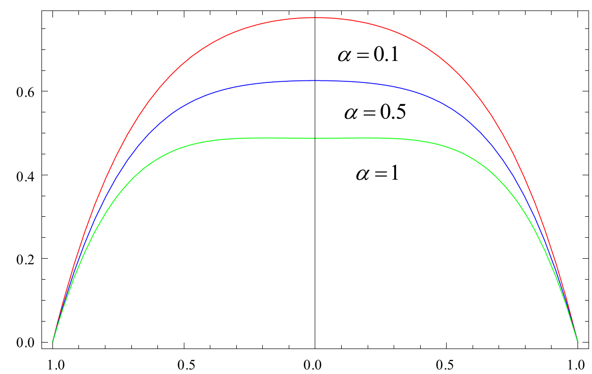

Figure 8 Optimal HAM solution $u(r)$ for $H^2 = 10$ and $\alpha = 0.1$ (red), 0.5 (blue), 1 (green).

Using Equations 40 and 41, we compute the first four iterates as we get a very satisfactory solution from these four iterates only. These are

$$u_1(r) = -\frac{1}{2}H^2 c_1(-1+c_2)(-1+r^2),$$

$$u_2(r) = -\frac{1}{24}H^2 c_1(-1+c_2)(-1+r^2) \times (12+(1+c_2-c_3)+c_1(-1+c_2) \times (-24+H^2(-5+r^2)(1+\alpha)),$$

$$\begin{aligned} u_3(r) = & -\frac{1}{720}H^2 c_1(-1+c_2)(-1+r^2) \\ & \times [360(1+c_2+c_2{}^2-c_3-c_2c_3) \\ & \times +60c_1(-1+c_2)(1+c_2-c_3) \\ & \times (-24+H^2(-5+r^2)(1+\alpha)) \\ & +c_2{}^2(-1+c_2)^2\{1,440+H^2(61-14r^2+r^4) \\ & \times(1+\alpha)^2-20H^2(-31-46\alpha+r^2(5+8\alpha))\}], \end{aligned}$$

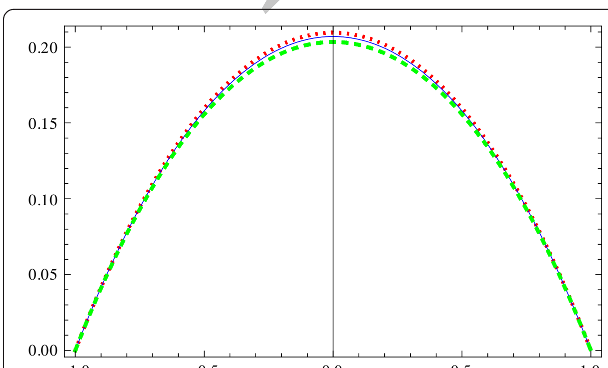

Figure 7 Optimal HAM solution $u(r)$ for $H^2 = 1$ and $\alpha = 0.1$ (red-dotted-top), 0.5 (blue-solid-middle), 1 (green-dashed-bottom).

$$\begin{aligned} u_4(r) = & -\frac{1}{604,800}H^2 c_1(-1+c_2)(-1+r^2) \\ & \times[302,400(1+c_2+c_2{}^3-c_2{}^2(-1+c_3)-c_3-c_2c_3) \\ & +25,200c_1(-1+c_2)(3+3c_2{}^2-4c_2{}^2(-1+c_3) \\ & -4c_3+c_3{}^2)(-24+H^2(-5+r^2)(1+\alpha)) \\ & +2,520(-1+c_2)^2(1+c_2-c_3)\{1,440 \\ & +H^2(61-14r^2+r^4)(1+\alpha)^2-20H^2(-31-46\alpha \\ & +r^2(5+8\alpha))\}+c_1{}^3(-1+c_2)^3\{-2,419,200 \\ & +15H^6(-1,385+323r^2-27r^4+r^6)(1+\alpha)^3 \\ & +11,200H^2(-143-281\alpha+r^2(19+43\alpha)) \\ & -112H^4(2,884+8,633\alpha+5,749\alpha^2 \\ & +2r^4(17+79\alpha+62\alpha^2) \\ & -r^2(566+1,867\alpha+1,301\alpha^2))\}]. \end{aligned} \quad (42)$$

The fourth-order optimal homotopy analysis method solution $\hat{u}_4(r)$ is obtained by substituting Equation 42 in Equation 38 and is given by

$$\hat{u}_4(r, c_1, c_2, c_3) = \sum_{i=0}^{4} u_i(r, c_1, c_2, c_3). \quad (43)$$

The optimal values of parameters c_1, c_2 and c_3 are computed by minimizing the square residual error E_m defined by Equation 19.

Convergence of the solutions

In this section, we discuss the convergence of the third-order OHAM solution and the fourth-order optimal homotopy analysis method solution given in Equations 28 and 43, respectively. The convergence of these two solutions

Table 4 CPU times incurred in calculating the iterations by OHAM, optimal homotopy analysis method and HAM

Iterations	CPU times (s)
4 iterates by optimal homotopy analysis method	2.108
3/4 iterates by OHAM	1.217/3.226
19 iterates by HAM	1,765

depend on their respective convergence control parameters C_1, C_2, C_3 and c_1, c_2, c_3. The values of these parameters are obtained by using the least square method given in the 'Analysis of the method' section. From Figures 1 and 2, we see that the square residual errors E_3/E_4 are stable even for larger deviations from the optimal values of C_1 (OHAM) or c_1 (optimal homotopy analysis method) as compared to the deviations in $\hbar$ (HAM) from its optimal value. From Figure two of [3], we see that when $\hbar$ moves towards the left of its optimal value −0.198 and approaches −0.30, the square residual error E_{20} shoots up from its minimum value 3.461×10^{-4} to a value greater than 10^7. So, for a relatively smaller variation of the order 10^{-1} in $\hbar$, there is a huge variation of the order 10^{11} in the square residual error E_{20}, whereas in our proposed algorithm based on OHAM, a similar variation in the value of C_1 about its optimal value −0.168637 causes no appreciable change in E_3. From Figure 2, we conclude that the sensitivity of E_4 (optimal homotopy analysis method) with respect to the variation in c_1 about its optimal value −0.0455211 is much lesser compared to the sensitivity of E_{20} (HAM) but is larger than the sensitivity of E_3 (OHAM).

The optimal values of the convergence control parameters for all the cases considered are obtained by minimizing Equation 19 using the Mathematica function Minimize and are given in Tables 1, 2 and 3. In addition, we plot the residual functions $R_4(r)$ for both the proposed algorithms in Figures 5 and 6 for the parameters $\alpha = 0.5$, $H^2 = 4$ and $\alpha = 1$, $H^2 = 4$, respectively. These plots demonstrate the accuracy of OHAM and optimal homotopy analysis method solutions over the HAM solution [3]. As was done by Mastroberardino in [3], the residuals have been plotted as a function of r for the optimal values of the convergence control parameters C_1, C_2, C_3 and c_1, c_2, c_3 (given in Tables 1, 2 and 3) and not as a function of these convergence control parameters for a fixed value of r as this is a better illustration of convergence.

Discussion of solution profiles

In this section, we give the various fourth-order solution profiles for small and large values of α. For $\alpha << 1$, the solution profiles obtained by the two proposed methods are similar to that obtained by Mckee et al. [1].

Case 1

For $\alpha << 1$, the solution profiles obtained by optimal homotopy analysis method are depicted in Figures 7 and 8. The similar profiles are obtained by using OHAM as well. From the figures, it is clear that the profiles obtained by our approach are the same as that obtained in [1].

Case 2

For $\alpha >> 1$, in Figures 3 and 4, we present OHAM solutions of the BVP (1) and (2) for values of $\alpha = 4$, 10 and $H^2 = 1$ and $\alpha = 4$, 10 and $H^2 = 10$, respectively. In the case of $\alpha = 4$ (respectively, $\alpha = 10$), the solutions are bounded above by $1 / (\alpha + 1) = 0.2$ (respectively, $1 / (\alpha + 1) = 0.09$) and are in agreement with those of Paullet [2]. As noted in [1], for large H^2, the solutions should tend to $u(r) = 1 / (\alpha + 1)$, and such behaviour is evident in Figures eight and nine. This is in contrast to Figures twelve and thirteen of [1] where for $H^2 = 10$ and $H^2 = 100$, the solutions do not exhibit the proper limiting behaviour and also cross the singularity at $1 / \alpha$ and thus are not C^2. So, our calculations support Paullet's numerical results for $\alpha >> 1$.

Comparison with HAM solutions

Tables 1, 2 and 3 display the square residual errors E_3 (OHAM), E_4 (optimal homotopy analysis method) and E_{20} (HAM) at various values of parameters α and H^2. It is observed that the square residual errors E_3/E_4 obtained by OHAM/optimal homotopy analysis method are comparable with E_{20} (obtained by HAM) for lower values of α and H^2 and are significantly smaller than E_{20} for higher values of α and H^2. Table 4 shows the ratio of CPU times incurred to find the 3, 4 and 19 iterates by OHAM, optimal homotopy analysis method and HAM, respectively. Further, the CPU time for HAM is very large compared to those for OHAM and optimal homotopy analysis method.

Conclusions

In this paper, we propose two semi-analytical algorithms based on OHAM and optimal homotopy analysis method to obtain semi-analytical solutions for a nonlinear boundary value problem governing electrohydrodynamic flow, though the nonlinearity confronted in this problem is in the form of a rational function posing a significant challenge in regard to obtaining analytical/semi-analytical solutions. Earlier in 1997, Mckee et al. [1] gave numerical solutions to the EHD flow in a circular cylindrical conduit described by Equations 1 and 2 for various values of H^2 and α with the perturbation expansions of the solutions for small and large values of parameter α. Later in 1999, Paullet [2] provided a rigorous result concerning the existence, uniqueness and qualitative properties of the solutions of Equations 1 and

2 for any $\alpha > 0$ and $H^2 \neq 0$. Paullet's solution [2] for $\alpha << 1$ matches with that of Mckee et al. [1], but there was a difference between the solutions of Paullet and Mckee for $\alpha >> 1$. Our solutions obtained from the two proposed algorithms support Paullet's solutions for $\alpha >> 1$. For $\alpha << 1$, all the solutions obtained by Mckee et al. [1], by Paullet [2] and from our proposed algorithms are in good agreement. For lower values of α and H^2, the solutions obtained by OHAM, optimal homotopy analysis method and HAM are compatible, whereas for higher values of α and H^2, the algorithm based on OHAM gives better results than the one based on optimal homotopy analysis method, and the optimal homotopy analysis method solutions are better than the HAM solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RKP proposed the algorithm and searched the EHD problem. RKP, VKB and CSS applied the algorithm to solve the problem. OPS supervised the work and along with RKP analysed the convergence of the solution for the larger values of the nonlinear parameters α and H^2. All authors have read and approved the final manuscript.

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