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Research Article

A numerical treatment based on Bernoulli Tau method for computing the open-loop Nash equilibrium in nonlinear differential games

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Abstract

The Tau method based on the Bernoulli polynomials is implemented efficiently to approximate the Nash equilibrium of open-loop kind in nonlinear differential games over a finite time horizon. By this treatment, the system of two-point boundary value problems of differential game extracted from Pontryagin's maximum principle is transferred to a system of algebraic equations that Newton's iteration method can be used for solving it. Also, for the mentioned approximation by the Bernoulli polynomials, the convergence analysis and the error upper bound are discussed. To demonstrate the applicably and accuracy of the proposed approach, some illustrated examples are presented at the final.

AMS subject classifications (2020): 35A01, 65L10, 65L12, 65L20, 65L70.

Keywords: Nonlinear differential games; Open-loop Nash equilibrium; Pontryagin's maximum principle; Bernoulli Tau method.

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1 Introduction

A differential game is an extension of optimal control theory that describes a conflict situation between some players, who seek their maximum or minimum own payoffs under a dynamical system. It has arisen in practical problems from economics to engineering applications in recent years [29, 30, 44, 11, 15, 31].

One of the most important and crucial solution concepts in game theory is the Nash equilibrium, in which players have no incentive to deviate from their original plans [23] and is classified into the following two cases in differential games based on the information of the state of the game that players know at different times of game:

- The players have no information during the game and only know the game state at the initial time. This kind of equilibrium is known as open-loop Nash equilibrium.
- The current game state is known to players. Such equilibrium is often called feedback Nash equilibrium.

The main approaches to computing the open-loop Nash equilibrium in differential games are indirect methods and direct methods. In indirect methods, the nonzero-sum differential game is reduced to a system of two-point boundary value problems (TPBVPs) by using the necessary optimality conditions of the Pontryagin's maximum principle that can be solved analytically or numerically [5]. In direct approaches that are optimization based methods, differential game problem is transferred to mathematical programming [12, 20]. However, the drawback of direct approaches is that there is no guarantee that the solution obtained is feasible for the original problem [27, 42].

Regarding the indirect methods, most researchers have focused on a special kind of differential game, namely linear-quadratic dynamic games that the state equation of the game is linear with respect to control and state variables, and both are quadratic in the performance indices. For this kind of differential game, the systems of TPBVPs are linear in general, and hence the open-loop Nash equilibrium can be obtained analytically based on solving Riccati equations [14, 17]. Indeed in practice, we face with differential games that their systems of TPBVPs are nonlinear generally. Therefore, using suitable numerical methods is necessary [35].

To the best of our knowledge, there are a few research works carried out to compute open-loop Nash differential games in nonlinear case. In [27], a pseudospectral method based on Chebyshev polynomials was applied for finding the players' open-loop strategies in nonlinear differential games. In [24], by Riccati equations, the open-loop Nash equilibrium of differential games in polynomial case was obtained. In [9], the coordinate transformation approach was extended for computing open-loop Nash equilibrium, and complementarity theory was applied for a class of zero-sum differential games to be solved in [43]. In [32], a combined quasilinearization method with exponential Bernstein functions was introduced for a numerical solution to TPBVPs.

There are several methods for solving differential equations numerically, such as spectral methods [8], shooting and multiple shooting methods [25], variational iteration method [16], and homotopy analysis method [1] that each of which has its own implementation. Spectral methods are based on the weighted residual method that has high accurate results in solving differential equations and are classified into three methods, namely collocation, Galerkin, and Tau methods [19, 22, 34, 41].

The Tau method is one of the most accurate spectral methods for a numerical solution to differential equations of different kinds [39, 3, 38, 18]. The goal of this paper is to propose an implementation of the Bernoulli Tau method (BTM), in which the solution functions are defined by means of a truncated Bernoulli series expansion, to compute the open-loop Nash equilibrium in nonlinear differential games with finite horizons.

The remainder of the paper is organized into the following sections. In Section 2, the nonzero-sum nonlinear differential games are defined, and the extraction of the systems of TPBVPs from Pontryagin's maximum principle is described. In Section 3, the Tau approach based on the Bernoulli polynomials is introduced and applied for computing the open-loop strategies of these differential games. In Section 4, some numerical examples are presented to validate the accuracy and applicability of the present method. Finally, conclusions are presented in section 5.

2 Problem statement

A family of nonzero-sum nonlinear differential games with finite horizon is described in the following definition.

Definition 1. A nonzero-sum nonlinear differential game is defined as follows [6]:

$$
\begin{aligned}
&\max_{u_i(\cdot)} J_i(u_i(\cdot), u_{-i}(\cdot)) = \int_0^T K_i(t, x(t), u_1(t), u_2(t), \ldots, u_m(t))\, dt + \Phi_i(x(T)),\\
&\dot{x}(t) = f(t, x(t), u_1(t), u_2(t), \ldots, u_m(t)), \qquad (1)\\
&x(0) = x_0 \in \mathbb{R},
\end{aligned}
$$

where $u_i(t) \in U_i \subset \mathbb{R}$ is the player i's control (strategy), $x(t) \in \mathbb{R}$ is the state vector of the differential game, and $M = \{1, 2, \ldots, m\}$ is the set of players. The functions $K_i(t, x(t), u_1(t), u_2(t), \ldots, u_m(t))$ and $\Phi_i(x(T)), i = 1, 2, \ldots, m$, are continuously differentiable functions that describe the player i's running payoff and terminal payoff, respectively. The goal of this differential game for each player i, $i = 1, 2, \ldots, m$, is to maximize his payoff by choosing a suitable strategy $u_i(t) \in U_i \subset \mathbb{R}$.

For differential game (1), the open-loop Nash equilibrium is described as follows.

Definition 2. The control actions $u_i^\star(\cdot), i = 1, 2, \ldots, m$, are considered as a Nash equilibrium for differential game (1), if the following inequalities hold:

$$J_i(u_i^\star(\cdot), u_{-i}^\star(\cdot)) \geqslant J_i(u_i(\cdot), u_{-i}^\star(\cdot)), \quad \text{for all } u_i \in U_i,$$

where u_i is the ith player's strategy and u_{-i} state the other players' strategies, that is, $u_{-i} = u_j, j \neq i$.

For deriving the first-order optimality necessary conditions of nonlinear differential game (1) and characterizing an open-loop strategy, the Hamiltonian functions are defined as follows:

$$H_i(t, x, u_i, u_{-i}, \lambda_i) = K_i(t, x, u_i, u_{-i}) + \lambda_i\, f(t, x, u_i, u_{-i}), \qquad i = 1, 2, \ldots, m,$$

where the variables λ_i, $i = 1, 2, \ldots, m$, are the adjoint functions.

Pontryagin's maximum principle provides a set of optimality conditions for control actions to construct an open-loop strategy in nonlinear differential game (1) as follows:

$$\begin{cases} \dot{x}(t) = f(t, x(t), u_1(t), \ldots, u_m(t)), \quad x(0) = x_0, & (2) \\ \dot{\lambda}_i(t) = -\dfrac{\partial H_i}{\partial x}(t, x(t), u_i(t), u_{-i}(t), \lambda_i(t)), \quad \lambda_i(T) = \dfrac{\partial \Phi_i(x(T))}{\partial x}, & (3) \\ \dfrac{\partial H_i}{\partial u_i}(t, x(t), u_i(t), u_{-i}(t), \lambda_i(t)) = 0, \quad i = 1, 2, \ldots, m. & (4) \end{cases}$$

An expression for $u_i(t)$, $i = 1, 2, \ldots, m$, with respect to $x(t)$ and $\lambda_i(t)$ can be obtained by solving the algebraic equations (4) as follows:

$$u_i = \Psi_i(t, x(t), \lambda_i(t)).$$

This expression is replaced in (2) and (3) to obtain the system of TPBVPs based on $x(t)$ and $\lambda_i(t)$, $i = 1, 2, \ldots, m$, as follows:

$$\begin{cases} \dot{x}(t) = f(t, x(t), \Psi_1(t), \Psi_2(t), \ldots, \Psi_m(t)), \\ \dot{\lambda}_i(t) = -\dfrac{\partial H_i}{\partial x}(t, x(t), \Psi_i(t), \Psi_{-i}(t), \lambda_i(t)), \\ x(0) = x_0, \\ \lambda_i(T) = \dfrac{\partial \Phi_i(x(T))}{\partial x}, \end{cases} \tag{5}$$

where $\Psi_i = \Psi_i(t, x(t), \lambda_i(t))$, $i = 1, 2, \ldots, m$.

This system of differential equations with split boundary conditions is nonlinear generally, which makes it difficult or impossible to be solved analytically. Therefore, using an appropriate numerical approach is required.

3 The Bernoulli Tau method for nonlinear differential games

In this part, an efficient formulation of the Tau method for a numerical solution to the system of TPBVPs is established by obtaining the open-loop strategy of nonlinear differential game (1).

The Tau method is a highly accurate spectral method for differential equations to be solved numerically. Implementing this method is based on expanding the solution functions $f(t)$ of differential equations in terms of suitable basis polynomials such as Bernoulli [40, 33], Jacobi [4, 28], and Bernstein polynomials [21] as follows:

$$f(t) = \sum_{i=0}^{\infty} f_i \, P_i(t),$$

where f_i and $P_i(t)$, $i = 0, 1, 2, \ldots$, are unknown coefficients and basis polynomials, respectively [7].

In practice, we use only a finite number of these basis polynomials, meaning that $f^n(t) = \sum_{i=0}^{n} f_i \, P_i(t)$ is a numerical approximation of the exact solution $f(t)$.

In this paper, the Bernoulli polynomials are considered as basis polynomials, in which the definition and properties of these polynomials in a function approximation are stated below.

Definition 3. Bernoulli polynomials of order n are defined on $[0, 1]$ by (see [26])

$$\beta_n(t) = \sum_{i=0}^{n} \binom{n}{i} \alpha_{n-i} t^i,$$

where $\alpha_i,\ i = 0, 1, \ldots, n$, are Bernoulli numbers and defined as

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!}.$$

The first few Bernoulli numbers are

$$\alpha_0 = 1, \quad \alpha_1 = \frac{-1}{2}, \quad \alpha_2 = \frac{1}{6}, \quad \alpha_4 = \frac{-1}{30}, \ldots,$$

with $\alpha_{2i+1} = 0$, for $i = 1, 2, 3, \ldots$.

The first seven Bernoulli polynomials are

$$\beta_0(t) = 1, \quad \beta_1(t) = t - \frac{1}{2}, \quad \beta_2(t) = t^2 - t + \frac{1}{6}, \quad \beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t,$$
$$\beta_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}, \quad \beta_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t,$$

$$\beta_6(t) = t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2 + \frac{1}{42}.$$

A complete basis is formed by these polynomials over the interval $[0, 1]$.

Any function $f(t)$ belonging to $L^2[0, 1]$ can be approximated by Bernoulli functions as follows:

$$f(t) \approx f^n(t) = \sum_{i=0}^{n} f_i \beta_i(t).$$

To apply the Tau method based on the Bernoulli polynomials for solving the system of TPBVPs (5), for simplification matters and without loss of generality, we consider $T = 1$, and then the unknown functions $x(t)$ and $\lambda_i(t)$, $i = 1, \ldots, m$, can be approximated as finite expansions of Bernoulli polynomials as follows:

$$x(t) \approx x^n(t) = \sum_{j=0}^{n} a_j \beta_j(t) = A^T \beta(t)$$

$$\lambda_i(t) \approx \lambda_i^n(t) = \sum_{j=0}^{n} b_{ij} \beta_j(t) = B_i^T \beta(t), \quad i = 1, 2, \ldots, m,$$

where $A^T = [a_0, a_1, \ldots, a_n]$ and $B_i^T = [b_{i0}, b_{i1}, \ldots, b_{in}]$, $i = 1, \ldots, m$, are the vectors of unknown coefficients and $\beta(t) = [\beta_0(t), \beta_1(t), \ldots, \beta_n(t)]^T$ is the vector of Bernoulli polynomials.

The residual functions are defined by substituting these expansions in the differential equations of the system of TPBVPs (5) as follows:

$$R_0(t) = \dot{x}^n(t) - f(t, x^n(t), \Psi_1^n(t), \Psi_2^n(t), \ldots, \Psi_m^n(t)),$$

$$R_i(t) = \dot{\lambda}_i^n(t) + \frac{\partial H}{\partial x^n}(t, x^n(t), \Psi_i^n(t), \Psi_{-i}^n(t)), \quad i = 1, \ldots, m.$$

Then, multiplying these residuals by $\beta_j(t)$, $j = 0, 1, \ldots, n-1$, integrating over the interval $[0, 1]$, and setting equal to zero, together with the boundary values, the following system of $(m + 1)(n + 1)$ algebraic equations is created, which Newton's iteration method can be applied to solve it and to determine the unknown vectors A^T *and* B_i^T, $i = 1, \ldots, m$:

$$\begin{cases} \int_0^1 R_0(t) B_j(t) dt = 0, \\ \int_0^1 R_i(t) B_j(t) dt = 0, \\ x^n(0) = x_0, \\ \lambda_i^n(1) = \dfrac{\partial \Phi_i(x^n(1))}{\partial x^n}. \end{cases}$$

By the following theorem, the convergence analysis and the error upper bound for the mentioned approximation obtained by the Bernoulli polynomials is discussed.

Theorem 1. Suppose that $x(t)$ and $\lambda_i(t), i = 1, \dots, m$, belong to $C^{n+1}[0,1]$ and that $S_n = span\{\beta_0(t), \beta_1(t), \dots, \beta_n(t)\}$. If $A^T\beta(t) \in S_n$ and $B_i^T\beta(t) \in S_n, i = 1, 2, \dots, m$, are the best approximations of $x(t)$ and $\lambda_i(t)$, $i = 1, \dots, m$, respectively, then

$$\|x(t) - A^T\beta(t)\|_{L^2[0,1]} \leq \frac{C}{(n+1)!\sqrt{2n+3}}$$

and

$$\|\lambda_i(t) - B_i^T\beta(t)\|_{L^2[0,1]} \leq \frac{C_i}{(n+1)!\sqrt{2n+3}}, \quad i = 1, 2, \dots, m,$$

where $C = \max_{t\in[0,1]} |x^{(n+1)}(t)|$ and $C_i = \max_{t\in[0,1]} |\lambda_i^{(n+1)}(t)|$, $i = 1, 2, \dots, m$.

Proof. The proof will be done for the first inequality and the other inequalities can be proved in a similar manner.

Since $x(t) \in C^{n+1}[0,1]$, there exists $C \in \mathbb{N}$ such that for every $t \in [0,1]$, we have $|x^k(t)| \leq C, k = 0, 1, \dots, n+1$, and $x(t)$ can be expanded into the Taylor formula as

$$x(t) = \sum_{k=0}^{n} \frac{x^{(k)}(0)}{k!}t^k + \frac{x^{(n+1)}(\xi)}{(n+1)!}t^{n+1} = \tilde{x}(t) + \frac{x^{(n+1)}(\xi)}{(n+1)!}t^{n+1},$$

where $\tilde{x}(t) = \sum_{i=0}^{n} \frac{x^{(k)}(0)}{k!}t^k$ and $\xi \in [0,t]$. Hence, we have

$$|x(t) - \tilde{x}(t)| = \frac{x^{(n+1)}(\xi)}{(n+1)!}t^{n+1}.$$

Because $A^T\beta(t)$ is the best approximation of $x(t)$ out of S_n, $\tilde{x}(t) \in S_n$, and considering the above equality, it is concluded that

$$\begin{aligned}
\|x(t) - A^T\beta(t)\|_{L^2[0,1]} \leq \|x(t) - \tilde{x}(t)\|_{L^2[0,1]} &= \left\| \frac{x^{(n+1)}(\xi)}{(n+1)!}t^{n+1} \right\|_{L^2[0,1]} \\
&= \left(\int_0^1 \left| \frac{x^{(n+1)}(\xi)}{(n+1)!}t^{n+1} \right|^2 dt \right)^{\frac{1}{2}} \\
&\leq \frac{C}{(n+1)!} \left(\int_0^1 t^{2n+2} dt \right)^{\frac{1}{2}}
\end{aligned}$$

$$= \frac{C}{(n+1)!\sqrt{2n+3}},$$

and this completes the proof. □

Remark 1. It should be noted that for practical use of Bernoulli polynomials on the interval $[a, b]$, it is necessary to shift the defining domain by the following variable substitution and construct the shifted Bernoulli polynomials:

$$t = \frac{x}{b-a} - \frac{a}{b-a}.$$

4 Numerical illustrations

In this part, three differential game problems are presented to illustrate the accuracy and efficiency of the proposed approach. Example 1 is a linear-quadratic differential game that its exact solution can be obtained. By this example and comparing it with the exact solution, we can verify and validate the proposed approach. Example 2 is also a linear quadratic differential game with attainable exact solution. In this example, we compare the results of the proposed method with the Chebyshev pseudospectral method (CPM) presented in [27]. Example 3 is a differential game arising from an economic model with a nonlinear system of TPBVPs that the exact solution is not available. To check the performance of the proposed method for this problem, a residual function is defined.

All the computations associated with the proposed method have been performed by Maple 17 software with 32 digits precision on a Core (TM) i7 PC with 2.70GHz of CPU and 16GB of RAM.

Example 1. For this differential game, the state equation is [13]

$$\dot{x}(t) = u_1(t) + u_2(t), \quad x(0) = 1,$$

and two players' performance indices are

$$\min_{u_1} J_1 = \int_0^1 (-x^2(t) + u_1^2(t))dt,$$
$$\min_{u_2} J_2 = \int_0^1 (2x^2(t) + u_2^2(t))dt + x^2(1).$$

The exact open-loop Nash equilibrium of this differential game is [13]

$$u_1^* = -\frac{1}{e} + e^{-t},$$
$$u_2^* = \frac{1}{e} - 2e^{-t}.$$

Hence, the exact values of players' performance indices are

$$J_1^* = -0.32975303263305,$$
$$J_2^* = 1.9344880850240.$$

The system of TPBVPs for the mentioned game is stated as

$$\begin{cases} \dot{x}(t) = -\dfrac{\lambda_1(t)}{2} - \dfrac{\lambda_2(t)}{2}, \\ \dot{\lambda}_1(t) = 2x(t), \\ \dot{\lambda}_2(t) = -4x(t), \\ x(0) = 1, \\ \lambda_1(1) = 0, \quad \lambda_2(1) = 2x(1). \end{cases}$$

The values of performance indices obtained by the proposed approach and the comparison of the analytical solutions are shown in Table 1. Also, the approximate solutions and the exact solutions with $n = 10$ together with absolute errors are plotted in Figure 1.

Table 1: Comparison of optimal payoff functionals J_1 and J_2 obtained by BTM with the exact solutions and also the CPU time(s) for Example 1.

n	J_{1BTM}	J_{2BTM}	$\lvert J_{1BTM} - J_1^*\rvert$	$\lvert J_{2BTM} - J_2^*\rvert$	CPU time(s)
4	−0.32975302954236861650	1.93448814833633875533	3.09×10^{-9}	5.23×10^{-8}	0.124
6	−0.32975303263303305145	1.93448808502434431993	1.35×10^{-14}	2.27×10^{-13}	0.156
8	−0.32975303263304656749	1.93448808502406878964	1.69×10^{-20}	2.84×10^{-19}	0.218
10	−0.32975303263304656750	1.93448808502406878929	8.17×10^{-27}	1.37×10^{-25}	0.328

Example 2. For this differential game, the state equation is [13]

$$\dot{x}(t) = 2x(t) + u_1(t) + u_2(t), \quad x(0) = 1,$$

and two players' performance indices are

$$\min_{u_1} J_1 = \int_0^3 (x^2(t) + u_1^2(t))dt,$$
$$\min_{u_2} J_2 = \int_0^3 (4x^2(t) + u_2^2(t))dt + 5x^2(3).$$

The exact open-loop Nash equilibrium of this differential game is [13]

$$u_1^* = -e^{-st} + \frac{1}{e^3}e^{-2t},$$
$$u_2^* = -4e^{-3t} - \frac{1}{e^3}e^{-2t}.$$

Hence, the exact values of players' performance indices are

$$J_1^* = 0.3140381912,$$

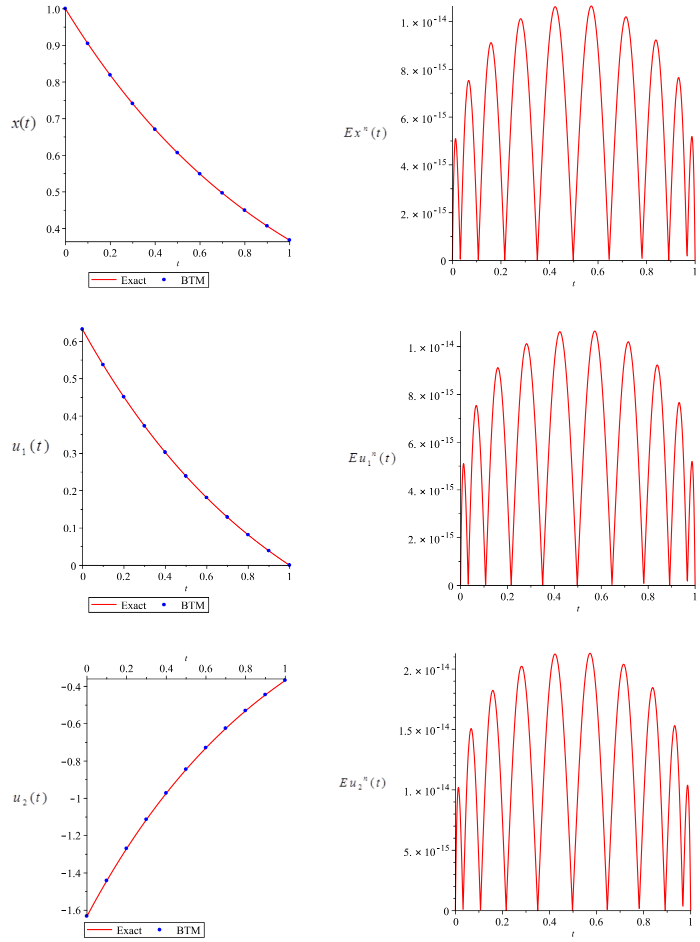

Figure 1: Plots of the approximate solutions and the analytical solutions together with absolute errors with $n = 10$ for Example 1.

$$J_2^* = 3.4136123279.$$

The system of TPBVPs for the mentioned game is stated as

$$\begin{cases} \dot{x}(t) = 2x(t) - \dfrac{\lambda_1(t)}{2} - \dfrac{\lambda_2(t)}{2}, \\ \dot{\lambda}_1(t) = -2x(t) - 2\lambda_1(t), \\ \dot{\lambda}_2(t) = -8x(t) - 2\lambda_2(t), \\ x(0) = 1, \\ \lambda_1(3) = 0, \quad \lambda_2(3) = 10x(3). \end{cases}$$

The values of performance indices obtained by the proposed approach and the comparison of the analytical solutions are shown in Table 2. Besides, to compare the BTM results with an existing approach, the results obtained by the CPM [27] are shown in Table 3.

Table 2: Comparison of optimal payoff functionals J_1 and J_2 obtained by BTM with the exact solutions and also the CPU time(s) for Example 2.

n	J_{1BTM}	J_{2BTM}	$\lvert J_{1BTM} - J_1^*\rvert$	$\lvert J_{2BTM} - J_2^*\rvert$	CPU time(s)
10	0.31403763402282	3.41361478021289	5.57×10^{-7}	2.45×10^{-6}	0.437
15	0.31403819123820	3.41361232797387	1.07×10^{-14}	4.58×10^{-14}	0.640
20	0.31403819123819	3.41361232797391	8.77×10^{-24}	3.85×10^{-23}	1.154

Table 3: Comparison of optimal payoff functionals J_1 and J_2 obtained by CPM with the exact solutions for Example 2.

n	J_{1CPM}	J_{2CPM}	$\lvert J_{1CPM} - J_1^*\rvert$	$\lvert J_{2CPM} - J_2^*\rvert$
10	0.3140689582	3.4134809955	0.0000307670	0.0001313324
15	0.3140381906	3.4136123306	6.1×10^{-10}	2.7×10^{-9}
20	0.3140381912	3.4136123279	5.2×10^{-15}	2.2×10^{-14}

Table 2 indicates that in the same situation in terms of the number of basis functions, the results obtained by the proposed method are more accurate than the results obtained by the CPM in this example.

Example 3. The following differential game describes the competition between two players in an effort for harvesting a natural renewable resource. The state equation of this game is expressed as

$$\dot{x}(t) = 0.1x(t) - 0.001x^2(t) - x(t)u_1(t) - x(t)u_2(t), \quad x(0) = 1.$$

The players' payoffs are given by

$$J_1(u_1, u_2) = \int_0^1 (3x(t)u_1(t) - \frac{1}{2}u_1^2(t))dt,$$
$$J_2(u_1, u_2) = \int_0^1 (2x(t)u_2(t) - \frac{1}{2}u_2^2(t))dt,$$

where the value $x(t) > 0$ is the resource level and the amounts $u_1(t) \geq 0$ and $u_2(t) \geq 0$ are the players' efforts for harvesting this resource, all at time t. Moreover, $\frac{1}{2}u_1^2$ and $\frac{1}{2}u_2^2$ indicate the costs for harvesting at effort levels u_1 and u_2, respectively [9].

Remark 2 (see [35])**.** By the linearity of the state equation of this differential game with respect to the control variables u_i, $i = 1, 2$, and the concavity of integrand of performance index J_i, $i = 1, 2$, with respect to u_i, $i = 1, 2$, (because $\dfrac{\partial^2 J_i}{\partial u_i^2} = -1 < 0, i = 1, 2$), it yields that the open-loop strategy exists and is unique for this dynamic game regrading the Filippov–Cesari existence theorem [10].

The nonlinear system of TPBVPs extracted from Pontryagin's maximum principle for this differential game is stated as follows:

$$\begin{cases} \dot{x} = 0.1x - 5.001x^2 + x^2\lambda_1 + x^2\lambda_2, \\ \dot{\lambda}_1 = -9x - 0.1\lambda_1 + 8.002x\lambda_1 - x\lambda_1^2 - x\lambda_1\lambda_2, \\ \dot{\lambda}_2 = -4x - 0.1\lambda_2 + 7.002x\lambda_2 - x\lambda_2^2 - x\lambda_1\lambda_2, \\ x(0) = 1, \\ \lambda_1(1) = 0, \quad \lambda_2(1) = 0. \end{cases}$$

The numerical results for various amounts of n are presented in Table 4. It is worth mentioning that since the exact solution to this differential game is not available, to check the accuracy and validity of the proposed method for the differential game under consideration, the error of residuals is defined as follows:

$$\|Res\|^2 = \int_0^1 (R_1^2(t) + R_2^2(t) + R_3^2(t))dt,$$

where $R_i(t), i = 1, 2, 3$, are the residuals defined in the previous section.

Table 4: Optimal payoff functionals J_1 and J_2 for Example 3 with error norms and also the CPU time(s).

n	J_{1BTM}	J_{2BTM}	$\|Res\|^2$	CPU time(s)
6	0.946161437829	0.452174552034	3.74×10^{-5}	3.931
8	0.946161294373	0.452174505059	5.44×10^{-7}	12.683
10	0.946161293220	0.452174504702	7.27×10^{-9}	35.334
12	0.946161293210	0.452174504699	9.16×10^{-11}	92.486

It is notable that due to the nonlinearity of the system of TPBVPs for this example and also the process of implementing the Tau method, we expect that it consumes more time than the previous examples to be solved. Table 4 verifies this matter.

5 Conclusions

In this paper, a formulation of the Bernoulli Tau method (BTM) was established efficiently for approximating the open-loop Nash equilibrium in nonlinear differential games over a finite time horizon. Using this approach, the system of TPBVPs extracted from Pontryagin's maximum principle was reduced to a system of algebraic equations by expanding the solution functions in terms of Bernoulli polynomials, which can be solved numerically to determine the unknown coefficients. At last, three examples were presented and solved by this approach to validate the applicably and accuracy of the present method. The approximate solutions were obtained with an excellent agreement with the exact solutions.

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