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Buckling Analysis of a Double-Walled Carbon Nanotube Embedded in an Elastic Medium Using the Energy Method

A. Ghorbanpour Arani*, M. Shokravi , M. Mohammadimehr

Department of Mechanical Engineering, University of Kashan, Ravand, Kashan, Iran

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ABSTRACT

The axially compressed buckling of a double-walled carbon nanotabe surrounded by an elastic medium using the energy and the Rayleigh-Ritz methods is investigated in this paper. In this research, based on the elastic shell models at nano scale, the effects of the van der Waals forces between the inner and the outer tubes, the small scale and the surrounding elastic medium on the critical buckling load are considered. Normal stresses at the outer tube medium interface are also included in the current analysis. An expression is derived relating the external pressure to the buckling mode number, from which the critical pressure can be obtained. It is seen from the results that the critical pressure is dependent on the outer radius to thickness ratio, the material parameters of the surrounding elastic medium such as Young's modulus and Poisson's ratio. Moreover, it is shown that the critical pressure descend very quickly with increasing the half axial wave numbers.

Keywords: Buckling; DWCNT; Elastic medium; Energy method

1 INTRODUCTION

EVER since Yakobson et al. [1] discovered the applicability of the continuum mechanics for the analysis of the carbon nanotubes (CNTs), researchers have intensively explored the behavior of single and multi-walled carbon nanotubes using continuum mechanics models. Continuum structural models can be used to simulate the single-walled carbon nanotubes (SWCNTs) directly, but they are not applicable to the multi-walled carbon nanotubes (MWCNTs) due to their multi-layered structure and the associated van der Waals forces.

In recent years, great efforts have been devoted to the improvement of continuum mechanics models to capture the effect of van der Waals forces which are significant in MWCNTs. The buckling instability of CNTs under axial compression is a research topic of primary interest owing to the hollow tube geometry of CNTs. Many researchers [2–3] proposed multi-beam and cylindrical shell models for the analysis of CNTs. The simple elastic beam models are, however, valid for only slender and long CNTs and the buckling mode is global in nature. When CNTs are short and thick local buckling predominates. In such cases, CNTs are to be modeled as cylindrical shells rather than beams. The effects of the surrounding elastic medium and the van der Waals forces between the inner and the outer nanotubes are taken into consideration. An elastic double shell model based on the continuum mechanics is presented for the torsional buckling of a double walled carbon nanotube (DWCNT). The critical buckling load can be predicted and the simplified analysis is also presented to estimate the critical torque for the DWCNT [4]. Recently, an elastic double shell model has been developed to study axially compressed buckling of a DWCNT embedded in an elastic medium based on the Donnell equations [5] of linear theory of cylindrical shells [6]. For simplicity, the elastic medium and the nanotube are assumed to possess the same Poisson's ratio and the shear stresses at the tube–medium interface is neglected. Ranjbartoreh et al. [7] obtained expressions for the critical axial forces and pressures of DWCNTs and calculated the critical axial forces and pressures for different axial half-sine

* Corresponding author. Tel.: +98 913 162 6594; fax: +98 361 555 9930.
E-mail address: aghorban@kashanu.ac.ir or a_ghorbanpour@yahoo.com (A. Ghorbanpour Arani).

and circumferential sine wave numbers. Ghorbanpour Arani et al. [8] investigated the torsional and the axially-compressed buckling of a MWCNT subjected to internal and external pressures. They considered the effects of the small scale and the surrounding elastic medium. Their results showed that the internal pressure increased the critical load; while, the external pressure tended to decrease it. Ghorbanpour Arani et al. [9] presented the transverse vibrations of the SWCNT and the DWCNT under axial load by applying the Euler–Bernoulli, the Timoshenko beam and the Donnell shell models. They concluded that the Euler–Bernoulli beam model and the Donnell shell model predictions have the lowest and the highest accuracies, respectively. In order to predict the vibration behavior of the carbon nanotube more accurately, they modified the current classical models using the non-local theory. Moreover, they obtained the natural frequencies and amplitude coefficient for the simply supported boundary conditions.

In this paper, the buckling of a short DWCNT embedded in an elastic medium and loaded by an external pressure and an axial stress is analyzed using the energy method [10]. This research is based on the elastic shell models. As a DWCNT is distinguished from traditional elastic shells by their hollow double-layer structure and the associated van der Waals forces between the inner and the outer tubes, the effects of the van der Waals forces, the small scale and the surrounding elastic medium on the critical buckling load are considered. The DWCNT is assumed to be thin and the tube is taken to be perfectly bonded to the surrounding medium. In the present model, both normal and shear stresses are included. It is assumed that the initial displacements up to the point of buckling are small so that the linear theory can be applied.

2 VAN DER WAALS INTERACTION

The van der Waals force between any two carbon atoms can be described by the Lennard Jones model [3]. The van der Waals force exerted on any atom in a tube can be estimated by adding up all forces between the atom and all atoms in the other tube. Fig. 1 shows a DWCNT embedded in an elastic medium. The subscripts 1 and 2 denote the quantities associated with the outer and the inner tubes, respectively. The outer tube is embedded in the elastic medium. Since the two nested tubes are originally concentric and the initial interlayer spacing is equal or very close to the equilibrium spacing, the initial van der Waals interaction between the two tubes of the DWCNTs can be neglected. When the external load is applied, the interlayer spacing changes and any increase (or decrease) in the interlayer spacing will cause an attractive (or repulsive) van der Waals interaction. For any point on the outer tube, the van der Waals interaction pressure depends linearly on the variation of the interlayer spacing at the point. Thus, the pressure p_1 which is positive inward on the outer tube due to the inner tube can be expressed by [11]

$$p_1 = c(w_1 - w_2) \tag{1}$$

where w_1 and w_2 are the radial displacements of the outer and inner tubes, respectively, which are positive outward. In the above relation, c denotes the van der Waals interaction coefficient, which is defined as

$$c = \frac{dG(\delta)}{d\delta}\Big|_{\delta=\delta_0} \tag{2}$$

where $G(\delta)$ is a nonlinear function of the inner tube spacing δ [12]. It is noted that c is a constant which is determined by the slope of the van der Waals law at the initial unbuckled interlayer spacing δ_0 (about 0.34 nm).

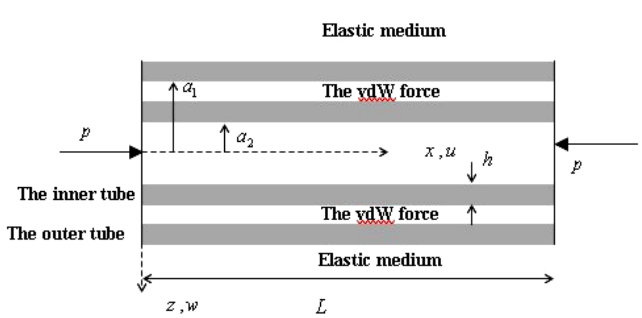

Fig. 1
A double-walled carbon nanotube embedded in an elastic medium.

According to the data provided by Saito et al. [11], the coefficient c can be estimated from

$$c = \frac{320\ erg/cm^2}{0.16d^2}, \qquad (d\text{=0.142 nm}) \tag{3}$$

Since the van der Waals forces between the inner and the outer tubes are equal and in opposite directions, the value of the pressure p_2 on the inner tube due to the outer tube can be obtained from

$$p_2 = -\frac{a_1}{a_2} p_1 \tag{4}$$

where a_1 and a_2 are the radii of the outer and the inner tubes, respectively.

3 PRE-BUCKLING ANALYSIS

In the Eringen nonlocal elasticity model [13], the stress state at a reference point in the body is regarded to be dependent not only on the strain state at the point but also on the strain states at all other points of the body. This is in accordance with the theory of lattice dynamics and experimental observations on the phonon dispersion. For homogeneous and isotropic elastic solids, the constitutive equation of the nonlocal elasticity can be given by [13]

$$(1 - e_0^2 a^2 \Delta^2)\sigma = \mathbf{C}_0 : \varepsilon \tag{5}$$

where symbols ':' denotes the inner product of tensors with double contraction, C_0 is the elastic stiffness tensor of classical (local) isotropic elasticity. σ and ε represent the nonlocal stress and strain tensors, respectively. e_0 is an internal characteristic length of the material (e.g., length of C-C bond, lattice spacing and granular distance). It is known that the middle-surface strains in the shell consist of two parts, namely, the primary strains caused by the applied loads which are present prior to buckling, and the secondary strains, which arise as a result of buckling. If u, v and w denote the additional middle-surface displacements due to buckling along the x, θ and the inward normal direction of the shell, respectively, the total middle-surface strains can be obtained as follows [4]

$$\begin{aligned}
\varepsilon_x &= \varepsilon_{x0} + \frac{\partial u}{\partial x} \\
\varepsilon_\theta &= \varepsilon_{\theta 0} + \frac{1}{a}\frac{\partial v}{\partial \theta} - \frac{w}{a} \\
\varepsilon_{x\theta} &= \frac{1}{2}\left(\frac{1}{a}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}\right)
\end{aligned} \tag{6}$$

where the symbols ε_x, ε_θ and $\varepsilon_{x\theta}$ denote strain components at any point through the shell wall thickness, while u, v and w refer to the displacement components of the middle surface of the shell in the axial, circumferential and radial directions, respectively. ε_{x0} and $\varepsilon_{\theta 0}$ are the middle-surface strains due to the radial pressure prior to buckling. Moreover, the thin shell can be considered as a two-dimensional stress problem with $\frac{\partial \sigma_x}{\partial \theta} = \frac{\partial \sigma_\theta}{\partial x} = 0$. Thus, it follows from Eq. (5) that the nonlocal normal and shear stresses can be written as [9]

$$\begin{aligned}
\sigma_x - (e_0 a)^2 \frac{\partial^2 \sigma_x}{\partial x^2} &= \frac{E}{1-\nu^2}(\varepsilon_x + \nu \varepsilon_\theta), \\
\sigma_\theta - (e_0 a)^2 \frac{1}{a^2}\frac{\partial^2 \sigma_\theta}{\partial \theta^2} &= \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu \varepsilon_x),
\end{aligned} \tag{7}$$

$$\sigma_{x\theta}-(e_0a)^2(\frac{\partial^2\sigma_{x\theta}}{\partial x^2}+\frac{1}{a^2}\frac{\partial^2\sigma_{x\theta}}{\partial\theta^2})=\frac{E}{1-\nu^2}\varepsilon_{x\theta}$$

where σ_x, σ_θ and $\sigma_{x\theta}$ are stress components. E and ν are the Young's modulus and the Poisson's ratio of the material, respectively. It can be observed that Eq. (7) reduces to the Hooke's law of classical elasticity for a two-dimensional stress problem when the small scale parameter vanishes. The resultant membrane forces are given by

$$N_x=\sigma_x h,\ \ N_\theta=\sigma_\theta h,\ \ N_{x\theta}=\sigma_{x\theta}h \tag{8}$$

$$\sigma_\theta^{k} = \frac{N_\theta^k}{h}=-\frac{P_k a_k}{h}$$
$$\sigma_x^k=\frac{N_x^k}{h} \tag{9}$$

where σ_x^k and σ_θ^k are the pre-buckling axial and circumferential membrane stresses in the kth tube layer, h is thickness of the inner and the outer tubes, p_k is the net (inward) pressure of the kth tube, and σ_x is the axial stress applied to the DWCNT .Substituting Eqs. (6) and (7) into Eq. (8) yields

$$\begin{aligned}
N_x&=C(\frac{\partial u}{\partial x}+\frac{\nu}{a}\frac{\partial v}{\partial\theta}-\nu\frac{w}{a})+(e_0a)^2\frac{\partial^2 N_x}{\partial x^2}+C(\varepsilon_{x0}+\nu\varepsilon_{\theta0})\\
N_\theta&=C(\nu\frac{\partial u}{\partial x}+\frac{1}{a}\frac{\partial v}{\partial\theta}-\frac{w}{a})+(e_0a)^2\frac{1}{a^2}\frac{\partial^2 N_\theta}{\partial\theta^2}+C(\nu\varepsilon_{x0}+\varepsilon_{\theta0})\\
N_{x\theta}&=\frac{1}{2}C(1-\nu)(\frac{\partial v}{\partial x}+\frac{1}{a}\frac{\partial u}{\partial\theta})+(e_0a)^2(\frac{1}{a^2}\frac{\partial^2 N_{x\theta}}{\partial\theta^2}+\frac{\partial^2 N_{x\theta}}{\partial x^2})+C(1-\nu)\varepsilon_{x\theta0}
\end{aligned} \tag{10}$$

where, $C=Eh/(1-\nu^2)$ and N_x, N_θ and $N_{x\theta}$ are the resultant forces.

4 BUCKLING ANALYSIS

Since the critical load is the load at which a system in equilibrium passes from a stable to an unstable conditions, it can be determined by finding the lowest load at which the second variation of the total potential energy of the system is non positive for at least one possible variation .In the following, the changes in the total potential energy V of the tube medium structure due to infinitesimal displacements from the above fundamental equilibrium state are considered.

4.1. Energy in the inner and the outer tube

The total energy can be expressed as [6]

$$V=U+\Omega+U_m \tag{11}$$

where

$$U=\frac{1}{2}\int\int\int(\sigma_x\varepsilon_x+\sigma_\theta\varepsilon_\theta)a\ \mathrm{d}x\ \mathrm{d}\theta\ \mathrm{d}z$$
$$\Omega=-\int\int(N_x u+N_\theta w)a\ \mathrm{d}x\ \mathrm{d}\theta \tag{12}$$

$$U_m = \frac{1}{2E_m}\int_0^{2\pi}\int_0^L [\sigma_{\theta m}^2 + \sigma_{xm}^2 - 2\nu_m(\sigma_{\theta m}\sigma_{xm})]\,\mathrm{d}x\,\mathrm{d}\theta$$

where U, Ω, and U_m are the strain energy, the potential energy of pressure P_1, P_2 and the strain energy of the surrounding elastic medium, respectively. Also, sub index m relates to elastic medium. It is known that the strain energy U consists of two parts, first part is the stretching U_{sn} of the outer and the inner tubes, and the second part involves the bending U_{bn} of the outer and the inner tubes. Hence, it can be written as [6]

$$U = U_{sn} + U_{bn}, \quad n = 1,2 \tag{13}$$

where

$$U_{sn} = \frac{a_n C}{2}\int\int((\varepsilon_{xm}^2)_n + (\varepsilon_{\theta m}^2)_n + 2\nu(\varepsilon_{xm})_n(\varepsilon_{\theta m})_n + \frac{1-\nu}{2}(\gamma_{x\theta m}^2)_n)\,\mathrm{d}x\,\mathrm{d}\theta \tag{14}$$

$$U_{bn} = \frac{a_n D}{2}\int\int((\kappa_x^2)_n + (\kappa_\theta^2)_n + 2\nu(\kappa_x)_n(\kappa_\theta)_n + 2(1-\nu)(\kappa_{x\theta}^2)_n)\,\mathrm{d}x\,\mathrm{d}\theta \tag{15}$$

$$\begin{aligned}
&\kappa_{xn} = -w_{n,xx}, \qquad \varepsilon_{xn} = u_{n,x}\\
&\kappa_{\theta n} = \frac{v_{n,\theta} - w_{n,\theta\theta}}{a_n^2}, \qquad \varepsilon_\theta = \frac{v_{n,\theta} + w_n}{a_n}\\
&\kappa_{x\theta n} = \frac{1}{2}\frac{v_{n,x} - 2w_{n,x\theta}}{a_n^2}, \qquad \gamma_{x\theta n} = v_{n,x} + \frac{u_{n,\theta}}{a_n}\\
&\beta_{\theta n} = \frac{v_n - w_{n,\theta}}{a_n}, \qquad \beta_{xn} = -w_{n,xx}
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\varepsilon_{xn} = \varepsilon_{xmn} + z\kappa_{xn}\\
&\varepsilon_{\theta n} = \varepsilon_{\theta mn} + z\kappa_{\theta n}\\
&\gamma_{x\theta n} = \gamma_{x\theta mn} + 2z\kappa_{x\theta n}
\end{aligned} \tag{17}$$

where $D = Eh^3/12(1-\nu^2)$ and u_n, v_n and w_n $(n = 1,2)$ are the axial, the circumferential and the radial displacements of the outer and the inner tubes, respectively. From the study of a SWCNT, the various values of the effective bending and in-plane stiffness and the corresponding effective Young's modulus and wall thickness have been found [11, 14-16]. In this study, the calculation of Yakobson et al. [1] is adopted for the values of these parameters. Using the data of Yakobson et al. [1], they found that the effective bending stiffness (D) is equal to $0.85\ ev$ while the effective plane stiffness (Eh) is equal to 360 J/m^2. Furthermore, they also showed that the Young's modulus and the Poisson's ratio are equal to 5.5 TPa and 0.19, respectively. To obtain the second variations in the strain energies associated with the stretching and the bending of the outer and the inner tube, it is assumed that

$$\begin{aligned}
&u_n \to u_{0n} + u_n\\
&v_n \to v_{0n} + v_n \qquad n = 1,2\\
&w_n \to w_{0n} + w_n
\end{aligned} \tag{18}$$

where (u_0, v_0, w_0) denotes the above fundamental equilibrium configuration, and the incremental displacements, such as, (u_1, v_1, w_1) and (u_2, v_2, w_2) for the outer and the inner tube, respectively are infinitesimally small. The change in the potential energy corresponding to the incremental displacement is written as

$$\Delta V = \delta V + \frac{1}{2!}\delta^2 V + \frac{1}{3!}\delta^3 V + ... \tag{19}$$

The first order term in Eq. (19) is zero because (w_0, v_0, u_0) is an equilibrium configuration. The secondary equilibrium paths of the governing equations may be obtained by application of the stationary potential energy criterion to this expression for ΔV . Substituting Eqs. (14) and (15) into Eq. (19), the second variation in the total energy is obtained as follows

$$\delta^2 V = F(u, v, w, h, \nu, a_1, a_2, E, \sigma_x, \sigma_\theta) + \delta^2 U_m \tag{20}$$

4.2. Second variation of the strain energy of the surrounding elastic medium

The difference between the work of the traction forces at the tube medium interface due to the displacement increments and that of the elastic medium on the tube through the effective pressure equals the strain energy increase in the medium. The surface traction at tube medium interface can be expressed as [17].

$$\begin{aligned}
&\frac{\partial \sigma_{xm}}{\partial x} + \frac{\partial \tau_{xrm}}{\partial r} + \frac{1}{r}\ \frac{\partial \tau_{\theta xm}}{\partial \theta} + \frac{\tau_{xr}}{r} = -F_x \\
&\frac{\partial \sigma_{rm}}{\partial r} + \frac{\sigma_{rm} - \sigma_{\theta m}}{r} + \frac{1}{r}\frac{\partial \tau_{r\theta m}}{\partial \theta} + \frac{\partial \tau_{rxm}}{\partial x} = -F_r \\
&\frac{\partial \tau_{r\theta m}}{\partial r} + \frac{1}{r}\ \frac{\partial \sigma_{\theta m}}{\partial \theta} + \frac{\partial \tau_{\theta xm}}{\partial x}\ + \frac{\tau_{r\theta m}}{r} = -F_\theta
\end{aligned} \tag{21}$$

$$G_m = \frac{E_m}{2(1+\nu_m)}, \qquad r = a_1 = \text{const.} \tag{22}$$

The relationships between the normal and the shear strains with the displacements are given in Appendix A. The axial and the circumferential membranes stresses of the elastic medium are expressed as

$$\sigma_{xm} = \frac{E_m \upsilon_m \varepsilon_\theta + E_m \varepsilon_x}{1-\upsilon_m^2}, \qquad \sigma_{\theta m} = \frac{\upsilon_m E_m \varepsilon_x\ + E_m \varepsilon_\theta}{1-\upsilon_m^2} \tag{23}$$

where E_m, υ_m are the Young's modulus and the Poisson's ratio of the elastic medium, respectively, and ε_θ, ε_x are the circumferential and the axial membrane strains of the elastic medium, respectively. The simply supported boundary conditions are considered for the DWCNT [16]. Therefore, buckling modes can be written as

$$\begin{aligned}
u &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} A_{rs} \sin n\theta . \cos\frac{m\pi}{l} x \\
v &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} B_{rs} \cos n\theta . \sin\frac{m\pi}{l} x \\
w &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} C_{rs} \sin n\theta . \sin\frac{m\pi}{l} x
\end{aligned} \tag{24}$$

where A_{rs}, B_{rs}, C_{rs} are the real coefficients, m and n are the axial half and the circumferential wave numbers and l is the length of the DWCNT. The work done by these surface tractions is given as [17]

$$M_1 = a\int_0^{2\pi}\int_0^L (F_r w + F_\theta v + F_x u)\ \mathrm{d}x\ \mathrm{d}\theta \tag{25}$$

Meanwhile, the work of the effective pressure is given by [17]

$$M_2 = -p\int_0^{2\pi}\int_0^{L}(v^2 + vw_{,\theta} - v_{,\theta}w + w^2)\ \mathrm{d}x\ \mathrm{d}\theta \tag{26}$$

The first variation, δU_m , is equal to zero due to the equilibrium consideration. The second variation of the strain energy of the elastic medium can be expressed as [18]

$$\delta^2 U_m = (M_1 - M_2) \tag{27}$$

4.3. Rayleigh-Ritz method

The Rayleigh-Ritz method together with the principle of minimum potential energy is employed to solve the problem. In this method, a series solution with coefficient $a_n\,(n = 1,2,\ldots)$ which satisfies the boundary conditions is considered. Minimizing the resulting expressions yields:

$$\frac{\partial V}{\partial a_1} = 0,\ldots.\frac{\partial V}{\partial a_n} = 0 \tag{28}$$

The general solution for the displacement variables can be expressed by the series for the outer and the inner tubes [19]

$$\begin{aligned} u_k &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{rsk} \sin n\theta\ \cos\frac{m\pi}{l}x \\ v_k &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{rsk} \cos n\theta\ \sin\frac{m\pi}{l}x, \qquad k = 1,2 \\ w_k &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{rsk} \sin n\theta\ \sin\frac{m\pi}{l}x \end{aligned} \tag{29}$$

where $A_{rsk}, B_{rsk}, C_{rsk}$ are the real coefficients. Substitution Eq. (29) into Eq. (20), one can obtain second variation of the total potential energy.

4.4. Determination of the critical load

The second variation of the total potential energy given by Eq. (20) is a function of six variables $A_{rsk}, B_{rsk}, C_{rsk}\,(k = 1,2)$. It can be written as [13]

$$\delta(\partial V^2) = \partial\frac{\delta^2 V}{\partial A_{rsk}}\delta A_{rsk} + \partial\frac{\delta^2 V}{\partial B_{rsk}}\delta B_{rsk} + \partial\frac{\delta^2 V}{\partial C_{rsk}}\delta C_{rsk} \tag{30}$$

Since δA_{rsk} , δB_{rsk} , δC_{rsk} are arbitrary ,the expressions in Eq. (30) can be written as

$$\partial\frac{\delta^2 V}{\partial A_{rsk}}\delta A_{rsk} = 0, \qquad \partial\frac{\delta^2 V}{\partial C_{rsk}}\delta C_{rsk} = 0, \qquad \partial\frac{\delta^2 V}{\partial B_{rsk}}\delta B_{rsk} = 0 \tag{31}$$

As the right hand side of each equation is zero, they are referred to as homogeneous equations. Obviously, $A_{rsk} = B_{rsk} = C_{rsk}$ =0 is a solution to Eq. (30). This is the trivial solution of equilibrium at all loads. The nontrivial solutions of such a problem are obtained by setting the determinant of governing equations equal to zero. It determines a relationship between pressure and the buckling mode wave number (n). The critical pressure for elastic buckling is defined by the lowest pressure with $n > 3$.

5 NUMERICAL RESULTS AND DISCUSSION

The shell dimensions and its mechanical properties are considered as follows [8, 9]

$$E_m = 2.06Gpa, \qquad v_m = 0.3, \qquad a_2 = 2.0nm, \qquad a_1 = 2.34, \qquad e_0 = 0.82 \tag{32}$$

As a result, the dependence of the pressure (*p*) on the circumferential wave number (*n*) is obtained, which is shown in Fig 2. As can be seen, the pressure reaches the minimum at *n*=3.Thus, the critical pressure p_{cr} is determined by *n*=3. Fig. 3 shows the relationship between the critical pressure and the Poisson's ratio of the elastic medium. It is seen from this figure that the critical pressure increases with the increment of the Poisson's ratio of the elastic medium. The relationship between the critical pressure and the Young's modulus of the elastic medium is determined for $a_1 / h = 22$, 34 and 45, and presented in Fig.4. It is seen that the critical buckling pressure increases with increasing the Young's modulus of the elastic medium for a fixed outer radius to thickness ratio (a_1 / h). It is seen from the Fig. 5 that for small outer radius to thickness ratios, the values of the critical pressure p_{cr} considering the van der Waals forces are larger than those ignoring them. Accordingly, it can be concluded that the influence of the van der Waals force on the critical pressure depends on the outer radius-to-thickness ratio.

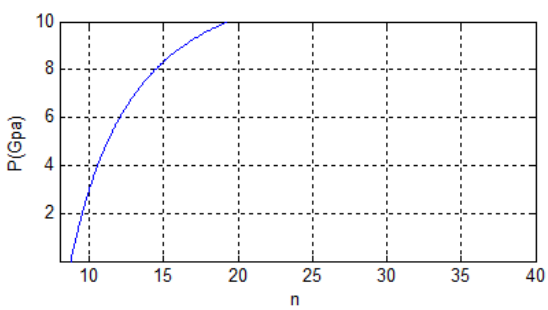

Fig. 2
The dependence of the pressure (*p*) on the circumferential wave number (*n*).

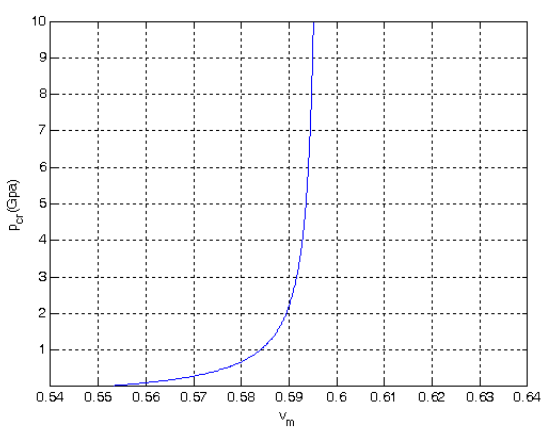

Fig. 3
The effect of Poisson's ratio of the elastic medium on the critical buckling load.

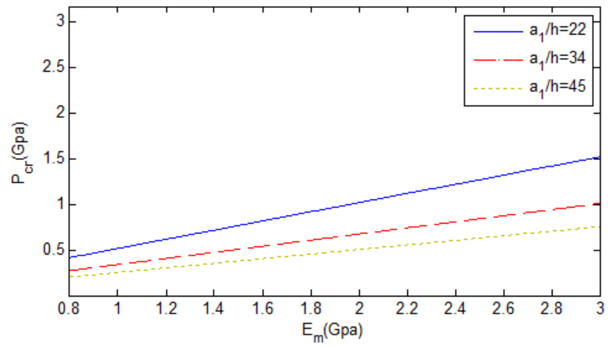

Fig. 4
The effect of Young’s modulus of the elastic medium on the critical pressure for different outer radius to thickness ratios.

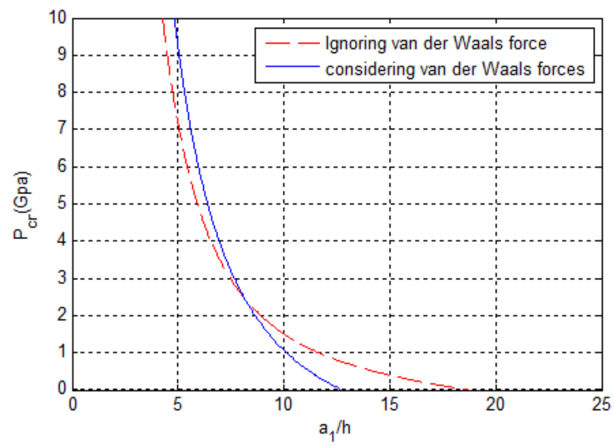

Fig. 5
Comparison of the critical pressure considering the effect of van der Waals force with that ignoring the effect of van der Waals force.

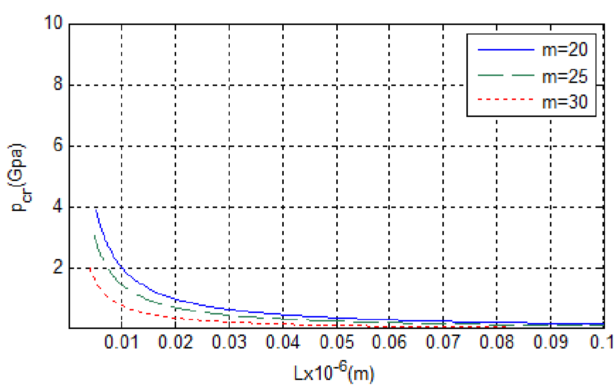

Fig. 6
The dependency of critical pressure on the half axial wave numbers for *n*=3.

Fig. 6 shows the variation of the critical buckling load p_{cr} with respect to the tube length (L) for various axial half wave numbers 20, 25 and 30 (n=3). As can be seen from this figure that the critical buckling load decreases with increasing of the length of the nanotube for a constant radius of the outer tube (a_1). It is also observed that the critical pressure descend very quickly with increasing the half axial wave numbers.

6 CONCLUSIONS

Using the energy and Rayleigh-Ritz methods, the elastic buckling of a small length DWCNT embedded in an elastic medium under axial loading is analyzed. The DWCNT is assumed to be thin and the tube is taken to be perfectly bonded to the surrounding medium. The difference between the Poisson’s ratio of the tube and that of the elastic medium is taken into account. The interaction of van der Waals force between the inner and the outer tubes and the effect of compressive axial load are incorporated in the formulation. The effects of the Young's modulus and the Poisson's ratio on the critical pressure of DWCNTs are discussed. The influence of the small scale on the critical axial buckling pressure of DWCNTs is considered. The following conclusions can be drawn from the present study:

1. The axial buckling pressure for a carbon nanotube can be overestimated by the classic shell model due to ignoring the effect of small length scale.
2. The critical pressure increases with the increment of the Poisson's ratio of the elastic medium.
3. The critical buckling pressure increases with increasing the Young's modulus of the elastic medium for a fixed outer radius to thickness ratio.
4. The critical buckling load decreases with increasing of the length of the nanotube.
5. The critical pressure descends very quickly with increasing the half axial wave numbers.

7 Appendix A

In Eq. (23), relationships between strain and displacement components can be written as

$$
\begin{aligned}
\varepsilon_\theta &= \frac{1}{r}\left(\frac{\partial v}{\partial \theta} + w\right) \\
\varepsilon_r &= \frac{\partial w}{\partial r} \\
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_{r\theta} &= \frac{1}{2}\left\{\frac{1}{r}\,\frac{\partial w}{\partial \theta} - \frac{v}{r} + \frac{\partial v}{\partial r}\right\} \\
\varepsilon_{rx} &= \frac{1}{2}\left\{\frac{\partial u}{\partial r} + \frac{\partial w}{\partial x}\right\} \\
\varepsilon_{\theta x} &= \frac{1}{2}\left\{\frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right\}
\end{aligned}
\tag{A.1}
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