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A Static Flexure of Thick Isotropic Plates Using Trigonometric Shear Deformation Theory

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ABSTRACT

A Trigonometric Shear Deformation Theory (TSDT) for the analysis of isotropic plate, taking into account transverse shear deformation effect as well as transverse normal strain effect, is presented. The theory presented herein is built upon the classical plate theory. In this displacement-based, trigonometric shear deformation theory, the in-plane displacement field uses sinusoidal function in terms of thickness coordinate to include the shear deformation effect. The cosine function in terms of thickness coordinate is used in transverse displacement to include the effect of transverse normal strain. It accounts for realistic variation of the transverse shear stress through the thickness and satisfies the shear stress free surface conditions at the top and bottom surfaces of the plate. The theory obviates the need of shear correction factor like other higher order or equivalent shear deformation theories. Governing equations and boundary conditions of the theory are obtained using the principle of virtual work. Results obtained for static flexural analysis of simply supported thick isotropic plates for various loading cases are compared with those of other refined theories and exact solution from theory of elasticity.

Keywords: Shear deformation; Isotropic thick plate; Flexure, Deflection; Normal and transverse shear stress

1 INTRODUCTION

THE well-known classical plate theory is due to Kirchhoff [1]. It is based on the Kirchhoff's hypothesis that straight lines normal to the undeformed midplane remain straight and normal to the deformed midplane and do not undergo thickness stretching. Since the transverse shear deformation is neglected in Kirchhoff theory, it can not be applied to thick plates wherein shear deformation effects are more significant. Thus, its suitability is limited to only thin plates. The first order shear deformation theory (FSDT) is considered as improvement over the classical plate theory. This is achieved by including gross transverse shear deformation in the kinematic assumptions. Reissner [2, 3] was the first to provide a consistent stress based plate theory which incorporates the effect of shear deformation. Mindlin [4] has provided displacement based theory. In this theory, the transverse shear strain is assumed to be constant across the thickness, and thus shear correction factor to correct the strain energy of shear deformation is required. These factors are problem dependent.

The second order shear deformation theories by Naghdi [5], Pister and Westmann [6], Whitney and Sun [7], Nelson and Lorch [8] give marginally improved results over FSDT, but suffer from the same drawbacks of FSDT. The limitations of classical plate theory and first order shear deformation theory forced the development of higher order shear deformation theories to avoid the use of shear correction factors, to include correct cross sectional warping and to get the realistic variation of the transverse shear strain and stresses through the thickness of plate. The higher order theories based on series expansions are developed by Reissner [9], Provan and Koeller [10], Lo et al. [11, 12] and are modified by Levinson [13], Reddy [14] to get the parabolic shear stress distribution through the

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thickness of plate and to satisfy the shear stress free surface conditions on the top and bottom surfaces of the plate to avoid the need of shear correction factor. The third order theories built upon classical plate theory are developed by Krishna Murty [15, 16], Savitri and Varadan [17], Soldatos [18] and are reviewed and generalized by Reddy [19]. Comprehensive reviews of these theories have been given by Noor and Burton [20] and Ghugal and Shimpi [21]. There exists another class of refined shear deformation theories, wherein use of trigonometric function is made to take into account shear deformation effect. In 1877 Levy [22] developed a refined theory for thick isotropic plate for the first time using sinusoidal functions in the displacement field interms of thickness coordinate. However, efficiency of this particular plate theory was not validated for more than a century. Stein [23] also proposed such theory and applied to isotropic plates in the modified form. The theories containing trigonometric functions involving thickness coordinate in the displacement fields are designated as trigonometric shear deformation theories. Shimpi [24] developed refined plate theory for the analysis of isotropic plate. Shimpi and Patel [25] presented two variable theories, for the analysis of orthotropic plates. Recently, Shimpi et al. [26], developed new first order shear deformation theories; however, second model of these authors is nothing but the classical plate theory. Hence, no improvement in the results of displacement and stresses is achieved.

In the present work, a trigonometric shear deformation theory for bidirectional bending of thick plate is presented. It has four variables and includes effect of transverse shear and transverse normal strain. Results obtained for various loading cases are compared with those of refined theories and exact elasticity theory [27] available in the literature.

2 THEORETICAL FORMULATION

2.1 *Plate under consideration*

Consider a plate (of length a, width b, and thickness h) of homogenous material. The plate occupies (in 0-x-y-z right-handed Cartesian coordinate system) a region

$$0 \le x \le a, \qquad 0 \le y \le b, \qquad -h/2 \le z \le h/2; \tag{1}$$

2.2 *Assumptions made in theoretical formulation*

1. The displacement components u and v are the in-plane displacements in x and y-directions respectively and w is the transverse displacement in z-direction. These displacements are small in comparison with the plate thickness.
2. The in-plane displacement u in x -direction and v in y-direction each consist of two parts: a displacement component analogous to displacement in classical plate theory of bending; displacement component due to shear deformation which is assumed to be sinusoidal in nature with respect to thickness coordinate.
3. The transverse displacement w in z -direction is assumed to be a function of x, y and z coordinates.
4. The body forces are ignored in the analysis.
5. The plate is subjected to transverse load only.

2.3 *The displacement field*

Based upon the before mentioned assumptions, the displacement field of the present plate theory is given as below

$$\begin{aligned} u &= -z\frac{\partial w(x,y)}{\partial x} + \frac{h}{\pi}\sin\frac{\pi z}{h}\,\phi(x,y) \\ v &= -z\frac{\partial w(x,y)}{\partial y} + \frac{h}{\pi}\sin\frac{\pi z}{h}\,\psi(x,y) \\ w &= w(x,y) + \frac{h}{\pi}\cos\frac{\pi z}{h}\,\xi(x,y) \end{aligned} \tag{2}$$

where u and v are the in-plane displacements and w is transverse displacement. The sinusoidal function in in-plane displacements is assigned according to the shear stress distribution through the thickness of the plate. The ϕ, ψ, and ξ are the unknown functions of position x and y in the region of plate.

2.4 *Strain displacement relationships*

Normal and shear strains are obtained within the framework of linear theory of elasticity using displacement field given by Eqs. (2).
Normal strains:

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} + \frac{h}{\pi}\sin\frac{\pi z}{h}\frac{\partial \phi}{\partial \mathrm{x}} \\
\varepsilon_y &= \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} + \frac{h}{\pi}\sin\frac{\pi z}{h}\frac{\partial \psi}{\partial y} \\
\varepsilon_z &= \frac{\partial w}{\partial z} = -\xi\sin\frac{\pi z}{h}
\end{aligned} \tag{3}$$

Shear strains:

$$\begin{aligned}
\gamma_{xy} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} + \frac{h}{\pi}\sin\frac{\pi z}{h}\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right) \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \cos\frac{\pi z}{h}\left(\frac{h}{\pi}\frac{\partial \xi}{\partial x} + \phi\right) \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \cos\frac{\pi z}{h}\left(\frac{h}{\pi}\frac{\partial \xi}{\partial y} + \psi\right)
\end{aligned} \tag{4}$$

2.5 *Stress-strain relationships*

The stress-strain relationships for the isotropic plate material can be written as

$$\begin{aligned}
\sigma_x &= \lambda\,(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_x \\
\sigma_y &= \lambda\,(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_y \\
\sigma_z &= \lambda\,(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_z \\
\tau_{xy} &= G\gamma_{xy}, \qquad \tau_{yz} = G\gamma_{yz}, \qquad \tau_{zx} = G\gamma_{zx}
\end{aligned} \tag{5}$$

where λ and G are the Lame's constants as given below

$$\lambda = \frac{\mu E}{(1-\mu)\,(1-2\mu)} \qquad \text{and} \qquad G = \frac{E}{2(1+\mu)}$$

The strain-displacement and stress-strain relations used in Eqs. (3) through (5) are well known in theory of elasticity and are discussed by Timoshenko and Goodier [28].

2.6 *Derivation of governing equations and boundary conditions*

Using Eq. (3) through (5) and principle of virtual work, variationally consistent differential equations and boundary conditions for the plate under consideration are obtained. The principle of virtual work when applied to the plate can be written as

$$\int_{-h/2}^{h/2}\int_0^b\int_0^a \left[\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_{yz}\,\delta\gamma_{yz} + \tau_{zx}\,\delta\gamma_{zx} + \tau_{xy}\,\delta\gamma_{xy}\right] dx\,dy\,dz - \int_0^b\int_0^a q(x,y)\delta w\,dx\,dy = 0 \tag{6}$$

where symbol δ denotes the variational operator. Employing Green's theorem in Eq. (6) successively, we obtain the coupled governing equations (Euler-Lagrange equations) and associated boundary conditions of the plate. The governing equations of the plate are as follows

$$D_1\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)-D_2\left(\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^3\phi}{\partial x\partial y^2}+\frac{\partial^3\psi}{\partial y^3}+\frac{\partial^3\psi}{\partial x^2\partial y}\right)+D_3\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}\right)=q \quad (7a)$$

$$D_2\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right)-D_4\frac{\partial^2\phi}{\partial x^2}-D_5\frac{\partial^2\phi}{\partial y^2}+D_6\phi-D_7\frac{\partial^2\psi}{\partial x\partial y}+D_8\frac{\partial\xi}{\partial x}=0 \quad (7b)$$

$$D_2\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2\partial y}\right)-D_4\frac{\partial^2\psi}{\partial y^2}-D_5\frac{\partial^2\psi}{\partial x^2}+D_6\psi-D_7\frac{\partial^2\phi}{\partial x\partial y}+D_8\frac{\partial\xi}{\partial y}=0 \quad (7c)$$

$$D_3\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-D_8\left(\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial y}\right)-D_5\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}\right)+D_9\xi=0 \quad (7d)$$

The associated boundary conditions are as follows

1. On edges $x=0$ and $x=a$, the following conditions hold

$$D_1\frac{\partial^3 w}{\partial x^3}+D_{10}\frac{\partial^3 w}{\partial x\partial y^2}-D_2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\psi}{\partial x\partial y}\right)-D_{11}\frac{\partial^2\phi}{\partial y^2}+D_3\frac{\partial\xi}{\partial x}=0 \quad \text{or} \quad w \quad \text{is prescribed} \quad (8a)$$

$$D_1\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}-D_2\frac{\partial\phi}{\partial x}-D_{13}\frac{\partial\psi}{\partial y}+D_3\xi=0 \quad \text{or} \quad \frac{\partial w}{\partial x} \quad \text{is prescribed} \quad (8b)$$

$$D_2\frac{\partial^2 w}{\partial x^2}+D_{13}\frac{\partial^2 w}{\partial y^2}-D_4\frac{\partial\phi}{\partial x}-D_{14}\frac{\partial\psi}{\partial y}+D_{15}\xi=0 \quad \text{or} \quad \phi \quad \text{is prescribed} \quad (8c)$$

$$D_{11}\frac{\partial^2 w}{\partial x\partial y}-D_{16}\left(\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\right)=0 \quad \text{or} \quad \psi \quad \text{is prescribed} \quad (8d)$$

$$D_{16}\frac{\partial\xi}{\partial x}+D_{17}\phi=0 \quad \text{or} \quad \xi \quad \text{is prescribed} \quad (8e)$$

2. On edges $y=0$ and $y=b$, the following conditions hold

$$D_1\frac{\partial^3 w}{\partial y^3}+D_{10}\frac{\partial^3 w}{\partial x^2\partial y}-D_5\left(\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\phi}{\partial x\partial y}\right)-D_{11}\frac{\partial^2\psi}{\partial x^2}+D_3\frac{\partial\xi}{\partial y}=0 \quad \text{or} \quad w \quad \text{is prescribed} \quad (9a)$$

$$D_1\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}-D_2\frac{\partial\psi}{\partial y}-D_5\frac{\partial\phi}{\partial x}+D_3\xi=0 \quad \text{or} \quad \frac{\partial w}{\partial y} \quad \text{is prescribed} \quad (9b)$$

$$D_{13}\frac{\partial^2 w}{\partial x^2}+D_2\frac{\partial^2 w}{\partial y^2}-D_{14}\frac{\partial\phi}{\partial x}-D_4\frac{\partial\psi}{\partial y}+D_{15}\xi=0 \quad \text{or} \quad \psi \quad \text{is prescribed} \quad (9c)$$

$$D_{11}\frac{\partial^2 w}{\partial x\partial y}-D_{16}\left(\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\right)=0 \quad \text{or} \quad \phi \quad \text{is prescribed} \quad (9d)$$

$$D_{16}\frac{\partial\xi}{\partial x}+D_{17}\psi=0 \quad \text{or} \quad \xi \quad \text{is prescribed} \quad (9e)$$

3. At corners $(x=0, y=0)$, $(x=0, y=b)$, $(x=a, y=0)$, and $(x=a, y=b)$ the following condition hold

$$D_{18}\frac{\partial^2 w}{\partial x\partial y}-D_{11}\left(\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial x}\right)=0 \quad \text{or} \quad w \quad \text{is prescribed} \tag{10}$$

where D_1 through D_{18} are the stiffnesses as given in Appendix A. Thus, the variationally consistent governing differential equations and boundary conditions are obtained. The flexural behavior of the plate is described by the solution satisfying these equations and the associated boundary conditions at each edge and corner of the plate.

3 ILLUSTRATIVE EXAMPLES

Example 1: A simply supported isotropic rectangular plate occupying the region given by the expression (1) is considered. The plate is subjected to uniformly distribute transverse load, $q(x,y)$ on surface z=-h/2 acting in the z-direction as given below

$$q\ x,y\ =\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{11}$$

where q_{mn} are the coefficients of Fourier expansion of load which are given by

$$q_{mn}=\frac{16q_0}{\pi^2 mn}, \quad \text{for} \quad m=1,\ 3,\ 5,\ldots, \text{ and } n=1,\ 3,\ 5,\ldots$$
$$q_{mn}=0, \quad \text{for} \quad m=2,\ 4,\ 6,\ldots, \text{ and } n=2,\ 4,\ 6,\ldots$$

The plate material are considered as E=210 GPa and μ=0.3, where E is the Young's modulus and μ is the Poisson's ratio. The governing differential equations and the associated boundary conditions for static flexure of rectangular plate under consideration can be obtained directly from Eqs. (7) through (10). The following are the boundary conditions of the simply supported isotropic plate on the edges x=0 and x=a

$$w=0 \tag{12a}$$
$$D_1\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}-D_2\frac{\partial\phi}{\partial x}-D_{13}\frac{\partial\psi}{\partial y}+D_3\xi=0 \tag{12b}$$
$$D_2\frac{\partial^2 w}{\partial x^2}+D_{13}\frac{\partial^2 w}{\partial y^2}-D_4\frac{\partial\phi}{\partial x}-D_{14}\frac{\partial\psi}{\partial y}+D_{15}\xi=0 \tag{12c}$$
$$D_{11}\frac{\partial^2 w}{\partial x\partial y}-D_{16}\left(\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\right)=0 \tag{12d}$$
$$D_{16}\frac{\partial\xi}{\partial x}+D_{17}\phi=0 \tag{12e}$$

and boundary conditions on the edges y=0 and y=b

$$w=0 \tag{13a}$$
$$D_1\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}-D_2\frac{\partial\psi}{\partial y}-D_5\frac{\partial\phi}{\partial x}+D_3\xi=0 \tag{13b}$$
$$D_{13}\frac{\partial^2 w}{\partial x^2}+D_2\frac{\partial^2 w}{\partial y^2}-D_{14}\frac{\partial\phi}{\partial x}-D_4\frac{\partial\psi}{\partial y}+D_{15}\xi=0 \tag{13c}$$
$$D_{11}\frac{\partial^2 w}{\partial x\partial y}-D_{16}\left(\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\right)=0 \tag{13d}$$

$$D_{16}\frac{\partial \xi}{\partial x}+D_{17}\psi = 0 \tag{13e}$$

Example 2: A simply supported rectangular plate is subjected to sinusoidal load in both *x* and *y* directions, on surface *z*=-*h*/2, acting in the downward *z* direction. The load is expressed as

$$q(x,y)=q_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) \tag{14}$$

where q_0 is the magnitude of the sinusoidal loading at the centre (*m* =1 and *n* = 1).

Example 3: A simply supported rectangular plate is subjected to linearly varying transverse load $(q_0\, x / a)$. The magnitude of coefficient of Fourier expansion of load in the Eq. (11) is given by $q_{mn}=(8q_0 / mn\pi^2)\cos(m\pi)$.

Example 4: A simply supported square plate is subjected to transverse concentrated load *P*. The magnitude of coefficient of Fourier expansion of load in the Eq. (11) is given by $q_{mn}=(4P/ab)\ \sin(m\pi\xi/a)\ \sin(n\pi\eta/b)$ where ξ and η represent distance of concentrated load from *x* and *y* axes respectively.

Example 5: A simply supported square plate is subjected to transverse line load *P* parallel to *y* axis. The magnitude of coefficient of Fourier expansion of load in the Eq. (11) is given by $q_{mn}=(2P/b)\sin(m\pi\xi/a)$, where, ξ represents a distance of line load from *y* axis.

The following is the solution form assumed for $w(x,y)$, $\phi(x,y)$, $\psi(x,y)$, and $\xi(x,y)$ in the examples 1 through 5, which satisfies boundary conditions exactly

$$\begin{aligned}
w\ x,y &= \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} w_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
\phi\ x,y &= \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \phi_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
\psi\ x,y &= \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \psi_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\\
\xi\ x,y &= \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \xi_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}
\end{aligned} \tag{15}$$

where w_{mn}, ϕ_{mn}, ψ_{mn}, and ξ_{mn} are the unknown coefficients of the respective Fourier expansions and *m, n* are positive integer**.** Substituting this form of solution (15) and the load $q(x,y)$ from Eq. (11) into the governing equations yields the four algebraic simultaneous equations, from which the unknown coefficients can be readily determined. Having obtained the values of w_{mn}, ϕ_{mn}, ψ_{mn}, and ξ_{mn} one can then calculate all the displacement and stress components within the plate using Eqs. (2) through (5). Transverse shear stresses can be obtained either by constitutive relations or by integrating equilibrium equations of three dimensional elasticity with respect to the thickness coordinate, satisfying shear stress free conditions on the top and bottom surfaces of the plate.

4 RESULTS AND DISCUSSION

The transverse displacement, in-plane and transverse stresses are presented in the following non-dimensional form for the purpose of presenting the results in this paper

$$\bar{w}=\frac{100Ew}{qhS^4},\qquad (\bar{\sigma}_x,\bar{\sigma}_y,\bar{\tau}_{xy})=\frac{(\sigma_x,\sigma_y,\tau_{xy})}{qS^2},\qquad (\bar{\tau}_{zx},\bar{\tau}_{yz})=\frac{(\tau_{zx},\tau_{yz})}{qS}$$

Further, it may be noted that τ_{zx} and $\bar{\tau}_{zx}$ obtained by constitutive relations are indicated by $\tau_{zx}{}^{CR}$ and $\bar{\tau}_{zx}{}^{CR}$ and they are indicated by $\tau_{zx}{}^{EE}$ and $\bar{\tau}_{zx}{}^{EE}$ when they are obtained by using equilibrium equations. Similar notations are also used for τ_{yz}. The percentage error in result of a particular theory is shown in parenthesis in subsequent tables with respect to the result of exact elasticity solution which is calculated as follows

$$\%\,\text{error} = \frac{\text{value by a particular theory} - \text{value by exact elasticity solution}}{\text{value by exact elasticity solution}} \times 100$$

Results obtained for displacements and stresses are compared and discussed with the corresponding results of classical plate theory (CPT), first order shear deformation theory (FSDT), higher order shear deformation theories (HSDTs), of various authors and the exact elasticity solution of plate. For the purpose of comparison, results were specially generated according to the exact elasticity solution [38].

Example 1: Table 1 shows comparison of deflection and stresses for the simply supported homogenous rectangular plate subjected to uniformly distributed load for *a/b* = 0.5 and *h/a* = 0.1. The present theory gives more accurate value of deflection than that is given by other refined theories as compared to exact value. The theory overestimates the value of in-plane normal stress $\bar{\sigma}_x$ and underestimates $\bar{\sigma}_y$ as compared to the value of exact solution. The value of in-plane shear stress obtained by present theory is in excellent agreement with the values of other refined theories. Transverse shear stress when obtained by constitutive relations using present theory is on higher side, however, use of equilibrium equations yield more accurate results in case of present theory.

Example 2: Table 2 shows the comparison of deflection and stresses for the simply supported homogenous square plate subjected to single sine load. The central deflection predicted by present theory is much closer to the exact value. The in-plane normal stress is overestimated by 6.00 % whereas Reddy's theory gives the exact value of this stress. The in-plane shear stresses obtained by other theories are in good agreement with the present theory. The transverse shear stress when predicted by equilibrium equation is underpredicted by 1.26 % as compared to value of exact solution.

Example 3: In case of simply supported homogenous isotropic square plate subjected to linearly varying load results are shown in Table 3 for *a*/*b* ratio 1. The transverse displacement predicted is 0.30 % less as compared to exact value, whereas Krishna Murty's theory yields much better value of this displacement as compared to present and Reddy's theory. The value of in-plane normal stress predicted is on higher side as compared to the value of exact solution and values of other refined theories. The values of in-plane shear stress obtained by present theory, HSDT of Krishna Murty, FSDT and CPT are identical; however, the value of this stress predicted by HSDT of Reddy is on higher side for this loading case. The error in the transverse shear stresses is reduced when predicted by equilibrium equations.

Table 1
Comparison of deflection $\bar{w}$ at $(x=a/2, y=b/2, z=0)$, in-plane normal stresses $\bar{\sigma}_x$ at $(x=a/2,\ y=b/2,\ z=h/2)$, $\bar{\sigma}_y$ at $(x=a/2,\ y=b/2,\ z=h/2)$, in-plane shear stress $\bar{\tau}_{xy}$ $(x=0,\ y=0,\ z=h/2)$ and transverse shear stress $\bar{\tau}_{zx}$ at $(x=0, y=b/2,\ z=0)$ in rectangular isotropic plate subjected to uniformly distributed load

a/b	*h/a*	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
0.5	0.1	Present	11.340	0.638	0.245	0.277	0.701	0.667
		(TSDT)	(-0.30)	(4.24)	(-12.81)		(3.240)	(-1.76)
		Krishna Murty [16]	11.310	0.613	0.310	0.278	0.682	0.667
		(HSDT)	(-0.57)	(0.163)	(10.32)		(0.441)	(-1.76)
		Reddy [14]	11.420	0.612	0.278	0.280	0.679	0.6776
		(HSDT)	(0.39)	(0.00)	(-1.06)		(0.00)	(-0.206)
		Mindlin [4]	11.420	0.610	0.277	0.276	0.545	0.6865
		FSDT	(0.39)	(-0.32)	(-1.42)		(-19.73)	(1.104)
		Kirchhoff [1]	11.060	0.610	0.278	0.277	0.686	0.6865
		CPT	(-2.78)	(-0.32)	(-1.06)		(1.03)	(1.104)
		Exact [27]	11.375	0.612	0.281	---	0.679	---

Table 2
Comparison of deflection $\bar{w}$ at $(x=a/2, y=b/2, z=0)$, in-plane normal stresses $\bar{\sigma}_x$ at $(x=a/2,\ y=b/2,\ z=h/2)$, $\bar{\sigma}_y$ at $(x=a/2,\ y=b/2,\ z=h/2)$, in-plane shear stress $\bar{\tau}_{xy}$ $(x=0,\ y=0,\ z=h/2)$ and transverse shear stress $\bar{\tau}_{zx}$ at $(x=0, y=b/2,\ z=0)$ in square isotropic plate subjected to single sine load

a/b	*h/a*	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
1	0.1	Present	2.933	0.307	0.307	0.110	0.245	0.235
		(TSDT)	(- 0.30)	(6.00)	(6.00)		(2.94)	(-1.26)
		Krishna Murty [16]	2.942	0.290	0.290	0.110	0.238	0.238
		(HSDT)	(0.00)	(0.34)	(0.34)		(0.00)	(0.00)
		Reddy [14]	2.960	0.289	0.289	0.107	0.238	0.228
		(HSDT)	(0.62)	(0.00)	(0.00)		(0.00)	(-4.20)
		Mindlin [4]	2.934	0.287	0.287	0.106	0.169	0.249
		FSDT	(0.27	(- 1.50)	(- 1.50)		(-29.0)	(4.62)
		Kirchhoff [1]	2.802	0.287	0.287	0.106	---	0.238
		CPT	(- 4.76)	(- 1.50)	(- 1.50)			(0.00)
		Exact [27]	2.942	0.289	0.289	---	0.238	----

Table 3
Comparison of deflection $\bar{w}$ at $(x=a/2, y=b/2, z=0)$, normal stresses $\bar{\sigma}_x$ at $(x=a/2,\ y=b/2,\ z=h/2)$, $\bar{\sigma}_y$ at $(x=a/2,\ y=b/2,\ z=h/2)$, in-plane shear stress $\bar{\tau}_{xy}$ $(x=0,\ y=0,\ z=h/2)$ and transverse shear stress $\bar{\tau}_{zx}$ at $\left(x=0, y=b/2,\ z=0\right)$ in square isotropic plate subjected to linearly varying load

a/b	*h/a*	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
1.0	0.1	Present	2.3125	0.1535	0.1535	0.0975	0.2522	0.2411
		(TSDT)	(-0.30)	(6.228)	(6.228)		(3.49)	(-1.148)
		Krishna Murty [16]	2.3199	0.1453	0.1453	0.0975	0.2454	0.240
		(HSDT)	(0.017)	(0.553)	(0.553)		(0.615)	(-1.44)
		Reddy [14]	2.3325	0.1445	0.1445	0.0995	0.2463	0.243
		(HSDT)	(0.560)	(0.00)	(0.00)		(0.984)	(-0.20)
		Mindlin [4]	2.3350	0.1435	0.1435	0.0975	0.1650	0.247
		FSDT	(0.668)	(-0.007)	(-0.007)		(-32.34)	(1.64)
		Kirchhoff [1]	2.2180	0.1435	0.1435	0.0975	---	0.247
		CPT	(-4.375)	(-0.007)	(-0.007)			(1.64)
		Exact [27]	2.3195	0.1445	0.1445	---	0.2439	---

Table 4
Comparison of deflection $\bar{w}$ at $(x=a/2, y=b/2, z=0)$, normal stresses $\bar{\sigma}_x$ at $(x=a/2,\ y=b/2,\ z=h/2)$, $\bar{\sigma}_y$ at $(x=a/2,\ y=b/2,\ z=h/2)$, in-plane shear stress $\bar{\tau}_{xy}$ $(x=0,\ y=0,\ z=h/2)$, and transverse shear stress $\bar{\tau}_{zx}$ at $(x=0, y=b/2,\ z=0)$ in rectangular isotropic plate subjected to linearly varying load

a/b	*h/a*	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
0.5	0.1	Present	5.6700	0.3194	0.1125	0.1385	0.350	(0.3335)
		(TSDT)	(-0.307)	(4.379)	(-12.81)		(3.240)	(-1.76)
		Krishna Murty [16]	5.6878	0.3067	0.155	0.1390	0.341	0.3375
		(HSDT)	(0.0017)	(0.228)	(10.32)		(0.441)	(-0.589)
		Reddy [14]	5.7100	0.3060	0.139	0.1400	0.3395	0.3388
		(HSDT)	(0.395)	(0.00)	(-1.06)		(0.00)	(-0.206)
		Mindlin [4]	5.7100	0.3048	0.1385	0.1380	0.2725	0.3433
		FSDT	(0.395)	(-0.39)	(-1.42)		(-19.73)	(1.104)
		Kirchhoff [1]	5.5300	0.3048	0.139	0.1385	0.343	0.3433
		CPT	(-2.772)	(-0.39)	(-1.06)		(1.03)	(1.104)
		Exact [27]	5.6875	0.3060	0.1405	---	0.3395	---

Table 4 shows comparison of deflection and stresses for the simply supported homogenous rectangular plate when subjected to linearly varying load for *a/b* = 0.5. The present theory gives the transverse displacement value closer to the exact value, however, theory of Krishna Murty yields more exact value compared to other refined theories. The in-plane normal stress $\bar{\sigma}_x$ when predicted by present theory is overestimated by 4.379 % and $\bar{\sigma}_y$ is underestimated by 12.81 %. The in-plane normal stress predicted by theory of Reddy is error free for *h*/*a* = 0.1. The in-plane shear stresses predicted by all the theories are in excellent agreement with each other. The transverse shear stress obtained by present theory is overestimated by 3.240 % when calculated by constitutive relation and underestimated by 1.76 % when obtained by equilibrium equations. In this case HSDT of Reddy yields the exact value transverse shear stress when obtained by constitutive relation and FSDT of Mindlin contains maximum error for this stress.

Example 4: Results for homogenous square plate when subjected to center concentrated load are shown in Table 5 for the *h/a* ratio 0.1. The present theory yields good results for deflection and stresses as compared to those of Reddy's and Krishna Murty's theories. Present theory underestimates deflection by 0.568 % and underestimates the in-plane normal stress by 4.024 %, whereas Krishna Murty's theory shows more error in in-plane stress and less error in deflection. Reddy's theory shows more error in deflection and transverse shear stress and less error in normal bending stress. Present theory overestimates transverse shear stress by 6 % which may be due to the effect of stress concentration when obtained by constitutive relation. At distance *x*=0.0113*a* away from the support present theory yields the exact value of transverse shear stress. The distribution of this shear stress through the thickness when obtained by equilibrium equation is shown in Fig. 2. This distribution shows that the maximum transverse shear stress occurs at *z*=± 0.34*h* instead of maximum value occurring at neutral plane when obtained by using equilibrium equations for *h/a*=0.1. It gives minimum value of this stress at the neutral plane. However, both these values are positive.

The through the thickness variation of normal bending stress is shown in Fig. 1 for *h*/*a* ratio 0.25. The variation of this stress under the concentrated load is non-linear through the thickness, which may be the effect of stress concentration, which can not be captured by the classical and first order shear deformation theories. The transverse shear stress ($\bar{\tau}_{zx}^{EE}$) variation through the thickness via equilibrium equations is shown in Fig. 3. The maximum value of shear stress obtained is 1.3729 (positive) which occurs at *z*=± 0.36*h* and at neutral plane (*z*=0) it shows minimum value with negative sign (0.1761). This anomalous behavior of shear stress is due to the effect of local stress concentration due to concentrated load. The distribution of shear stress $\bar{\tau}_{zx}^{EE}$ obtained by FSDT and CPT coincides with each other for both the *h*/*a* ratios as shown in Fig. 2 and Fig. 3.

Example 5: As seen from Table 6, for a plate subjected to line load, for *h*/*a*=0.1, deflection predicted by present theory is underestimated by 0.333 %, whereas Krishna Murty's theory shows minimum error and Reddy's theory overestimates it by 0.763 %. Also, the deflection predicted by FSDT of Mindlin is in excellent agreement with the exact solution for this loading case. Among the higher order theories, present theory shows less error in case of in-plane normal stress. However, value of this stress by theory of Reddy contains more error as compared to value of HSDT of Krishna Murty, FSDT of Mindlin and CPT of Kirchhoff.

Table 5
Comparison of deflection $\bar{w}$ at $(x=a/2, y=b/2, z=0)$, normal stress $\bar{\sigma}_x$ at $(x=a/2,\ y=b/2,\ z=h/2)$ and transverse shear stress $\bar{\tau}_{zx}$ at $(x=0, y=b/2,\ z=0)$ in square isotropic plate subjected to center concentrated load

a/b	*h/a*	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$
1.0	0.1	Present (TSDT)	14.0062 (-0.568)	2.4561 (-4.024)	0.7959 (6.00)
		Present (TSDT) at *x/a* = 0.0113	---	---	0.7359 (0.00)
		Krishna Murty [16] (HSDT)	14.0821 (-0.063)	2.3902 (-6.609)	0.7936 (7.840)
		Reddy [14] (HSDT)	14.4717 (2.701)	2.4956 (-2.481)	0.9174 (24.66)
		Mindlin [4] FSDT	14.2216 (0.926)	1.8672 (-27.03)	0.6672 (-9.335)
		Kirchhoff [1] CPT	12.6575 (-10.173)	1.8672 (-27.03)	---
		Exact [27]	14.0910	2.5591	0.7359

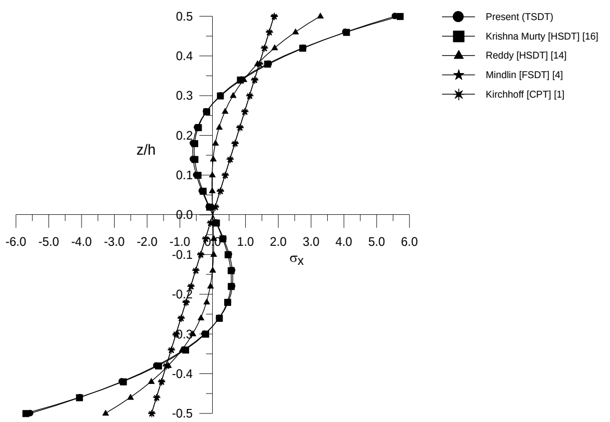

Fig. 1
Variation of in-plane normal stress at ($x = a/2$, $y = b/2$, z) through thickness of square plate subjected to center concentrated load for $h/a = 0.25$.

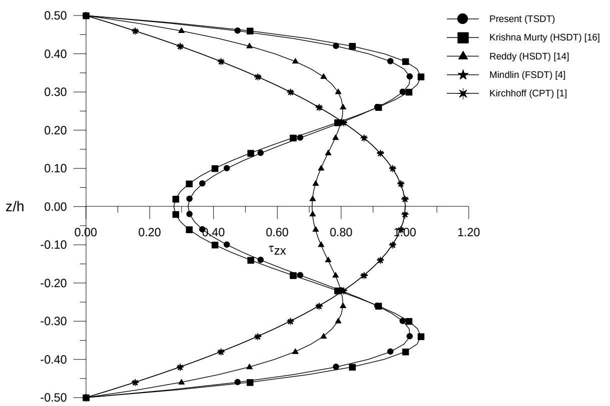

Fig. 2
Variation of transverse shear stress at ($x = 0$, $y = b/2$, z) through the thickness of square plate subjected to center concentrated load for $h/a = 0.1$.

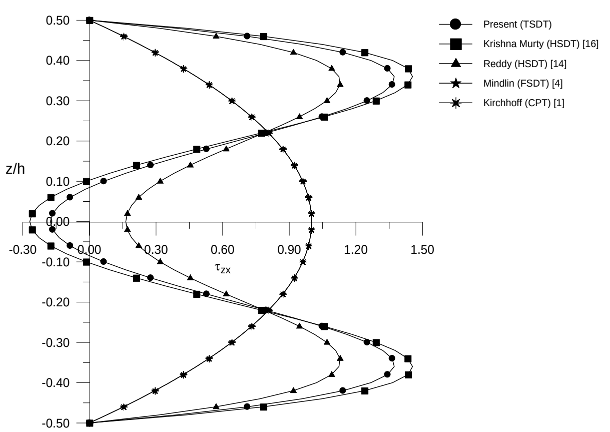

Fig. 3
Variation of transverse shear stress at ($x = 0$, $y = b/2$, z) through the thickness of square plate subjected to center concentrated load for $h/a = 0.25$.

Transverse shear stress when obtained by constitutive relations shows more error compared to the results of other higher order theories. The FSDT of Mindlin contains maximum error in the value of shear stress (31.11 %). The theory of Krishna Murty yields more accurate value of this stress when obtained by constitutive relation, whereas, Reddy's theory yields more accurate value of this stress when obtained by equilibrium equations. The FSDT and CPT yield identical results of transverse shear stress when obtained by integration of equilibrium equations.

Table 6
Comparison of deflection $\bar{w}$ at $(x=a/2, y=b/2, z=0)$, normal stress $\bar{\sigma}_x$ at $(x=a/2,\ y=b/2,\ z=h/2)$ and transverse shear stress $\bar{\tau}_{zx}$ at $(x=0,\ y=b/2,\ z=0)$ in square isotropic plate subjected to line load

a/b	*h/a*	Theory	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}^{CR}$	$\bar{\tau}_{zx}^{EE}$
1.0	0.1	Present	5.8862	0.6476	2.9980	2.7447
		(TSDT)	(-0.333)	(-4.229)	(3.357)	(-5.374)
		Krishna Murty[16]	5.9056	0.6294	2.921	2.7307
		(HSDT)	(-0.005)	(-6.921)	(0.703)	(-5.867)
		Reddy [14]	5.9510	0.4955	2.966	2.8879
		(HSDT)	(0.763)	(-26.72)	(2.254)	(-0.437)
		Mindlin [4]	5.8954	0.5115	1.998	2.9976
		FSDT	(-0.177)	(-24.35)	(-31.11)	(3.344)
		Kirchhoff [1]	5.6058	0.5115	---	2.9976
		CPT	(-5.081)	(-24.35)		(3.344)
		Exact [27]	5.9059	0.6762	2.9006	---

5 CONCLUSIONS

A displacement based, refined shear deformation theory includes both the effects of transverse shear and transverse normal deformations. The constitutive relations are satisfied in respect of in-plane stress and transverse shear stress. The transverse shear stresses satisfy shear stress free boundary conditions at the top and bottom faces of the plate. The theory obviates the need of shear correction factor. The governing differential equations and associated boundary conditions obtained are variationally consistent. The results of transverse displacement obtained by the present theory are in excellent agreement with those of exact theory due to inclusion of transverse normal strain effect as seen from the comparisons. The stresses obtained by present theory are in good agreement with the other higher order theories. The present theory is capable of predicting the effect of stress concentration due to concentrated load on stresses, the effect of which can not be captured by classical and first order shear deformation plate theories. This validates the efficacy and credibility of the present trigonometric shear deformation theory.

APPENDIX A

The stiffness coefficients appeared in Eqs. (7) through (10) are as follows

$$D_1=(\lambda+2G)\frac{h^3}{12},\quad D_2=(\lambda+2G)\frac{2h^3}{\pi^3},\quad D_3=\lambda\frac{2h^2}{\pi^2},\quad D_4=(\lambda+2G)\frac{h^3}{2\pi^2};$$
$$D_5=G\frac{h^3}{2\pi^2},\quad D_6=G\frac{h}{2},\quad D_7=(\lambda+G)\frac{h^3}{2\pi^2},\quad D_8=(\lambda+G)\frac{h^2}{2\pi};$$
$$D_9=(\lambda+G)\frac{h}{2},\quad D_{10}=(\lambda+4G)\frac{h^3}{12},\quad D_{11}=4G\frac{h^3}{\pi^3},\quad D_{12}=\lambda\frac{h^3}{12};$$
$$D_{13}=2\lambda\frac{h^3}{\pi^3},\quad D_{14}=\lambda\frac{h^3}{2\pi^2},\quad D_{15}=\lambda\frac{h^2}{2\pi},\quad D_{16}=G\frac{h^3}{2\pi^2},$$
$$D_{17}=G\frac{h}{2\pi},\quad D_{18}=4G\frac{h^3}{12}; \tag{A.1}$$

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