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An Exact Solution for Classic Coupled Thermoporoelasticity in Axisymmetric Cylinder

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ABSTRACT
In this paper, the classic coupled poro-thermoelasticity model of hollow and solid cylinders under radial symmetric loading condition is considered. A full analytical method is used and an exact unique solution of the classic coupled equations is presented. The thermal and pressure boundary conditions, the body force, the heat source and the injected volume rate per unit volume of a distribute water source are considered in the most general forms and no limiting assumption is used. This generality allows simulation of several of the applicable problems.

Keywords: Coupled thermoporoelasticity; Hollow cylinder; Exact solution

1 INTRODUCTION

COUPLED thermal and poro-mechanical processes play an important role in a number of problems of interest in the geomechanics such as stability of boreholes and permeability enhancement in geothermal reservoirs or high temperature petroleum bearing formations. A thermoporoelastic approach combines the theory of heat conduction with poroelastic constitutive equations and coupling the temperature field with the stresses and pore pressure. There is a limited number of papers that present the closed-form or analytical solution for the coupled porothermoelasticity problems. Bai [1] investigated the response of saturated porous media subjected to local thermal loading on the surface of semi-infinite space. He used the numerical integral methods for calculating the unsteady temperature, pore pressure, and displacement fields. This author also studied the fluctuation responses of saturated porous media subjected to cyclic thermal loading [2]. In the mentioned paper, an analytical solution was deduced using the Laplace transform and the Gauss-Legendre method and Laplace transform inversion. Droujinine [3] investigated dispersion and attenuation of body waves in a wide range of materials representing realistic rock structures. He used the time-domain asymptotic ray theory to a new generalized coordinate-free wave equation with an arbitrary tensor relaxation function. Bai and Li [4] found a solution for cylindrical cavity in saturated thermoporoelastic medium using Laplace transform and numerical Laplace transform inversion.

The numbers of papers that present the closed-form or analytical solutions for the coupled thermoelasticity problems are limited. Hetnarski [5] found the solution of the coupled thermo elasticity in the form of a series function. Hetnarski and Ignaczak presented a study of the one-dimensional thermo elastic waves produced by an instantaneous plane source of heat in homogeneous isotropic infinite and semi-infinite bodies of the Green-Lindsay type [6]. These authors also presented an analysis for laser-induced waves propagating in an absorbing thermo elastic semi-space of the Green-Lindsay type [7]. Georgiadis and Lykotrafitis obtained a three-dimensional transient thermo elastic solution for Rayleigh-type disturbances propagating on the surface of a half-space [8]. Wagner [9] presented the fundamental matrix of a system of partial differential operators that governs the diffusion of heat and

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the strains in elastic media. This method can be used to predict the temperature distribution and the strains by an instantaneous point heat, point source of heat, or by a suddenly applied dilate force.

Here, a full analytical method is used to obtain the response of the governing equations. An exact solution is presented. The method of solution is based on the Fourier's expansion and eigenfunction methods which are traditional and routine methods in solving the partial differential equations. Since the coefficients of equations are not functions of the time variable (t), an exponential form is considered for the general solution matched with the physical wave properties of thermal and mechanical waves. For the particular solution, that is the response to mechanical and thermal shocks, the eigenfuncion method and Laplace transformation is used. This work is following the previous work which was presented for radially symmetric cylindrical coordinates [10].

2 GOVERNING EQUATIONS

A short hollow cylinder with inner and outer radius r_i and r_o, respectively, and length l made of isotropic material subjected to radial-symmetric mechanical, thermal, and pressure shocks is considered. The classic theory of porothermoelasticity for wave propagation is taken to allow coupling between deformation, thermal energy and pressure fields and to describe the physical behavior of the elastic domain to mechanical, thermal and pressure shock loads. Navier equation in terms of the displacement components is obtained as [4]

$$u,_{rr}+\frac{1}{r}u,_{r}-\frac{1}{r^2}u+\frac{1}{2}\left(\frac{1-2\nu}{1-\nu}\right)u,_{zz}+\frac{1}{2}\frac{1}{(1-\nu)}w,_{rz}-\frac{(1+\nu)}{(1-\nu)}\alpha T,_{r}$$
$$-\alpha\frac{(1+\nu)(1-2\nu)}{(1-\nu)^2E}p,_{r}-\beta\frac{(1+\nu)(1-2\nu)}{(1-\nu)^2E}T,_{r}-\rho\frac{(1+\nu)(1-2\nu)}{(1-\nu)^2E}\ddot{u}=-\frac{(1+\nu)(1-2\nu)}{(1-\nu)^2E}F(r,z,t) \tag{1}$$

$$w,_{rr}+\frac{1}{r}w,_{r}+2\frac{(1-\nu)}{(1-2\nu)}w,_{zz}+\frac{1}{(1-2\nu)}u,_{zr}+\frac{1}{(1-2\nu)}\frac{1}{r}u,_{z}-2\frac{(1+\nu)}{(1-2\nu)}\alpha T,_{z}$$
$$-2\alpha\frac{(1+\nu)}{(1-\nu)E}p,_{z}-2\beta\frac{(1+\nu)}{(1-\nu)E}T,_{z}-2\rho\frac{(1+\nu)}{(1-\nu)E}\ddot{w}=-2\frac{(1+\nu)}{(1-\nu)E}R(r,z,t) \tag{2}$$

Heat conduction equation in radial-symmetric direction with the mechanical coupling term is

$$T,_{rr}+\frac{1}{r}T,_{r}+T,_{zz}-Z\frac{T_\circ}{K}\dot{T}+Y\frac{T_\circ}{K}\dot{p}-\beta\frac{T_\circ}{K}(\dot{u},_{r}+\frac{1}{r}\dot{u}+\dot{w},_{z})=-\frac{1}{K}Q(r,z,t) \tag{3}$$

According to Darcy's law and continuity condition of seepage, the equation of mass conservation can be written as

$$p,_{rr}+\frac{1}{r}p,_{r}+p,_{zz}-\alpha_p\frac{\gamma_w}{k}\dot{p}+Y\frac{\gamma_w}{k}\dot{T}-\alpha\frac{\gamma_w}{k}(\dot{u},_{r}+\frac{1}{r}\dot{u}+\dot{w},_{z})=-\frac{\gamma_w}{k}W(r,z,t) \tag{4}$$

where (,) denotes partial derivative, u is the displacement component in the radial direction, p is the pore pressure, ρ is bulk mass density, $\alpha=1-C_s/C$ is the Biot's coefficient, $C_s=3(1-2\nu_s)E_s$ is the coefficient of volumetric compression of the solid grains, with E_s and ν_s being the elastic modulus and Poisson's ratio of solid grains and $C=3(1-2\nu)E$ is the coefficient of volumetric compression of solid skeleton, with E and ν being the elastic modulus and Poisson's ratio of solid skeleton, $T_\circ$ is initial reference temperature, $\beta=3\alpha_s/C$ is the thermal expansion factor, α_s is the coefficient of linear thermal expansion of solid grains, $Y=3(n\alpha_w+(\alpha-n)\alpha_s)$ and $\alpha_p=n(C_w-C_s)+\alpha C_s$ are coupling parameters, α_w and C_w are the coefficients of linear thermal expansion and volumetric compression of pure water, n is the porosity, k is the hydraulic conductivity, γ_w is the unit of pore water and $Z=((1-n)\rho_s c_s+n\rho_w c_w)/T_\circ-3\beta\alpha_s$ is coupling parameter, ρ_w and ρ_s are the densities of pore water and solid grains and c_w and c_s are the heat capacities of pore water and solid grains and K is the coefficient of heat conductivity. Here, $F(r,z,t)$, $R(r,z,t)$, $Q(r,z,t)$ and $W(r,z,t)$ are the body forces, heat generation source and the injected volume rate per unit volume of a distribute water source, respectively. The mechanical, thermal and pressure boundary conditions for inner and outer surface of cylinder are

$$\begin{aligned}
&C_{11}u(r_i,z,t)+C_{12}u_{,r}(r_i,z,t)+C_{13}T(r_i,z,t)+C_{14}p(r_i,z,t)=f_1(t)\\
&C_{21}u(r_o,z,t)+C_{22}u_{,r}(r_o,z,t)+C_{23}T(r_o,z,t)+C_{24}p(r_o,z,t)=f_2(t)\\
&C_{31}w(r_i,z,t)+C_{32}w_{,r}(r_i,z,t)+C_{33}T(r_i,z,t)+C_{34}p(r_i,z,t)=f_3(t)\\
&C_{41}w(r_o,z,t)+C_{42}w_{,r}(r_o,z,t)+C_{43}T(r_o,z,t)+C_{44}p(r_o,z,t)=f_4(t)\\
&C_{51}T(r_i,z,t)+C_{52}T_{,r}(r_i,z,t)=f_5(t)\\
&C_{61}T(r_o,z,t)+C_{62}T_{,r}(r_o,z,t)=f_6(t)\\
&C_{71}p(r_i,z,t)=f_7(t)\\
&C_{81}p(r_o,z,t)=f_8(t)
\end{aligned} \tag{5}$$

where C_{ij} are the mechanical, thermal and pressure coefficients, and which by assigning different values for them, different types of mechanical, thermal, and pressure boundary conditions may be obtained. These boundary conditions include the displacement, strain, stress, specified temperature, convection, pressure, and heat flux condition. The boundary conditions at the ends of cylinder are considered in simply support as [11]

$$\begin{aligned}
&u(r,0,t)=0, \quad u(r,l,t)=0, \quad w_z(r,0,t)=0, \quad w_z(r,l,0)=0\\
&T(r,0,t)=0, \quad T(r,l,t)=0, \quad p(r,0,t)=0, \quad p(r,l,t)=0
\end{aligned} \tag{6}$$

The initial boundary conditions are assumed in the following general form

$$\begin{aligned}
&u(r,z,0)=f_9(r,z), \quad u_{,t}(r,z,0)=f_{10}(r,z), \quad w(r,z,0)=f_{11}(r,z), \quad w_{,t}(r,z,0)=f_{12}(r,z)\\
&T(r,z,0)=f_{13}(r,z), \quad p(r,z,0)=f_{14}(r,z)
\end{aligned} \tag{7}$$

3 SOLUTION

Eqs. (1) to (4) constitute a system of non-homogeneous partial differential equations with non-constant coefficients (functions of the radius variable *r* only) has general and particular solutions.

3.1 *General solution with homogeneous boundary conditions*

A form of solution can be suitable for Eqs. (1) to (4) and the boundary conditions (6) may be assumed for the general solution as

$$\begin{aligned}
u(r,z,t)&=\sum_{n=1}^{\infty} U(r)e^{\lambda t}\sin(\zeta_n z)\\
w(r,z,t)&=\sum_{n=1}^{\infty} W(r)e^{\lambda t}\cos(\zeta_n z)\\
T(r,z,t)&=\sum_{n=1}^{\infty} \theta(r)e^{\lambda t}\sin(\zeta_n z)\\
p(r,z,t)&=\sum_{n=1}^{\infty} P(r)e^{\lambda t}\sin(\zeta_n z)
\end{aligned} \tag{8}$$

where ζ_n is $n\pi/l$. Substituting Eq. (8) into homogeneous parts of Eqs. (1) to (4), yields

$$U''+\frac{1}{r}U'-\frac{1}{r^2}U+d_1{\zeta_n}^2U+d_2\zeta_nW'+d_3P'+d_4\theta'+d_5\lambda^2U=0$$
$$W''+\frac{1}{r}W'+d_6{\zeta_n}^2W+d_7\zeta_nU'+d_8\zeta_n\frac{1}{r}U+d_9\zeta_nP+d_{10}\zeta_n\theta+d_{11}\lambda^2W=0$$
$$\theta''+\frac{1}{r}\theta'-{\zeta_n}^2\theta+d_{12}\lambda\theta+d_{13}\lambda P+d_{14}\lambda(U'+\frac{1}{r}U-W\zeta_n)=0$$
$$P''+\frac{1}{r}P'-{\zeta_n}^2P+d_{15}\lambda P+d_{16}\lambda\theta+d_{17}\lambda(U'+\frac{1}{r}U-W\zeta_n)=0 \tag{9}$$

Eqs. (9) are system of ordinary equations with non-constant coefficients and are related to Bessel differential equations and the solution should be in Bessel functions. The first solutions of U_1, W_1, θ_1 and P_1 are considered as

$$U_1(r)=A_1J_1(\beta r),\qquad W_1(r)=B_1J_0(\beta r),\qquad \theta_1(r)=C_1J_0(\beta r),\qquad P_1(r)=D_1J_0(\beta r) \tag{10}$$

Substituting Eqs. (10) into Eqs. (19) yields

$$\{(-\beta^2+d_1{\zeta_n}^2+d_5\lambda^2)A_1-d_2\zeta_n\beta B_1-d_4\beta C_1-d_3\beta D_1\}\,J_0(\beta r)=0$$
$$\{-d_7\zeta_n\beta A_1+(\beta^2-d_6{\zeta_n}^2-d_{11}\lambda^2)B_1-d_{10}\zeta_nC_1-d_9\zeta_nD_1\}\,J_1(\beta r)=0$$
$$\{-d_{14}\lambda\beta A_1+\zeta_nd_{14}\lambda B_1+(\beta^2+{\zeta_n}^2-d_{12}\lambda)C_1-d_{13}\lambda D_1\}\,J_1(\beta r)=0$$
$$\{-d_{17}\lambda\beta A_1+d_{17}\zeta_n\lambda B_1+d_{16}\lambda C_1+(\beta^2+{\zeta_n}^2-d_{15}\lambda)D_1\}\,J_1(\beta r)=0 \tag{11}$$

Eqs. (11) show that U_1, W_1, θ_1 and P_1 can be the solutions of Eqs. (9), if and only if

$$\begin{bmatrix}(-\beta^2+d_1{\zeta_n}^2+d_5\lambda^2) & -d_2\zeta_n\beta & -d_4\beta & -d_3\beta\\ -d_7\zeta_n\beta & (\beta^2-d_6{\zeta_n}^2-d_{11}\lambda^2) & -d_{10}\zeta_n & -d_9\zeta_n\\ -d_{14}\lambda\beta & d_{14}\lambda\zeta_n & (\beta^2+{\zeta_n}^2-d_{12}\lambda) & -d_{13}\lambda\\ -d_{17}\lambda\beta & d_{17}\lambda\zeta_n & d_{16}\lambda & (\beta^2+{\zeta_n}^2-d_{15}\lambda)\end{bmatrix}\begin{Bmatrix}A_1\\B_1\\C_1\\D_1\end{Bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{12}$$

The non-trivial solution of Eq. (12) is obtained by equating the determinant of this equation to zero and brings the first characteristic equation. The second solutions of U_2, W_2, θ_2 and P_2 are considered as

$$U_2(r)=[A_2J_1(\beta r)+A_3rJ_2(\beta r)]$$
$$W_2(r)=[B_2J_0(\beta r)+B_3rJ_1(\beta r)]$$
$$\theta_2(r)=[C_2J_0(\beta r)+C_3rJ_1(\beta r)]$$
$$P_2(r)=[D_2J_0(\beta r)+D_3rJ_1(\beta r)] \tag{13}$$

The expressions for U_2, W_2, θ_2 and P_2 can be the solutions of Eqs. (9), if and only if

$$\begin{bmatrix}(-\beta^2+d_1{\zeta_n}^2+d_5\lambda^2) & -d_2\zeta_n\beta & -d_4\beta & -d_3\beta\\ -d_7\zeta_n\beta & (\beta^2-d_6{\zeta_n}^2-d_{11}\lambda^2) & -d_{10}\zeta_n & -d_9\zeta_n\\ -d_{14}\lambda\beta & d_{14}\lambda\zeta_n & (\beta^2+{\zeta_n}^2-d_{12}\lambda) & -d_{13}\lambda\\ -d_{17}\lambda\beta & d_{17}\lambda\zeta_n & d_{16}\lambda & (\beta^2+{\zeta_n}^2-d_{15}\lambda)\end{bmatrix}\begin{Bmatrix}A_3\\B_3\\C_3\\D_3\end{Bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{14}$$

$$(d_5\lambda^2-\beta^2+d_1{\zeta_n}^2)A_2+(d_5\lambda^2+d_1{\zeta_n}^2)\frac{2}{\beta}A_3-d_2\zeta_n\beta B_2-d_4\beta C_2-d_3D_2\beta=0 \tag{15}$$

$$d_7\zeta_n\beta A_2+(-\beta^2+d_{11}\lambda^2+d_6\zeta_n{}^2)B_2+2\beta B_3+d_{10}\zeta_n C_2+d_9\zeta_n D_2=0 \tag{16}$$

$$-C_2\beta^2+2\beta C_3-\zeta_n{}^2C_2+d_{12}\lambda C_2+d_{13}\lambda D_2-\zeta_n d_{14}\lambda B_2+d_{14}\lambda A_2\beta=0 \tag{17}$$

$$d_{17}\lambda\beta A_2-d_{17}\lambda\zeta_n B_2+d_{16}\lambda C_2+(-\zeta_n{}^2-\beta^2+d_{15}\lambda)D_2+2\beta D_3=0 \tag{18}$$

The non-trivial solution of Eqs. (14) is obtained by equating the determinant of coefficients matrix of this equation to zero as second characteristic equation and it is completely similar to the first characteristic equation. Eqs. (15) to (18) give the relations between A_2,B_2,B_3,C_2,C_3,D_2 and D_3. They play as the balancing ratios that make Eq. (13) to be the second solution of the system of Eqs. (9). The third solution of the system of the ordinary differential equations with non-constant coefficients (9) must be considered as

$$\begin{aligned}U_3(r)&=[A_4J_1(\beta r)+A_5rJ_2(\beta r)+A_6r^2J_3(\beta r)]\\W_3(r)&=[B_4J_0(\beta r)+B_5rJ_1(\beta r)+B_6r^2J_2(\beta r)]\\\theta_3(r)&=[C_4J_0(\beta r)+C_5rJ_1(\beta r)+C_6r^2J_2(\beta r)]\\P_3(r)&=[D_4J_0(\beta r)+D_5rJ_1(\beta r)+D_6r^2J_2(\beta r)]\end{aligned} \tag{19}$$

The expressions for U_3, W_3, θ_3 and P_3 can be solutions of Eq. (9), if and only if

$$\begin{bmatrix}(-\beta^2+d_1\zeta_n{}^2+d_5\lambda^2) & -d_2\zeta_n\beta & -d_4\beta & -d_3\beta\\ -d_7\zeta_n\beta & (\beta^2-d_6\zeta_n{}^2-d_{11}\lambda^2) & -d_{10}\zeta_n & -d_9\zeta_n\\ -d_{14}\lambda\beta & d_{14}\lambda\zeta_n & (\beta^2+\zeta_n{}^2-d_{12}\lambda) & -d_{13}\lambda\\ -d_{17}\lambda\beta & d_{17}\lambda\zeta_n & d_{16}\lambda & (\beta^2+\zeta_n{}^2-d_{15}\lambda)\end{bmatrix}\begin{Bmatrix}A_6\\B_6\\C_6\\D_6\end{Bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{20}$$

$$(d_5\lambda^2-\beta^2+d_1\zeta_n{}^2)A_4+(d_5\lambda^2+d_1\zeta_n{}^2)\frac{2}{\beta}A_5+(d_5\lambda^2+d_1\zeta_n{}^2)\frac{8}{\beta^2}A_6-d_2\zeta_n\beta B_4-d_4\beta C_4-d_3\beta D_4=0 \tag{21}$$

$$(-d_5\lambda^2+\beta^2-d_1\zeta_n{}^2)A_5-(d_5\lambda^2+d_1\zeta_n{}^2)\frac{4}{\beta}A_6+d_2\zeta_n\beta B_5+d_4\beta C_5+d_3\beta D_5=0 \tag{22}$$

$$d_7\zeta_n\beta A_4+(-\beta^2+d_{11}\lambda^2+d_6\zeta_n{}^2)B_4+2\beta B_5+d_{10}\zeta_n C_4+d_9\zeta_n D_4=0 \tag{23}$$

$$\begin{aligned}&d_7\zeta_n A_5\beta+2d_7\zeta_n A_6+(-\beta^2+d_6\zeta_n{}^2+d_{11}\lambda^2)B_5+\left(2\beta+d_6\zeta_n{}^2\frac{2}{\beta}+d_{11}\lambda^2\frac{2}{\beta}\right)B_6\\&\qquad+d_{10}\zeta_n C_5+d_{10}\zeta_n C_6\frac{2}{\beta}+d_9\zeta_n D_5+d_9\zeta_n D_6\frac{2}{\beta}=0\end{aligned} \tag{24}$$

$$-\beta^2C_4+2\beta C_5-\zeta_n{}^2C_4+d_{12}\lambda C_4+d_{13}\lambda D_4-\zeta_n d_{14}\lambda B_4+d_{14}\lambda\beta A_4=0 \tag{25}$$

$$\begin{aligned}&d_{14}\lambda\beta A_5+2d_{14}\lambda A_6-\zeta_n d_{14}\lambda B_5-\frac{2}{\beta}\zeta_n d_{14}\lambda B_6+(-\beta^2-\zeta_n{}^2+d_{12}\lambda)C_5\\&\qquad+\left(\beta-\zeta_n{}^2\frac{2}{\beta}+\beta+d_{12}\lambda\frac{2}{\beta}\right)C_6+d_{13}\lambda D_5+d_{13}\lambda\frac{2}{\beta}D_6=0\end{aligned} \tag{26}$$

$$d_{17}\lambda\beta A_4-d_{17}\lambda\zeta_n B_4+d_{16}\lambda C_4+(-\zeta_n{}^2-\beta^2+d_{15}\lambda)D_4+2\beta D_5=0 \tag{27}$$

$$\begin{aligned}&d_{17}\lambda A_5\beta+2d_{17}\lambda A_6-d_{17}\zeta_n\lambda B_5-d_{17}\zeta_n\lambda B_6\frac{2}{\beta}+d_{16}\lambda C_5+d_{16}\lambda C_6\\&\qquad+\frac{2}{\beta}\left(-\beta^2+d_{15}\lambda-\zeta_n{}^2\right)D_5+\left(d_{15}\lambda\frac{2}{\beta}+2\beta-\zeta_n{}^2\frac{2}{\beta}\right)D_6=0\end{aligned} \tag{28}$$

The non-trivial solution of Eqs. (20) is obtained by equating the determinant to zero of this equation and brings the third characteristic equation and it is as same as first and second characteristic equations. Eqs.(21) to (28) gives the relation between $A_4, A_5, B_4, B_5, B_6, C_4, C_5, C_6, D_4, D_5$ and D_6. They play as the balancing ratios that help Eq. (19) to be the third solution of the system of equations (9). The fourth solutions of U_1, W_1, θ_1 and P_1 are considered as

$$\begin{aligned}
U_4(r) &= [A_7 J_1(\beta r) + A_8 r J_2(\beta r) + A_9 r^2 J_3(\beta r) + A_{10} r^3 J_4(\beta r)] \\
W_4(r) &= [B_7 J_0(\beta r) + B_8 r J_1(\beta r) + B_9 r^2 J_2(\beta r) + B_{10} r^3 J_3(\beta r)] \\
\theta_4(r) &= [C_7 J_0(\beta r) + C_8 r J_1(\beta r) + C_9 r^2 J_2(\beta r) + C_{10} r^3 J_3(\beta r)] \\
P_4(r) &= [D_7 J_0(\beta r) + D_8 r J_1(\beta r) + D_9 r^2 J_2(\beta r) + D_{10} r^3 J_3(\beta r)]
\end{aligned} \tag{29}$$

The expressions for U_4, W_4, θ_4 and P_4 can be solutions of Eq. (9), if and only if

$$\begin{bmatrix} (-\beta^2 + d_1\zeta_n^2 + d_5\lambda^2) & -d_2\zeta_n\beta & -d_4\beta & -d_3\beta \\ -d_7\zeta_n\beta & (\beta^2 - d_6\zeta_n^2 - d_{11}\lambda^2) & -d_{10}\zeta_n & -d_9\zeta_n \\ -d_{14}\lambda\beta & d_{14}\lambda\zeta_n & (\beta^2 + \zeta_n^2 - d_{12}\lambda) & -d_{13}\lambda \\ -d_{17}\lambda\beta & d_{17}\lambda\zeta_n & d_{16}\lambda & (\beta^2 + \zeta_n^2 - d_{15}\lambda) \end{bmatrix} \begin{Bmatrix} A_{10} \\ B_{10} \\ C_{10} \\ D_{10} \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

$$\begin{aligned}
&(d_5\lambda^2 - \beta^2 + d_1\zeta_n^2)A_7 + (d_5\lambda^2 + d_1\zeta_n^2)\frac{2}{\beta}A_8 + (d_5\lambda^2 + d_1\zeta_n^2)\frac{8}{\beta^2}A_9 \\
&\qquad + (d_5\lambda^2 + d_1\zeta_n^2)\frac{48}{\beta^3}A_{10} - d_2\zeta_n B_7\beta - d_4 C_7\beta - d_3 D_7\beta = 0
\end{aligned} \tag{31}$$

$$\begin{aligned}
&(-d_5\lambda^2 + \beta^2 - d_1\zeta_n^2)A_8 - (d_5\lambda^2 + d_1\zeta_n^2)\frac{4}{\beta}A_9 - (d_5\lambda^2 + d_1\zeta_n^2)\frac{24}{\beta^2}A_{10} \\
&\qquad + d_2\zeta_n B_8\beta + d_4 C_8\beta + d_3 D_8\beta = 0
\end{aligned} \tag{32}$$

$$\begin{aligned}
&(-d_5\lambda^2 + \beta^2 - d_1\zeta_n^2)A_9 + \left(-d_5\lambda^2\frac{8}{\beta} + 2\beta - d_1\zeta_n^2\frac{8}{\beta}\right)A_{10} + d_2\zeta_n\beta B_9 \\
&\qquad + 2d_2\zeta_n B_{10} + d_4 C_9\beta + 2d_4 C_{10} + d_3\beta D_9 + 2d_3 D_{10} = 0
\end{aligned} \tag{33}$$

$$\left\{d_7\zeta_n\beta A_7 + (-\beta^2 + d_{11}\lambda^2 + d_6\zeta_n^2)B_7 + 2\beta B_8 - B_{10}r^2\beta + d_{10}\zeta_n C_7 + d_9\zeta_n D_7\right\} = 0 \tag{34}$$

$$\begin{aligned}
&d_7\zeta_n A_8\beta + 2d_7\zeta_n A_9 + \frac{8}{\beta}d_7\zeta_n A_{10} + (-\beta^2 + d_6\zeta_n^2 + d_{11}\lambda^2)B_8 + \left(2\beta + d_6\zeta_n^2\frac{2}{\beta} + d_{11}\lambda^2\frac{2}{\beta}\right)B_9 \\
&\qquad + \left(4 + d_6\zeta_n^2\frac{8}{\beta^2} + d_{11}\lambda^2\frac{8}{\beta^2}\right)B_{10} + d_{10}\zeta_n C_8 + d_{10}\zeta_n C_9\frac{2}{\beta} + d_{10}\zeta_n C_{10}\frac{8}{\beta^2} + d_9\zeta_n D_8 \\
&\qquad + d_9\zeta_n D_9\frac{2}{\beta} + d_9\zeta_n D_{10}\frac{8}{\beta^2} = 0
\end{aligned} \tag{35}$$

$$\begin{aligned}
&-d_7\zeta_n\beta A_9 + (-5d_7\zeta_n + d_8\zeta_n)A_{10} + (B_9\beta^2 - d_{11}\lambda^2 B_9 - d_6\zeta_n^2)B_9 - \left(\beta + d_{11}\lambda^2\frac{4}{\beta} + d_6\zeta_n^2\frac{4}{\beta}\right)B_{10} \\
&\qquad - d_{10}\zeta_n\left\{C_9 + C_{10}\frac{4}{\beta}\right\} - d_9\zeta_n\left\{D_9 + D_{10}\frac{4}{\beta}\right\} = 0
\end{aligned} \tag{36}$$

$$\left\{-C_7\beta^2 + 2C_8\beta - \zeta_n^2 C_7 + d_{12}\lambda C_7 + d_{13}\lambda D_7 - \zeta_n d_{14}\lambda B_7 + d_{14}\lambda A_7\beta\right\} = 0 \tag{37}$$

$$d_{14}\lambda A_8\beta+2d_{14}\lambda A_9+d_{14}\lambda\frac{8}{\beta}A_{10}-B_8\zeta_n d_{14}\lambda-B_9\frac{2}{\beta}\zeta_n d_{14}\lambda-B_{10}\frac{8}{\beta^2}\zeta_n d_{14}\lambda$$
$$+(-\beta^2-\zeta_n{}^2+d_{12}\lambda)C_8+\left(\beta-\zeta_n{}^2\frac{2}{\beta}+\beta+d_{12}\lambda\frac{2}{\beta}\right)C_9+\left(4-\zeta_n{}^2\frac{8}{\beta^2}+d_{12}\lambda\frac{8}{\beta^2}\right)C_{10}$$
$$+d_{13}\lambda D_8+d_{13}\lambda\frac{2}{\beta}D_9+d_{13}\lambda\frac{8}{\beta^2}D_{10}=0 \tag{38}$$

$$-d_{14}\lambda\beta A_9-4d_{14}\lambda A_{10}+\zeta_n d_{14}\lambda B_9+\zeta_n d_{14}\lambda B_{10}\frac{4}{\beta}+(\beta^2+\zeta_n{}^2-d_{12}\lambda)C_9$$
$$-\left(2\beta+d_{12}\lambda\frac{4}{\beta}+\zeta_n{}^2\frac{4}{\beta}\right)C_{10}-d_{13}\lambda D_9-d_{13}\lambda D_{10}\frac{4}{\beta}=0 \tag{39}$$

$$\left\{d_{17}\lambda\beta A_7-d_{17}\lambda\zeta_n B_7+d_{16}\lambda C_7+(-\zeta_n{}^2-\beta^2+d_{15}\lambda)D_7+2\beta D_8\right\}=0 \tag{40}$$

$$d_{17}\lambda A_8\beta+2d_{17}\lambda A_9+d_{17}\lambda\frac{8}{\beta}A_{10}-d_{17}\zeta_n\lambda B_8-d_{17}\zeta_n\lambda B_9\frac{2}{\beta}-d_{17}\zeta_n\lambda B_{10}\frac{8}{\beta^2}+d_{16}\lambda C_8$$
$$+d_{16}\lambda C_9\frac{2}{\beta}+d_{16}\lambda C_{10}\frac{8}{\beta^2}+(-\beta^2+d_{15}\lambda-\zeta_n{}^2)D_8$$
$$+\left(d_{15}\lambda\frac{2}{\beta}+2\beta-\zeta_n{}^2\frac{2}{\beta}\right)D_9+\left(4+d_{15}\lambda\frac{8}{\beta^2}-\zeta_n{}^2\frac{8}{\beta^2}\right)D_{10}=0 \tag{41}$$

$$-4d_{17}\lambda A_{10}-d_{17}\lambda A_9\beta+d_{17}\zeta_n\lambda B_9+d_{17}\zeta_n\lambda B_{10}\frac{4}{\beta}-d_{16}\lambda C_9-d_{16}\lambda C_{10}\frac{4}{\beta}$$
$$+\left(-\beta+\zeta_n{}^2\frac{4}{\beta}-\beta-d_{15}\lambda\frac{4}{\beta}\right)D_{10}+(\zeta_n{}^2+\beta^2-d_{15}\lambda)D_9=0 \tag{42}$$

The non-trivial solution of Eqs. (30) is obtained by equating the determinant of coefficients matrix of this equation to zero as fourth characteristic equation and it is completely similar to the first, second and third characteristic equations. This equality is interesting as it prevents mathematical dilemma and complexity and a single value for the eigenvalue β simultaneously satisfies three characteristic equations. Eqs. (31) to (42) gives the relation between $A_7, A_8, A_9, B_7, B_8, B_9, B_{10}, C_7, C_8, C_9, C_{10}, D_7, D_8, D_9$ and D_{10} and they play as the balancing ratios that help Eq. (29) to be the third solution of the system of Eqs. (9). Therefore, the complete general solutions for the solid cylinder are

$$U^g(r)=A_1J_1(\beta r)+A_3[\zeta_1J_1(\beta r)+rJ_2(\beta r)]+A_6[\zeta_2J_1(\beta r)+\zeta_3rJ_2(\beta r)+r^2J_3(\beta r)]$$
$$+A_{10}[\zeta_4J_1(\beta r)+\zeta_5rJ_2(\beta r)+\zeta_6r^2J_3(\beta r)+r^3J_4(\beta r)]$$
$$W^g(r)=A_1\zeta_7J_0(\beta r)+A_3[\zeta_8J_0(\beta r)+\zeta_9rJ_1(\beta r)]+A_6[\zeta_{10}J_0(\beta r)+\zeta_{11}rJ_1(\beta r)+\zeta_{12}r^2J_2(\beta r)]$$
$$+A_{10}[\zeta_{13}J_0(\beta r)+\zeta_{14}rJ_1(\beta r)+\zeta_{15}r^2J_2(\beta r)+\zeta_{16}r^3J_3(\beta r)]$$
$$\theta^g(r)=A_1\zeta_{17}J_0(\beta r)+A_3[\zeta_{18}J_0(\beta r)+\zeta_{19}rJ_1(\beta r)]+A_6[\zeta_{20}J_0(\beta r)+\zeta_{21}rJ_1(\beta r)+\zeta_{22}r^2J_2(\beta r)]$$
$$+A_{10}[\zeta_{23}J_0(\beta r)+\zeta_{24}rJ_1(\beta r)+\zeta_{25}r^2J_2(\beta r)+\zeta_{26}r^3J_3(\beta r)]$$
$$P^g(r)=A_1\zeta_{27}J_0(\beta r)+A_3[\zeta_{28}J_0(\beta r)+\zeta_{29}rJ_1(\beta r)]+A_6[\zeta_{30}J_0(\beta r)+\zeta_{31}rJ_1(\beta r)+\zeta_{32}r^2J_2(\beta r)]$$
$$+A_{10}[\zeta_{33}J_0(\beta r)+\zeta_{34}rJ_1(\beta r)+\zeta_{35}r^2J_2(\beta r)+\zeta_{36}r^3J_3(\beta r)] \tag{43}$$

where ζ_1 to ζ_{36} are ratios obtained from Eqs (31) to (42), (21) to (28), (15) to (18) and(12). Substituting U^g, W^g, θ^g and P^g in the homogeneous form of the boundary conditions (5), three linear algebraic equations are obtained. They are the coefficients depending on λ and β. Setting the determinant of the coefficients equal to zero, the fifth characteristic equation is obtained. Simultaneous solution of this equation and Eq. (15) results into infinite number of two eigenvalues β_n and λ_n. Therefore, U^g, W^g, θ^g and P^g are rewritten as

$$\begin{aligned} U^g(r) &= A_1[J_1(\beta r)+\zeta_{37}[\zeta_1 J_1(\beta r)+rJ_2(\beta r)]+\zeta_{38}[\zeta_2 J_1(\beta r)+\zeta_3 rJ_2(\beta r)+r^2J_3(\beta r)] \\ &\quad +\zeta_{39}[\zeta_4 J_1(\beta r)+\zeta_5 rJ_2(\beta r)+\zeta_6 r^2J_3(\beta r)+r^3J_4(\beta r)]] \\ W^g(r) &= A_1[\zeta_7 J_0(\beta r)+\zeta_{37}[\zeta_8 J_0(\beta r)+\zeta_9 rJ_1(\beta r)]+\zeta_{38}[\zeta_{10} J_0(\beta r)+\zeta_{11} rJ_1(\beta r)+\zeta_{12} r^2J_2(\beta r)] \\ &\quad +\zeta_{39}[\zeta_{13} J_0(\beta r)+\zeta_{14} rJ_1(\beta r)+\zeta_{15} r^2J_2(\beta r)+\zeta_{16} r^3J_3(\beta r)]] \\ \theta^g(r) &= A_1[\zeta_{17} J_0(\beta r)+\zeta_{37}[\zeta_{18} J_0(\beta r)+\zeta_{19} rJ_1(\beta r)]+\zeta_{38}[\zeta_{20} J_0(\beta r)+\zeta_{21} rJ_1(\beta r)+\zeta_{22} r^2J_2(\beta r)] \\ &\quad +\zeta_{39}[\zeta_{23} J_0(\beta r)+\zeta_{24} rJ_1(\beta r)+\zeta_{25} r^2J_2(\beta r)+\zeta_{26} r^3J_3(\beta r)]] \\ P^g(r) &= A_1[\zeta_{27} J_0(\beta r)+\zeta_{37}[\zeta_{28} J_0(\beta r)+\zeta_{29} rJ_1(\beta r)]+\zeta_{38}[\zeta_{30} J_0(\beta r)+\zeta_{31} rJ_1(\beta r)+\zeta_{32} r^2J_2(\beta r)] \\ &\quad +\zeta_{39}[\zeta_{33} J_0(\beta r)+\zeta_{34} rJ_1(\beta r)+\zeta_{35} r^2J_2(\beta r)+\zeta_{36} r^3J_3(\beta r)]] \end{aligned} \tag{44}$$

where ζ_{37} to ζ_{39} are constant parameters. Let us show the functions in the brackets of Eq. (44) by functions H_0, H_1, H_2 and H_3 as

$$\begin{aligned} H_0(r) &= [J_1(\beta r)+\zeta_{37}[\zeta_1 J_1(\beta r)+rJ_2(\beta r)]+\zeta_{38}[\zeta_2 J_1(\beta r)+\zeta_3 rJ_2(\beta r)+r^2J_3(\beta r)] \\ &\quad +\zeta_{39}[\zeta_4 J_1(\beta r)+\zeta_5 rJ_2(\beta r)+\zeta_6 r^2J_3(\beta r)+r^3J_4(\beta r)]] \\ H_1(r) &= [\zeta_7 J_0(\beta r)+\zeta_{37}[\zeta_8 J_0(\beta r)+\zeta_9 rJ_1(\beta r)]+\zeta_{38}[\zeta_{10} J_0(\beta r)+\zeta_{11} rJ_1(\beta r)+\zeta_{12} r^2J_2(\beta r)] \\ &\quad +\zeta_{39}[\zeta_{13} J_0(\beta r)+\zeta_{14} rJ_1(\beta r)+\zeta_{15} r^2J_2(\beta r)+\zeta_{16} r^3J_3(\beta r)]] \\ H_2(r) &= [\zeta_{17} J_0(\beta r)+\zeta_{37}[\zeta_{18} J_0(\beta r)+\zeta_{19} rJ_1(\beta r)]+\zeta_{38}[\zeta_{20} J_0(\beta r)+\zeta_{21} rJ_1(\beta r)+\zeta_{22} r^2J_2(\beta r)] \\ &\quad +\zeta_{39}[\zeta_{23} J_0(\beta r)+\zeta_{24} rJ_1(\beta r)+\zeta_{25} r^2J_2(\beta r)+\zeta_{26} r^3J_3(\beta r)]] \\ H_3(r) &= [\zeta_{27} J_0(\beta r)+\zeta_{37}[\zeta_{28} J_0(\beta r)+\zeta_{29} rJ_1(\beta r)]+\zeta_{38}[\zeta_{30} J_0(\beta r)+\zeta_{31} rJ_1(\beta r)+\zeta_{32} r^2J_2(\beta r)] \\ &\quad +\zeta_{39}[\zeta_{33} J_0(\beta r)+\zeta_{34} rJ_1(\beta r)+\zeta_{35} r^2J_2(\beta r)+\zeta_{36} r^3J_3(\beta r)]] \end{aligned} \tag{45}$$

According to the Sturm-Liouville theorem, these functions are orthogonal with respect to the weight function $p(r)=r$ such as

$$\int_{r_i}^{r_o} H(\beta_n r)H(\beta_m r)r \, \mathrm{d}r = \begin{Bmatrix} 0 & n \neq m \\ \|H(\beta_n r)\|^2 & n = m \end{Bmatrix} \tag{46}$$

where $\|H(\beta_n r)\|$ is norm of the H function and equals

$$\|H(\beta_n r)\| = \left(\int_{r_i}^{r_o} rH^2(\beta_n r) \, \mathrm{d}r \right)^{0.5} \tag{47}$$

Due to the orthogonality of function H, every piece-wise continuous function, such as $f(r)$, can be expanded in terms of the function H (either H_0 or H_1 or H_2 or H_3) and is called the H-Fourier series as

$$f(r)=\sum_{n=1}^{\infty} e_n \quad H(\beta_n r) \tag{48}$$

where e_n equals

$$e_n=\frac{1}{\left\|H(\beta_n r)\right\|^2}\int_{r_i}^{r_o} f(r)H(r)\, r\,\mathrm{d}r \tag{49}$$

Using Eqs. (8), (44), and (45) the displacement and temperature distributions due to the general solution become

$$\begin{aligned}
u^g(r,t,z) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{\sum_{K=1}^{6} a_{nmk}e^{\lambda_{nmk}t}\}H_0(\beta_{mn}r)\sin(\zeta_n z)\\
w^g(r,t,z) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{\sum_{K=1}^{6} B_{nmk}a_{nmk}e^{\lambda_{nmk}t}\}H_1(\beta_{mn}r)\cos(\zeta_n z)\\
T^g(r,t,z) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{\sum_{K=1}^{6} N_{nmk}a_{nmk}e^{\lambda_{nmk}t}\}H_2(\beta_{mn}r)\sin(\zeta_n z)\\
p^g(r,t,z) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{\sum_{K=1}^{6} M_{nmk}a_{nmk}e^{\lambda_{nmk}t}\}H_3(\beta_{mn}r)\sin(\zeta_n z)
\end{aligned} \tag{50}$$

where B_{mn}, N_{mn} and M_{nm} are ratios obtained by substituting Eqs. (1) to (4). Using the initial conditions (7) and with the help of Eqs. (47) to (49) and (45) and (50), six unknown constants are obtained.

3.4 *Particular solution with non-homogeneous boundary conditions*

The general solutions may be used as proper functions for guessing the particular solution adopted to the non-homogeneous parts of the Eqs. (1) to (4) and the non-homogeneous boundary conditions (5) as

$$\begin{aligned}
u^p(r,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{\left[G_1(t)J_1(\beta_{nm}r)+G_2(t)rJ_2(\beta_{nm}r)+G_3(t)r^2J_3(\beta_{nm}r)+G_4(t)r^3J_4(\beta_{nm}r)\right]+r^2G_5(t)\}\sin(\zeta_n z)\\
w^p(r,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{\left[G_6(t)J_0(\beta_{nm}r)+G_7(t)rJ_1(\beta_{nm}r)+G_8(t)r^2J_2(\beta_{nm}r)+G_9(t)r^3J_3(\beta_{nm}r)\right]+r^2G_{10}(t)\}\cos(\zeta_n z)\\
T^p(r,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{\left[G_{11}(t)J_0(\beta_{nm}r)+G_{12}(t)rJ_1(\beta_{nm}r)+G_{13}(t)r^2J_2(\beta_{nm}r)+G_{14}(t)r^3J_3(\beta_{nm}r)\right]+r^2G_{15}(t)\}\sin(\zeta_n z)\\
p^p(r,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{\left[G_{16}(t)J_0(\beta_{nm}r)+G_{17}(t)rJ_1(\beta_{nm}r)+G_{18}(t)r^2J_2(\beta_{nm}r)+G_{19}(t)r^3J_3(\beta_{nm}r)\right]+r^2G_{20}(t)\}\sin(\zeta_n z)
\end{aligned} \tag{51}$$

It is necessary and suitable to expand the body force $F(r,t)$, $R\,(z,t)$, heat source $Q(r,t)$ and porosity function $W(r,t)$ in H-Fourier expansion form as

$$F(r,z,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} F_{nm}(t)H_0(\beta_{nm}r)\ \sin(\zeta_n z)$$

$$R(r,z,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} R_{nm}(t)H_1(\beta_{nm}r)\cos(\zeta_n z)$$
$$Q(r,z,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Q_{nm}(t)H_2(\beta_{nm}r)\sin(\zeta_n z)$$
$$P(r,z,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} P_{nm}(t)H_3(\beta_{nm}r)\sin(\zeta_n z) \tag{52}$$

where $F_{nm}(t)$, $R_{nm}(t)$, $Q_{nm}(t)$ and $P_{nm}(t)$ are

$$F_{nm}(t)=\frac{2}{\left\|H_0(\beta_n r)\right\|^2 l}\int_0^l\int_{r_i}^{r_o} F(r,t)H_0(\beta_n r)r\sin(\zeta_n z)\,\mathrm{d}r\,\mathrm{d}z$$
$$R_{nm}(t)=\frac{2}{\left\|H_1(\beta_n r)\right\|^2 l}\int_0^l\int_{r_i}^{r_o} Q(r,t)H_1(\beta_n r)r\cos(\zeta_n z)\,\mathrm{d}r\,\mathrm{d}z$$
$$Q_{nm}(t)=\frac{2}{\left\|H_2(\beta_n r)\right\|^2 l}\int_0^l\int_{r_i}^{r_o} Q(r,t)H_2(\beta_n r)r\sin(\zeta_n z)\,\mathrm{d}r\,\mathrm{d}z$$
$$P_{nm}(t)=\frac{2}{\left\|H_3(\beta_n r)\right\|^2 l}\int_0^l\int_{r_i}^{r_o} P(r,t)H_3(\beta_n r)r\sin(\zeta_n z)\,\mathrm{d}r\,\mathrm{d}z \tag{53}$$

Substituting Eqs. (51) and (53) into non-homogeneous form of equations (1) into (4) yield

$$\begin{aligned}&G_2(t)\beta^2-\ddot{G}_2(t)d_3-4\ddot{G}_3(t)\frac{1}{\beta}d_3-d_{13}G_4(t)C_1-4d_{13}G_4(t)C_2\frac{1}{\beta}-d_{16}\ddot{G}_4(t)C_1-4d_{16}\ddot{G}_4(t)C_2\frac{1}{\beta}\\&\quad+G_6(t)d_2\beta-d_{15}G_8(t)C_1-4d_{15}G_8(t)C_2\frac{1}{\beta}+G_{10}(t)d_1\beta-d_{14}G_{12}(t)C_1\\&\quad-16d_{14}G_{12}(t)C_2\frac{1}{\beta}-d_{25}d_{10}FC_1-4d_{25}d_{10}FC_2\frac{1}{\beta}=0\end{aligned} \tag{54a}$$

$$\begin{aligned}&-G_1(t)\beta^2+\ddot{G}_1(t)d_3+2\ddot{G}_2(t)\frac{1}{\beta}d_3+8\ddot{G}_3(t)\frac{1}{\beta^2}d_3d_{13}G_4(t)C_0+2d_{13}G_4(t)C_1\frac{1}{\beta}+8d_{13}G_4(t)C_2\frac{1}{\beta^2}\\&\quad+d_{16}\ddot{G}_4(t)C_0+2d_{16}\ddot{G}_4(t)C_1\frac{1}{\beta}+8d_{16}\ddot{G}_4(t)C_2\frac{1}{\beta^2}-G_5(t)\beta d_2+d_{15}G_8(t)C_0\\&\quad+2d_{15}G_8(t)C_1\frac{1}{\beta}+8d_{15}G_8(t)C_2\frac{1}{\beta^2}-G_9(t)\beta d_1+d_{14}G_{12}(t)C_0+2d_{14}G_{12}(t)C_1\frac{1}{\beta}\\&\quad+8d_{14}G_{12}(t)C_2\frac{1}{\beta^2}+d_{25}d_{10}FC_0+2d_{25}d_{10}FC_1\frac{1}{\beta}+8d_{25}d_{10}FC_2\frac{1}{\beta^2}=0\end{aligned} \tag{54b}$$

$$\begin{aligned}&G_3(t)\beta^2-\ddot{G}_3(t)d_3-d_{13}G_4(t)C_2-d_{16}\ddot{G}_4(t)C_2+G_7(t)\beta d_2-d_{15}G_8(t)C_2+G_{11}(t)\beta d_1\\&\quad-d_{14}G_{12}(t)C_2-d_{25}d_{10}FC_2=0\end{aligned} \tag{54c}$$

$$\begin{aligned}&d_6\beta\dot{G}_1(t)+d_{18}E_0\dot{G}_4(t)-\beta^2G_5(t)+d_4\dot{G}_5(t)+2\beta G_6(t)+d_{17}E_0G_8(t)+d_{19}E_0\dot{G}_8(t)+d_5\dot{G}_9(t)\\&\quad+d_{20}E_0\dot{G}_{12}(t)+d_{26}d_{11}E_0Q_n(t)=0\end{aligned} \tag{54d}$$

$$\begin{aligned}&-d_6\beta\dot{G}_3(t)-E_2d_{18}\dot{G}_4(t)+\beta^2G_7(t)-d_4\dot{G}_7(t)-d_{17}E_2G_8(t)-d_{19}E_2\dot{G}_8(t)-d_5\dot{G}_{11}(t)\\&\quad-d_{20}E_2\dot{G}_{12}(t)-d_{26}E_2Q_n(t)=0\end{aligned} \tag{54e}$$

$$d_6\beta\dot{G}_2(t)+2d_6\dot{G}_3(t)+\left(d_{18}E_1+\frac{2}{\beta}d_{14}E_2\right)\dot{G}_4(t)-\beta^2G_6(t)+d_4\dot{G}_6(t)+2\beta G_7(t)+\frac{2}{\beta}d_4\dot{G}_7(t)$$
$$+\left(d_{17}E_1+\frac{2}{\beta}d_{17}E_2\right)G_8(t)+\left(d_{19}E_1+\frac{2}{\beta}E_2d_{19}\right)\dot{G}_8(t)+\dot{G}_{10}(t)d_5+\frac{2}{\beta}\dot{G}_{11}(t)d_5$$
$$+\left(d_{20}E_1+\frac{2}{\beta}d_{20}E_2\right)\dot{G}_{12}(t)+\left(d_{26}E_1+\frac{2}{\beta}d_{26}E_2\right)Q_n(t)=0 \tag{54f}$$

$$d_9\beta\dot{G}_1(t)+d_{22}D_0\dot{G}_4(t)+d_7\dot{G}_5(t)+d_{23}D_0\dot{G}_8(t)-\beta^2G_9(t)+d_8\dot{G}_9(t)+2G_{10}(t)\beta+d_{21}G_{12}(s)D_0$$
$$+d_{24}\dot{G}_{12}(t)+d_{27}d_{12}D_0W_n(t)=0 \tag{54g}$$

$$-d_9\beta\dot{G}_3(t)-d_{22}D_2\dot{G}_4(t)-d_7\dot{G}_7(t)-d_{23}D_2\dot{G}_8(t)+\beta^2G_{11}(t)-d_8\dot{G}_{11}(t)-d_{21}D_2G_{12}(t)$$
$$-d_{24}\dot{G}_{12}(t)D_2-d_{27}d_{12}E_2W_n(t)=0 \tag{54h}$$

$$d_9\dot{G}_2(t)\beta+d_9 2\dot{G}_3(t)+\left(d_{22}D_1+\frac{2}{\beta}d_{22}D_2\right)\dot{G}_4(t)+d_7\dot{G}_6(t)+\frac{2}{\beta^2}d_7\dot{G}_7(t)$$
$$+\left(d_{23}D_1+\frac{2}{\beta}d_{23}D_2\right)\dot{G}_8(t)-\beta^2G_{10}(t)+d_8\dot{G}_{10}(t)+2\beta G_{11}(t)+\frac{2}{\beta}d_8\dot{G}_{11}(t)$$
$$+\left(d_{21}D_1+\frac{2}{\beta}d_{21}D_2\right)G_{12}(t)+\left(d_{24}D_1+\frac{2}{\beta}d_{24}D_2\right)\dot{G}_{12}(t)=0 \tag{54i}$$

where d_{10} to d_{27} are the coefficients of the *H*-expansion and constant parameters. By taking Laplace transform of Eq. (54) and using four boundary conditions of Eq. (5) (for solid cylinder only second, fourth, sixth and eighth boundary conditions are applicable), a system of algebraic equations is obtained and solved by Cramer's methods in the Laplace domain where by the inverse Laplace transform the functions are transformed into the real time domain and finally $G_1(t)$ to $G_{20}(t)$ are calculated. In this process, it is necessary to consider the following points:

1. The initial conditions (7) are considered only for the general solutions and the, initial conditions of $G_1(t)$ to $G_{20}(t)$ for the particular solutions are considered equal to zero.
2. Laplace transform of Eqs. (54) are in terms of polynomial function form of the Laplace parameters (not the Bessel functions form of s). Therefore, the exact inverse Laplace transform is possible and somehow simple.

4 RESULTS AND DISCUSSIONS

As an example, a solid cylinder with r_i=0, r_o=1 m and L=π m is considered. The material properties are listed in Table 1. The initial temperature $T_\circ$ is considered to be 293°K. Now, an instantaneous hot spot $T(1,z,t)=10^{-3}T_\circ\delta(t)\ \sin(z)$, where $\delta(t)$ is unit Dirac function, is considered and the outside radius of the cylinder is assumed to be fixed ($u(1,z,$ t) =0). For drawing the graphs, a nondimensional time $\hat{t}=vt/r_\circ$ is considered where $v=\sqrt{E(1-\nu)/\rho(1+\nu)(1-2\nu)}$ is dilatational -wave velocity.

Table1
Material Parameters

Parameters	Value	Unit	Parameters	Value	Unit
n	0.4	-	α_s	1.5×10^{-5}	1/^{o}C
E	6×10^5	Pa	α_w	2×10^{-4}	1/^{o}C
υ	0.3	-	c_s	0.8	J/g^{o}C
$T_\circ$	293	$^\circ$K	c_w	4.2	J/g^{o}C
K_s	2×10^{10}	Pa	ρ_s	2.6×10^6	g/m^3
K_w	5×10^9	Pa	ρ_w	1×10^6	g/m^3
K	0.5	W/m^{o}C	α	1	-

Figs. 1 and 2 show wave-front for radial and longitudinal displacements due to thermal shock. Figs. 3 and 4 respectively show shock wave at cylinder and the response of pressure due to thermal shock. For the second example, mechanical shock is applied to the outside surface of the cylinder given as $u(1,z,t) = 10^{-12} u_{\circ} \delta(t)\ \sin(z)$ and the surface is assumed to be at zero temperature ($T(1,z,t)=0$).

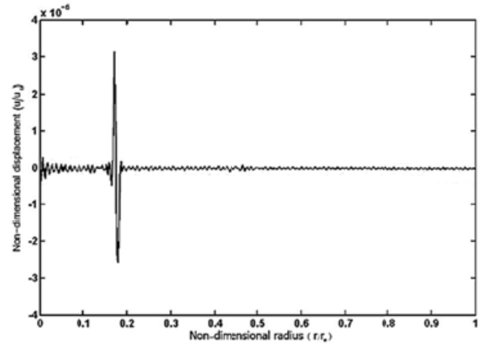

Fig. 1
Non-dimensional displacement distribution due to input $u(1,z,t) = 10^{-12} u_0 \delta(t) \sin(z)$ at non-dimensional time $\hat{t} = 0.8$.

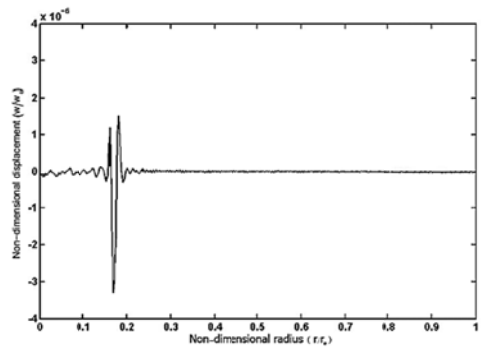

Fig. 2
Non-dimensional displacement distribution due to input $u(1,z,t) = 10^{-12} u_0 \delta(t) \sin(z)$ at non-dimensional time $\hat{t} = 0.8$.

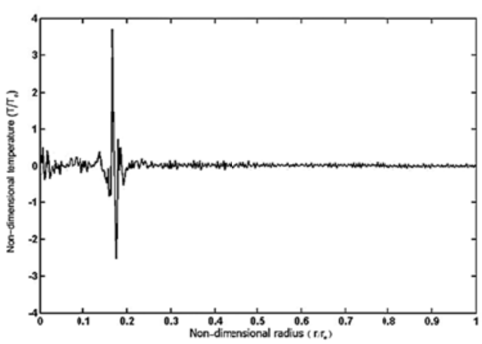

Fig. 3
Non-dimensional displacement distribution due to input $u(1,z,t) = 10^{-12} u_0 \delta(t) \sin(z)$ at non-dimensional time $\hat{t} = 0.8$.

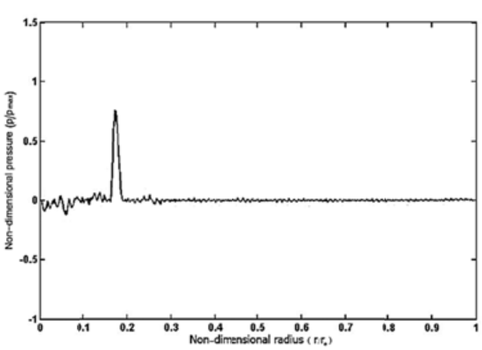

Fig. 4
Non-dimensional displacement distribution due to input $u(1,z,t) = 10^{-12} u_0 \delta(t) \sin(z)$ at non-dimensional time $\hat{t} = 0.8$.

Figs. 5 and 6 show wave-front for radial and longitudinal displacements. Fig. 7 shows thermal response due to mechanical shock. Fig. 8 shows wave front movement of pressure in radius direction. For all graphs z is considered equal $\pi/2$. The convergence of the solutions for these examples is achieved by consideration of 1200 eigenvalues used for the H-Fourier expansion.

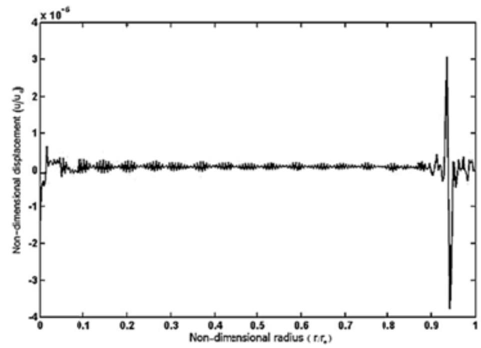

Fig. 5
Non-dimensional displacement distribution due to input $T(1,z,t)=10^{-3}T_0\delta(t)\sin(z)$ at non-dimensional time $\hat{t}=0.1$.

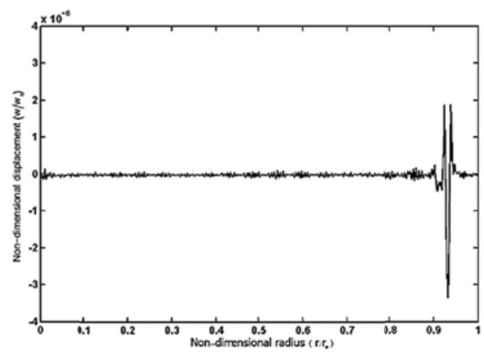

Fig. 6
Non-dimensional displacement distribution due to input $T(1,z,t)=10^{-3}T_0\delta(t)\sin(z)$ at non-dimensional time $\hat{t}=0.1$.

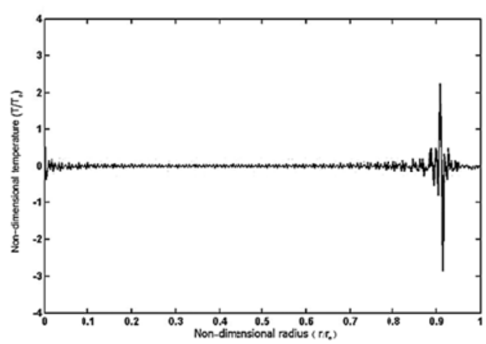

Fig. 7
Non-dimensional displacement distribution due to input $T(1,z,t)=10^{-3}T_0\delta(t)\sin(z)$ at non-dimensional time $\hat{t}=0.1$.

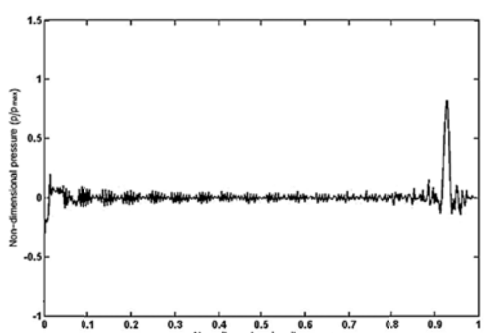

Fig. 8
Non-dimensional displacement distribution due to input $T(1,z,t)=10^{-3}T_0\delta(t)\sin(z)$ at non-dimensional time $\hat{t}=0.1$.

More than this number for eigenvalues, the increased round-off and truncation errors effect the quality of the graphs. The convergence of solution is better for displacement in comparison with the temperature and pressure.

5 CONCLUSIONS

In this paper, analytical solution for the coupled porothermoelasticity of thick cylinders under radial temperature is presented. The method is based on the eigenfunctions Fourier expansion, which is a classical and traditional method of solution of the typical initial and boundary value problems. The non-competetive strength of this method is its ability to reveal the fundamental mathematical and physical properties and interpretations of the problem under study. In the coupled porothermoelastic problem of radial-symmetric cylinder, the governing equations are a system of partial differential equations with two independent variables, radius (r) and time (t). The traditional procedure to solve this class of problems is to eliminate the time variable by using the Laplace transform. The resulting system is a set of ordinary differential equations in terms of the radius variable, which falls in the Bessel functions family. This method of analysis brings the Laplace parameter (s) in the argument of the Bessel functions, causing hardship or impossibility in carrying out the exact inverse of the Laplace transformation. As a result, the numerical inverse of the Laplase transformation is used in the papers dealing with this type of problems in literature. To prevent this problem, in the present paper, when the Laplace transform is applied to the particular solution, it is postponed after eliminating the radius variable r by H-Fourier Expansion. Thus, the Laplace parameter (s) appears in polynomial function forms and hence the exact Laplace inversion transformation is possible.

6 ACKNOWLEDGMENT

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APPENDIX

$$d_1=-\frac{1}{2}\left(\frac{1-2\nu}{1-\nu}\right),\quad d_2=-\frac{1}{2}\frac{1}{(1-\nu)},\quad d_3=-\beta\frac{(1+\nu)(1-2\nu)}{(1-\nu)^2E}-\frac{(1+\nu)}{(1-\nu)}\alpha,\quad d_4=-\beta\frac{(1+\nu)(1-2\nu)}{(1-\nu)^2E},$$
$$d_5=-\rho\frac{(1+\nu)(1-2\nu)}{(1-\nu)^2E},\quad d_6=-2\frac{(1-\nu)}{(1-2\nu)},\quad d_7=\frac{1}{(1-2\nu)},\quad d_8=\frac{1}{(1-2\nu)},\quad d_9=-2\alpha\frac{(1+\nu)}{(1-\nu)E},$$
$$d_{10}=-2\beta\frac{(1+\nu)}{(1-\nu)E}-2\frac{(1+\nu)}{(1-2\nu)}\alpha,\quad d_{11}=-2\rho\frac{(1+\nu)}{(1-\nu)E},\quad d_{12}=-Z\frac{T_\circ}{K},\quad d_{13}=Y\frac{T_\circ}{K},\quad d_{14}=-\beta\frac{T_\circ}{K},$$
$$d_{15}=-\alpha_p\frac{\gamma_w}{k},\quad d_{16}=Y\frac{\gamma_w}{k},\quad d_{17}=-\alpha\frac{\gamma_w}{k},\quad d_{18}=-\frac{(1+\nu)(1-2\nu)}{(1-\nu)^2E},\quad d_{19}=-2\frac{(1+\nu)}{(1-\nu)E},$$
$$d_{20}=-\frac{1}{K}Q,\quad d_{21}=-\frac{\gamma_w}{k} \tag{A.1}$$

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