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Free Vibration of Thick Isotropic Plates Using Trigonometric Shear Deformation Theory

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ABSTRACT

In this paper a variationally consistent trigonometric shear deformation theory is presented for the free vibration of thick isotropic square and rectangular plate. In this displacement based theory, the in-plane displacement field uses sinusoidal function in terms of thickness coordinate to include the shear deformation effect. The cosine function in terms of thickness coordinate is used in transverse displacement to include the effect of transverse normal strain. Governing equations and boundary conditions of the theory are obtained using the principle of virtual work. Results of frequency of bending mode, thickness-shear mode and thickness-stretch mode are obtained from free vibration of simply supported isotropic square and rectangular plates and compared with those of other refined theories and frequencies from exact theory. Present theory yields exact dynamic shear correction factor $\pi^2/12$ from thickness shear motion of the plate.

Keywords: Shear deformation; Thick isotropic plate; Shear correction factor; Transverse normal strain; Free vibration; Thickness shear frequencies

1 INTRODUCTION

PLATES are the basic structural components that are widely used in various engineering disciplines such as aerospace, civil, marine and mechanical engineering. The transverse shear and transverse normal deformation effects are more pronounced in shear flexible plates which may be made up of isotropic, orthotropic, anisotropic or laminated composite materials. In order to address the correct structural behavior of structural elements made up of these materials; development of refined theories, which take into account refined effects in static and dynamic analysis of structural elements, becomes necessary.

The study of plate vibration dates back to the early eighteen century, with the German physicist, Chladni (1787), who observed the nodal patterns for a flat square plate. Since then there has been a tremendous research interest in the subject of plate vibrations. Several thin plate vibration solutions based on Kirchhoff's plate theory are available in the literature. The classical plate theory based on Kirchhoff's hypothesis [1, 2] is not adequate for the analysis of shear flexible plates due to the neglect of transverse shear deformation and the rotary inertia in the theory; as a consequence, it under predicts deflections and over predicts all the vibration frequencies for thick plates, and the higher frequencies for the thin plates. The most suitable starting point for the analysis of both thin and thick plates seems to be a theory in which the classical hypothesis of zero transverse shear strains is relaxed. At first, Reissner [3, 4] proposed that the rotations of the normal to the plate mid-surface in the transverse plane could be introduced as independent variables in the plate theory. Reissner has developed a stress based theory which incorporates the effect of shear. Mindlin [5] simplified Reissner's assumption that normal to the plate mid-surface before deformation remains straight but not necessarily normal to the plate mid-surface after deformation and the stress

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normal to the mid-surface is disregarded as in the case of classical plate theory of Kirchhoff. Mindlin employed displacement based approach. In Mindlin's theory, transverse shear stress is assumed to be constant through the thickness of the plate, but this assumption violates the shear stress free surface conditions. The theory includes both the shear deformation and rotary inertia effects. Both effects decrease the frequencies. There are still other effects not accounted for by the Mindlin are stretching in the thickness direction and the warping of the normal to the mid-plane, which are more important in case of thick plates. Mindlin's theory satisfies constitutive relations for transverse shear stresses and shear strains by using shear correction factor. The value of this factor is not unique but depends on the material, geometry, loading and boundary condition parameters. Wang et al. [6] discussed these theories in detail and developed the relationships between bending solutions of Reissner and Mindlin plate theories. Reissner's formulation comes out as special case of Librescu's approach [7] presented for elastostatic analysis of anisotropic shell type structures. Donnell [8] make correction to the classical plate theory by assuming uniform distribution of shear stress across the thickness of the plate, and, rectify the effect of this assumption.

Usually, in two dimensional plate theories, displacement components are considered power series expressions in thickness coordinate (z). Depending on the number of terms retained in the power series expressions, various higher order theories for homogeneous and laminated plates can be developed. Detailed discussion on the applicability and accuracy of these theories is presented by Lo et al. [9, 10]. It is observed that most of the displacement based theories such as theories of Levinson [11], Murty and Vellaichamy [12] and Reddy [13, 14, and 15] utilize some simplification of the generalized displacement function given by Lo et al. The simplified higher order theories, generally third order shear deformation theories give parabolic variation of transverse shear stress through the thickness of the plate satisfying the shear stress free boundary conditions on the top and bottom surfaces of the plate. Thus, these theories do not require shear correction factors. Levinson [9] formulated a theory based on displacement approach which does not require shear correction factor. However, Levinson's theory is variationally inconsistent since the field equations and boundary conditions are not derived using principle of virtual work. Srinivas et al. [16, 17] developed exact elasticity solutions for the flexure and free vibration of simply supported homogeneous, isotropic, thick rectangular plates. The exact elasticity solutions play important role in validation of results of two dimensional thick plate theories. Comprehensive reviews of these theories have been given by Noor and Burton [18], Reddy [19], Reddy and Robbins [20], and Liu and Li [21] whereas Liew et al. [22] surveyed plate theories particularly applied to thick plate vibration problems. In the development of such theories use of polynomials, trigonometric functions, hyperbolic functions and exponential functions in terms of thickness coordinate is widely and wisely made. A recent reviews such refined shear deformation theories are presented by Ghugal and Shimpi [23] and Kreja [24].

Levy [25] developed a refined theory for thick isotropic plate for the first time using sinusoidal functions in the displacement field to take into account shear deformation effect. Stein [26] has used trigonometric functions in terms of thickness coordinates for the analysis of laminated beams and plates, but the shear stress free conditions at the top and bottom surfaces are not satisfied. Shimpi et al. [27] have developed a new refined theory for the analysis of isotropic and orthotropic plate using trigonometric function in terms of thickness coordinates which includes the effect of transverse normal strain. Recently, Shimpi and Patel [28] developed two variable refined plate theory for the analysis of isotropic plates; however, theory of these authors yields the frequencies identical to those of Mindlin's theory. Ghugal and Pawar [29, 30] have developed hyperbolic shear deformation theory for the flexure, buckling and vibration analysis of thick shear flexible plates. Ghugal and Sayyad [31, 32] have used trigonometric shear deformation theory for the free vibration analysis of orthotropic plates and Ghugal and Kulkarni [33] employed it for the thermal analysis of isotropic, orthotropic and laminated composite plates.

In the present paper, a variationally consistent trigonometric shear deformation theory for free vibration of homogenous, isotropic plate is developed. It has four variables and includes effects of transverse shear and transverse normal strain. The theory satisfies the tangential traction free boundary conditions (zero shear stress conditions) on the top and bottom surfaces of the plate. The primary objective of this investigation is to present the frequencies of flexural mode, thickness shear and thickness stretch modes of free vibration of thick plates.

2 THEORETICAL FORMULATION

Consider an undeformed rectangular plate of length a, width b, and thickness h composed of an isotropic homogenous material. The rectangular Cartesian coordinate system is such that the middle plane of the plate coincides with the xy plane, and z axis is normal to the middle plane. The plate is bounded by the coordinate planes x=0, a and y=0, b. The reference surface is the middle surface of the plate defined by z=0 and z represents the thickness coordinate measured from the undeformed middle surface. Thus, the plate occupies in O - x - y - z right-handed rectangular Cartesian coordinate system, a region:

$$0 \le x \le a, \qquad 0 \le y \le b, \qquad -h/2 \le z \le h/2 \tag{1}$$

The displacement field of the present plate theory is of the following form [32]

$$\begin{aligned} u(x,y,z,t) &= -z\frac{\partial w(x,y,t)}{\partial x} + \frac{h}{\pi}\sin\frac{\pi z}{h}\,\phi(x,y,t) \\ v(x,y,z,t) &= -z\frac{\partial w(x,y,t)}{\partial y} + \frac{h}{\pi}\sin\frac{\pi z}{h}\,\psi(x,y,t) \\ w(x,y,z,t) &= w(x,y,t) + \frac{h}{\pi}\cos\frac{\pi z}{h}\,\xi(x,y,t) \end{aligned} \tag{2}$$

where u and v are the inplane displacements in x- and y- directions respectively and w is transverse displacement in z- direction. The sinusoidal function is assigned according to the shear stress distribution through the thickness of the plate. The ϕ, ψ and ξ represent rotations of the plate at neutral surface with respect to yz, xz and xy planes due to bending, respectively, which are unknown functions to be determined. The generalized displacements w, ϕ, ψ and ξ on the right hand side of Eqs. (2) are the independent variables.

The normal strains $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$ and shear strains $(\gamma_{xy}, \gamma_{xz}, \gamma_{yz})$ are obtained within the framework of linear theory of elasticity. The infinitesimal strains associated with the displacement field (2) are as follows

$$\begin{aligned} \varepsilon_x &= \frac{\partial u(x,y,z,t)}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} + \frac{h}{\pi}\sin\frac{\pi z}{h}\frac{\partial \phi}{\partial x} \\ \varepsilon_y &= \frac{\partial v(x,y,z,t)}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} + \frac{h}{\pi}\sin\frac{\pi z}{h}\frac{\partial \psi}{\partial y} \\ \varepsilon_z &= \frac{\partial w(x,y,z,t)}{\partial z} = -\xi\sin\frac{\pi z}{h} \end{aligned} \tag{3}$$

$$\begin{aligned} \gamma_{xy} &= \frac{\partial u(x,y,z,t)}{\partial y} + \frac{\partial v(x,y,z,t)}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} + \frac{h}{\pi}\sin\frac{\pi z}{h}\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right) \\ \gamma_{xz} &= \frac{\partial u(x,y,z,t)}{\partial z} + \frac{\partial w(x,y,z,t)}{\partial x} = \cos\frac{\pi z}{h}\left(\frac{h}{\pi}\frac{\partial \xi}{\partial x} + \phi\right) \\ \gamma_{yz} &= \frac{\partial v(x,y,z,t)}{\partial z} + \frac{\partial w(x,y,z,t)}{\partial y} = \cos\frac{\pi z}{h}\left(\frac{h}{\pi}\frac{\partial \xi}{\partial y} + \psi\right) \end{aligned} \tag{4}$$

The stress strain relationships for the linear isotropic elastic plate can be written as

$$\begin{aligned} \sigma_x &= \lambda\,(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\,\varepsilon_x \\ \sigma_y &= \lambda\,(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\,\varepsilon_y \\ \sigma_z &= \lambda\,(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\,\varepsilon_z \\ \tau_{xy} &= G\gamma_{xy}, \qquad \tau_{yz} = G\gamma_{yz}, \qquad \tau_{zx} = G\gamma_{zx} \end{aligned} \tag{5}$$

where λ and G are the Lame's constants as given below $\lambda = \mu E/(1-\mu)(1-2\mu)$ and $G = E/2(1+\mu)$ in which E is the Young's modulus, G is the shear modulus and μ is the Poisson's ratio. The Strain-displacement and stress-strain relations used in Eqs. (3) through (5) are discussed by Timoshenko and Goodier [34].

2.1 Derivation of governing equations and boundary conditions

Using Eq. (3) through (5) and principle of virtual work, variationally consistent differential equations and boundary conditions for the plate under consideration are obtained. The principle of virtual work when applied to the plate leads to:

$$\int_{z=-h/2}^{z=h/2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\sigma_x\,\delta\varepsilon_x+\sigma_y\,\delta\varepsilon_y+\sigma_z\,\delta\varepsilon_z+\tau_{yz}\,\delta\gamma_{yz}+\tau_{zx}\,\delta\gamma_{zx}+\tau_{xy}\,\delta\gamma_{xy}\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$-\int_{y=0}^{y=b}\int_{x=0}^{x=a}q(x,y)\delta w\,\mathrm{d}x\,\mathrm{d}y+\rho\int_{z=-h/2}^{z=h/2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\left[\frac{\partial^2 u}{\partial t^2}\delta u+\frac{\partial^2 v}{\partial t^2}\delta v+\frac{\partial^2 w}{\partial t^2}\delta w\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0 \tag{6}$$

Employing Green's theorem in Eq. (6) successively, we obtain the coupled *Euler-Lagrange* equations of the plate and the associated boundary conditions of the plate. The governing equations in terms of stress resultants are as follows:

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+q=I_1\frac{\partial^2 w}{\partial t^2}-I_2\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right)+I_3\left(\frac{\partial^3\phi}{\partial x\partial t^2}+\frac{\partial^3\psi}{\partial y\partial t^2}\right)+I_4\frac{\partial^2\xi}{\partial t^2} \tag{7a}$$

$$\frac{\partial M_x^s}{\partial x}+\frac{\partial M_{xy}^s}{\partial y}-V_x^s=I_5\frac{\partial^2\phi}{\partial t^2}-I_3\frac{\partial^3 w}{\partial x\partial t^2} \tag{7b}$$

$$\frac{\partial M_y^s}{\partial y}+\frac{\partial M_{xy}^s}{\partial x}-V_{yz}^s=I_5\frac{\partial^2\psi}{\partial t^2}-I_3\frac{\partial^3 w}{\partial y\partial t^2} \tag{7c}$$

$$\frac{\partial V_{xz}^s}{\partial x}+\frac{\partial V_{yz}^s}{\partial y}-\frac{\pi}{h}V_{zz}^s=I_4\frac{\partial^2 w}{\partial t^2}+I_6\frac{\partial^2\xi}{\partial t^2} \tag{7d}$$

The boundary conditions along edges $x=0$ and $x=a$ obtained are of the following form:

$$\left.\begin{array}{lll}V_x=\partial M_x/\partial x+2\partial M_{xy}/\partial y=0 & \text{or} & w \text{ is specified}\\ M_x=0 & \text{or} & \partial w/\partial x \text{ is specified}\\ M_x^s=0 & \text{or} & \phi \text{ is specified}\\ M_{xy}^s=0 & \text{or} & \psi \text{ is specified}\\ V_{xz}^s=0 & \text{or} & \xi \text{ is specified}\end{array}\right\} \tag{8}$$

and along $y=0$ and $y=b$ edges, the boundary conditions are as follows:

$$\left.\begin{array}{lll}V_y=\partial M_y/\partial y+2\partial M_{xy}/\partial x=0 & \text{or} & w \text{ is specified}\\ M_y=0 & \text{or} & \partial w/\partial y \text{ is specified}\\ M_y^s=0 & \text{or} & \psi \text{ is specified}\\ M_{xy}^s=0 & \text{or} & \phi \text{ is specified}\\ V_{yz}^s=0 & \text{or} & \xi \text{ is specified}\end{array}\right\} \tag{9}$$

At corners $(x=0,y=0)$, $(x=a,y=0)$, $(x=0,y=b)$, $(x=a,y=b)$ boundary condition is:

$$M_{xy}=0 \quad \text{or} \quad w \text{ is specified.} \tag{10}$$

The stress resultants appeared in the governing equations and boundary conditions are defined as follows:

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_x, \tau_{xy}) z \,\mathrm{d}z, \qquad (M_x^s, M_y^s, M_{xy}^s) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_x, \tau_{xy}) f(z) \,\mathrm{d}z \tag{11}$$

$$(V_{xz}^s, V_{yz}^s) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) f'(z) \,\mathrm{d}z, \qquad V_{zz}^s = \int_{-h/2}^{h/2} \sigma_{zz} g'(z) \,\mathrm{d}z \tag{12}$$

where M_x, M_y, M_{xy} are the bending and twisting moment resultants or the stress couples analogous to classical plate theory, M_x^s, M_y^s, M_{xy}^s are refined moments or stress couples due to transverse shear deformation effects and $V_{xz}^s, V_{yz}^s, V_{zz}^s$ are the transverse shear and transverse normal stress resultants and $f(z) = (h/\pi)\sin \pi z/h$, $g(z) = (h/\pi)\cos \pi z/h$ and the prime $()'$ indicates the differentiation of function with respect to *z*. The inertia terms I_i appeared in the governing equations and boundary conditions are expressed as follows:

$$\left[I_1, I_2, I_3, I_4, I_5, I_6\right]^T = \rho \int_{-h/2}^{h/2} \left\{1, z^2, z f(z), f^2(z), g(z), g^2(z)\right\} \mathrm{d}z \tag{13}$$

The governing equations in terms of displacement variables can be expressed as follows:

$$\begin{aligned} &D_1\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - D_2\left(\frac{\partial^3 \phi}{\partial x^3} + \frac{\partial^3 \phi}{\partial x \partial y^2} + \frac{\partial^3 \psi}{\partial y^3} + \frac{\partial^3 \psi}{\partial x^2 \partial y}\right) + D_3\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right) + I_1 \frac{\partial^2 w}{\partial t^2} \\ &\quad - I_2\left(\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2}\right) + I_3\left(\frac{\partial^3 \phi}{\partial x \partial t^2} + \frac{\partial^3 \psi}{\partial y \partial t^2}\right) + I_4 \frac{\partial^2 \xi}{\partial t^2} = q \end{aligned} \tag{14a}$$

$$D_2\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right) - D_4 \frac{\partial^2 \phi}{\partial x^2} - D_5 \frac{\partial^2 \phi}{\partial y^2} + D_6 \phi - D_7 \frac{\partial^2 \psi}{\partial x \partial y} + D_8 \frac{\partial \xi}{\partial x} - \left(I_3 \frac{\partial^3 w}{\partial x \partial t^2} - I_5 \frac{\partial^2 \phi}{\partial t^2}\right) = 0 \tag{14b}$$

$$D_2\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}\right) - D_4 \frac{\partial^2 \psi}{\partial y^2} - D_5 \frac{\partial^2 \psi}{\partial x^2} + D_6 \psi - D_7 \frac{\partial^2 \phi}{\partial x \partial y} + D_8 \frac{\partial \xi}{\partial y} - \left(I_3 \frac{\partial^3 w}{\partial y \partial t^2} - I_5 \frac{\partial^2 \psi}{\partial t^2}\right) = 0 \tag{14c}$$

$$D_3\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - D_8\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) - D_5\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right) + D_9 \xi + \left(I_4 \frac{\partial^2 w}{\partial t^2} + I_5 \frac{\partial^2 \xi}{\partial t^2}\right) = 0 \tag{14d}$$

and the associated boundary conditions can be expressed as follows:

1. On edges $x = 0$ and $x = a$, the following conditions hold:

$$D_1 \frac{\partial^3 w}{\partial x^3} + D_{10} \frac{\partial^3 w}{\partial x \partial y^2} - D_2\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}\right) - D_{11} \frac{\partial^2 \phi}{\partial y^2} + D_3 \frac{\partial \xi}{\partial x} + I_2 \frac{\partial^3 w}{\partial x \partial t^2} - I_3 \frac{\partial^2 \phi}{\partial t^2} = 0 \quad \text{or } w \text{ is prescribed} \tag{15a}$$

$$D_1 \frac{\partial^2 w}{\partial x^2} + D_{12} \frac{\partial^2 w}{\partial y^2} - D_2 \frac{\partial \phi}{\partial x} - D_{13} \frac{\partial \psi}{\partial y} + D_3 \xi = 0 \qquad \text{or } \frac{\partial w}{\partial x} \text{ is prescribed} \tag{15b}$$

$$D_2 \frac{\partial^2 w}{\partial x^2} + D_{13} \frac{\partial^2 w}{\partial y^2} - D_4 \frac{\partial \phi}{\partial x} - D_{14} \frac{\partial \psi}{\partial y} + D_{15} \xi = 0 \qquad \text{or } \phi \text{ is prescribed} \tag{15c}$$

$$D_{11} \frac{\partial^2 w}{\partial x \partial y} - D_{16}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\right) = 0 \qquad \text{or } \psi \text{ is prescribed} \tag{15d}$$

$$D_5 \frac{\partial \xi}{\partial x} + D_{16} \phi = 0 \qquad \text{or } \xi \text{ is prescribed} \tag{15e}$$

2. On edges $y = 0$ and $y = b$, the following conditions hold:

$$D_1\frac{\partial^3 w}{\partial y^3}+D_{10}\frac{\partial^3 w}{\partial x^2\partial y}-D_5\left(\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\phi}{\partial x\partial y}\right)-D_{11}\frac{\partial^2\psi}{\partial x^2}+D_3\frac{\partial\xi}{\partial y}+I_2\frac{\partial^3 w}{\partial y\partial t^2}-I_3\frac{\partial^2\psi}{\partial t^2}=0 \text{ or } w \text{ is prescribed} \quad (16a)$$

$$D_1\frac{\partial^2 w}{\partial y^2}+D_{12}\frac{\partial^2 w}{\partial x^2}-D_2\frac{\partial\psi}{\partial y}-D_{13}\frac{\partial\phi}{\partial x}+D_3\xi = 0 \qquad \text{or } \frac{\partial w}{\partial x} \text{ is prescribed} \quad (16b)$$

$$D_{13}\frac{\partial^2 w}{\partial x^2}+D_2\frac{\partial^2 w}{\partial y^2}-D_{14}\frac{\partial\phi}{\partial x}-D_4\frac{\partial\psi}{\partial y}+D_{15}\xi = 0 \qquad \text{or } \psi \text{ is prescribed} \quad (16c)$$

$$D_{11}\frac{\partial^2 w}{\partial x\partial y}-D_{16}\left(\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\right) = 0 \qquad \text{or } \phi \text{ is prescribed} \quad (16d)$$

$$D_5\frac{\partial\xi}{\partial x}+D_{16}\psi = 0 \qquad \text{or } \xi \text{ is prescribed} \quad (16e)$$

3. At Corners $(x=0,\ y=0),(x=0,\ y=b),(x=a,\ y=0)$ and $(x=a,\ y=b)$ the following condition hold:

$$D_{17}\frac{\partial^2 w}{\partial x\partial y}-D_{11}\left(\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\right) = 0 \qquad \text{or } w \text{ is prescribed} \quad (17)$$

where constants D_1 through D_{17} and I_1 through I_6 appeared in the governing equations and boundary conditions are given in Appendix A.

2.2 Superiority of the present theory

The present theory is a displacement-based refined theory, and refined shear deformation theories are known to be successful techniques for improving the accuracy of displacement and stresses [21]. The kinematics of the present theory is much richer than those of the higher order shear deformation theories available in the literature, because if the trigonometric term (involving thickness coordinate *z*) is expanded in power series, the kinematics of higher order theories (which are usually obtained by power series in thickness coordinate z) are implicitly taken into account to good deal of extent. Also, it needs to be noted that every additional power of thickness coordinate in the displacement field of other higher-order theories of Lo et al. [9, 10] type not only introduces additional unknown variables in those theories but these variables are also difficult to interpret physically [20]. Thus use of the sine term in the thickness coordinate (in the kinematics) enhances the richness of the theory, and also results in the reduction of the number of unknown variables as compared to other theories [35, 36] without loss of physics of the problem in modeling. The theory gives the realistic variation of transverse shear stress through the thickness of plate which is governed by a cosine-law distribution and satisfies the shear stress free boundary conditions on the top and bottom surfaces of the plate.The theory obviates the need of a shear correction factor due to the realistic variation of transverse shear stress. The present theory yields the exact value of dynamic shear correction factor $(\pi^2/12)$. Thus, the displacement field chosen is superior to those of others. The boundary value problem of the theory is derived using principle of virtual work; hence the present theory is variationally consistent.

3 ILLUSTRATIVE EXAMPLES

Simply supported isotropic square and rectangular plates occupying the region given by the Eq. (1) are considered for detail numerical study. The governing differential equations (14) in terms of displacement variables by setting the external transverse load *q* equal to zero are used. The associated boundary conditions for free vibration of plates under consideration can be obtained directly from Eqs. (8) and (9). The following boundary conditions are imposed at the simply supported edges:

$$w=\psi=\xi=M_x=M_x^s=0 \qquad \text{at} \quad x=0 \ \text{and} \ x=a \quad (18)$$

$$w = \phi = \xi = M_y = M_y^s = 0 \quad \text{at} \quad y=0 \text{ and } y=b \tag{19}$$

A solution to resulting governing equations, which satisfies the associated boundary conditions (time dependent), is of the form:

$$\begin{aligned}
w(x,y) &= \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\omega_{mn} t \\
\phi(x,y) &= \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \phi_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\omega_{mn} t \\
\psi(x,y) &= \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \psi_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \sin\omega_{mn} t \\
\xi(x,y) &= \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \xi_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\omega_{mn} t
\end{aligned} \tag{20}$$

where w_{mn} is the amplitude of translation and ϕ_{mn}, ψ_{mn} and ξ_{mn} are the amplitudes of rotation and $\bar{\omega}_{mn}$ is the natural frequency of *m*th and *n*th mode of vibration. Substitution of this solution form into the governing equations of free vibration of plate results in following standard eigen value problem.

$$\left([K] - \omega_{mn}^2 [M]\right)\{\Delta\} = 0 \tag{21}$$

where $[K]$ is the stiffness matrix, $[M]$ is the mass matrix and $\{\Delta\}$ is the vector of amplitudes of translation and rotation. The elements of these matrices are given in Appendix 2. From this solution lowest natural frequency for all modes of vibration can be obtained. Following material properties of plate given by Ghugal and Sayyad [31] are used.

$$E = 210,\ \mu = 0.3,\ G = \frac{E}{2(1+\mu)} \text{ and } \rho = 7800 \text{ Kg/m}^3$$

where *E* is the Young's modulus, *G* is the shear modulus, μ is the Poisson's ratio and ρ is density of the material.

4 NUMERICAL RESULTS

In this paper, free vibration analysis of simply supported square and rectangular plates for aspect ratio (side to thickness ratio, *a*/*h*) 10 is attempted. The simply supported plates considered are composed of isotropic material. The results obtained using trigonometric shear deformation theory are compared with exact results and with those of other refined theories available in literature. Following non-dimensional form is used for the purpose of presenting the results in this paper.

$$\bar{\omega}_{mn} = \omega_{mn} h \sqrt{\rho / G} \tag{22}$$

The non-dimensional frequency corresponding to flexural (bending) mode is denoted by $\bar{\omega}_w$ and frequencies corresponding to thickness shear modes are denoted by $\bar{\omega}_\phi$, $\bar{\omega}_\psi$ and that of thickness stretch mode is denoted by $\bar{\omega}_\xi$. The percentage error in results obtained using a particular model with respect to the results of exact elasticity solutions is calculated as follows:

$$\%\,\text{error} = \frac{\text{value by a particular theory} - \text{value by exact theory}}{\text{value by exact theory}} \times 100 \tag{23}$$

Table 1
Comparison of non-dimensional natural frequencies of isotropic square plate (b/a = 1) for aspect ratio 10.

(m, n)	Exact [17]			Present				Reddy [13]			Mindlin [5]			CPT [1]
	$\bar{\omega}_w$	ω_ϕ	$\bar{\omega}_\psi$	$\bar{\omega}_w$	$\bar{\omega}_\phi$	$\bar{\omega}_\psi$	$\bar{\omega}_\xi$	$\bar{\omega}_w$	ω_ϕ	$\bar{\omega}_\psi$	$\bar{\omega}_w$	ω_ϕ	$\bar{\omega}_\psi$	$\bar{\omega}_w$
(1, 1)	0.0932	3.1729	3.2465	0.0933	3.1729	3.2469	13.5360	0.0931	3.1749	3.2555	0.0930	3.1730	3.2538	0.0955
(1, 2)	0.2226	3.2192	3.3933	0.2231	3.2191	3.3940	13.5460	0.2219	3.2212	3.4125	0.2219	3.2193	3.4112	0.2360
(1, 3)	0.4171	3.2949	3.6160	0.4184	3.2949	3.6178	13.5517	0.4150	3.2969	3.6517	0.4149	3.2951	3.6510	0.4629
(2, 2)	0.3421	3.2648	3.5298	0.3431	3.2648	3.5312	13.5567	0.3406	3.2668	3.5589	0.3406	3.2650	3.5580	0.3732
(2, 3)	0.5239	3.3396	3.7393	0.5258	3.3396	3.7414	13.5626	0.5208	3.3415	3.7848	0.5206	3.3397	3.7842	0.5951
(2, 4)	0.7511	3.4414	4.0037	0.7542	3.4414	4.0082	13.5649	0.7453	3.4433	4.0720	0.7446	3.4416	4.0720	0.8926
(3, 3)	0.6889	3.4126	3.9310	0.6917	3.4126	3.9351	13.5694	0.6839	3.4145	3.9928	0.6834	3.4128	3.9926	0.8090
(4, 4)	1.0889	3.6094	4.4013	1.0945	3.6094	4.4102	13.5773	1.0785	3.6112	4.5092	1.0764	3.6096	4.5098	1.3716

Table 2
Comparison of non-dimensional natural frequencies of isotropic rectangular plate ($b/a = \sqrt{2}$) for aspect ratio 10 .

(m, n)	Exact [17]	Present				Reddy [13]			Mindlin [5]			CPT [1]
	$\bar{\omega}_w$	$\bar{\omega}_w$	$\bar{\omega}_\phi$	$\bar{\omega}_\psi$	$\bar{\omega}_\xi$	$\bar{\omega}_w$	ω_ϕ	$\bar{\omega}_\psi$	$\bar{\omega}_w$	ω_ϕ	$\bar{\omega}_\psi$	$\bar{\omega}_w$
(1, 1)	0.0704	0.0705	3.2212	3.1652	13.5284	0.0704	3.2283	3.1672	0.0703	3.2265	3.1652	0.0718
(1, 2)	0.1376	0.1393	3.1885	3.2973	13.5522	0.1374	3.1905	3.3094	0.1373	3.1885	3.3077	0.1427
(1, 3)	0.2431	0.2438	3.2270	3.4178	13.5548	0.2426	3.2289	3.4378	0.2424	3.2270	3.4365	0.2591
(1, 4)	0.3800	0.3811	3.2801	3.5750	13.5365	0.3789	4.0917	3.4506	0.3782	3.2801	3.6051	0.4182
(2, 1)	0.2018	0.2023	3.3704	3.2116	13.5388	0.2041	3.3873	3.2136	0.2012	3.3859	3.2116	0.2128
(2, 2)	0.2634	0.2642	3.4410	3.2346	13.5628	0.2628	3.4626	3.2366	0.2625	3.4613	3.2346	0.2821
(2, 3)	0.3612	0.3623	3.2725	3.5534	13.5655	0.3601	3.2745	3.5827	0.3595	3.2725	3.5817	0.3958
(2, 4)	0.4890	0.4906	3.3249	3.7013	13.5475	0.4874	3.2820	3.6060	0.4861	3.3249	3.7407	0.5513
(3, 1)	0.3987	0.3999	3.5966	3.2876	13.5445	0.3975	3.6291	3.2895	0.3967	3.6282	3.2876	0.4406
(3, 2)	0.4535	0.4550	3.6602	3.3100	13.5684	0.4520	3.6971	3.3120	0.4509	3.6963	3.3100	0.5073
(3, 3)	0.5411	0.5431	3.7622	3.3471	13.5715	0.5392	3.8065	3.3490	0.5375	3.8060	3.3471	0.6168

4.1 Discussion of numerical results

Results obtained for frequencies are compared and discussed with the corresponding results of classical, refined theories of various researchers and exact theory.

a) Bending frequency $(\bar{\omega}_w)$: Table 1 shows comparison of bending frequencies for higher modes when b/a=1.0 and a/h=10. It can be observed from Table 1 that the present theory yields excellent values of frequencies for all modes of vibration. The minimum % error predicted by present theory is 0.10 % when m=1, n=1 whereas maximum % error is 0.51 % when m=4, n=4. Theory of Reddy [17] underestimates the value of bending frequency by 0.10 % and 0.96 % when m=1, n=1 and m=4, n=4 respectively. Mindlin's theory [5] yields the lower values of bending frequency for all modes of vibration compared to those of higher order and exact theories, whereas classical plate theory (CPT) of Kirchhoff [1] yields the higher values for this frequency. Table 2 shows comparison of bending frequency for rectangular plate $(b/a = \sqrt{2})$. Results of present theory are in close agreement with those of exact results whereas Reddy's theory [13] shows exact value for the same when m=1, n=1. Mindlin's theory [5] underestimates it by 0.14 % and CPT overestimates it by 1.99 % when m=1, n=1 (fundamental mode).

b) Thickness shear mode frequency $(\bar{\omega}_\phi)$: From Table 1 it is observed that, for square plate (b/a=1.0) present theory shows exact result for thickness shear mode frequency for higher modes. Reddy's theory [13] yields the higher values of shear mode frequency compared to those of present and exact theories. Results obtained by Mindlin's theory [5] and CPT [1] are not satisfactory for higher modes. Results of frequency obtained for rectangular plate $(b/a=\sqrt{2})$ are shown in Table 2. The frequencies obtained by present theory and theory of Mindlin for this plate are more or less identical with each other. However, theory of Reddy predicts higher values of this frequency.

c) Thickness shear mode frequency $(\bar{\omega}_{\psi})$: For square plate (*b*/*a*=1.0) present theory shows good accuracy of result when *m*=1, *n*=1 whereas Reddy [13] and Mindlin's theory [5] overestimates the same by 0.28 % and 0.22 % respectively when *m*=1, *n*=1. In case of rectangular plate $(b/a=\sqrt{2})$ results obtained by present theory and Mindlin's theory [5] are identical for several modes as can be seen from Table 2.

The solution for the circular frequency of thickness shear motion (*m*=0, *n*=0) for infinitely long thin rectangular plate according to present theory is given by

$$\omega_{\phi}=\sqrt{\frac{K_{22}}{M_{22}}}=\sqrt{\frac{Gh}{2}\times\frac{2\pi^2}{\rho h^3}}=\sqrt{\frac{\pi^2}{12}\frac{Gh}{\rho I}}=\sqrt{K_d\frac{Gh}{\rho I}} \tag{24}$$

where $K_d=\pi^2/12$ is the dynamic shear correction factor given by trigonometric shear deformation theory which is matching with the exact value obtained by Lamb [37]. Thus the present theory yields the exact value for the dynamic shear correction factor which is a most important parameter in the dynamic analysis of plates.

d) Thickness stretch mode frequency $(\bar{\omega}_{\xi})$: Using the present theory, thickness stretch mode frequency can also be obtained as can be seen from Table 1 and 2. For comparison, values for this frequency are not available in the literature. The two dimensional theories which do not include the effect of transverse normal strain, do not provide these frequencies. Hence, the results of this frequency can serve as a benchmark solution for the purpose of comparison of results by other two dimensional plate theories.

5 CONCLUSIONS

In this paper, a variationally consistent trigonometric shear deformation theory is applied to free vibration of isotropic square and rectangular plates. The effects of transverse shear and transverse normal deformation are both included in the present theory. The theory gives realistic variation of transverse shear stress through the thickness of plate and satisfies the shear stress free boundary conditions on the top and bottom planes of the plate. The theory requires no shear correction factor. The results of frequencies of bending and thickness-shear motions are compared with exact frequencies and those of other higher order theories. It is observed that the frequencies obtained by present theory are in excellent agreement with the frequencies of exact theory. The present theory is capable to produce frequencies of thickness-stretch mode of vibration $(\bar{\omega}_{\xi})$. The theory yields the exact dynamic shear correction factor from the thickness shear motion which is a most important factor in the dynamic analysis of plates.

Appendix A

The constants appeared in the governing equations and boundary conditions are as under:

$$D_1=(\lambda+2G)\frac{h^3}{12},\quad D_2=(\lambda+2G)\frac{2h^3}{\pi^3},\quad D_3=\frac{2\lambda h^2}{\pi^2},\quad D_4=(\lambda+2G)\frac{h^3}{2\pi^2},\quad D_5=\frac{Gh^3}{2\pi^2},$$
$$D_6=\frac{Gh}{2},\quad D_7=(\lambda+G)\frac{h^3}{2\pi^2},\quad D_8=(\lambda+G)\frac{h^2}{2\pi},\quad D_9=(\lambda+G)\frac{h}{2},\quad D_{10}=(\lambda+4G)\frac{h^3}{12},$$
$$D_{11}=\frac{4Gh^3}{\pi^3},\quad D_{12}=\frac{\lambda h^3}{12},\quad D_{13}=\frac{2\lambda h^3}{\pi^3},\quad D_{14}=\frac{\lambda h^3}{2\pi^2},\quad D_{15}=\frac{\lambda h^2}{2\pi},\quad D_{16}=\frac{Gh}{2\pi},$$
$$D_{17}=\frac{Gh^3}{3},\quad I_1=\rho h,\quad I_2=\frac{\rho h^3}{12},\quad I_3=\frac{2\rho h^3}{\pi^3},\quad I_4=\frac{2\rho h^2}{\pi^2},\quad I_5=\frac{\rho h^3}{2\pi^2} \tag{A.1}$$

Appendix B

The elements of stiffness matrix $[K]$ are as under:

$$K_{11}=D_1\pi^4\left(\frac{m^4}{a^4}+\frac{2m^2n^2}{a^2b^2}+\frac{n^4}{b^4}\right),\quad K_{12}=-D_2\pi^3\left(\frac{m^3}{a^3}+\frac{mn^2}{ab^2}\right),\quad K_{13}=-D_2\pi^3\left(\frac{n^3}{b^3}+\frac{m^2n}{a^2b}\right),$$
$$K_{14}=-D_3\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right),\quad K_{21}=K_{12},\quad K_{22}=\left(D_7\frac{m^2\pi^2}{a^2}+D_{16}\frac{n^2\pi^2}{b^2}+D_6\right),\quad K_{23}=D_7\frac{mn\pi^2}{ab},$$
$$K_{24}=D_8\frac{m\pi}{a},\quad K_{31}=K_{13},\quad K_{32}=K_{23},\quad K_{33}=\left(D_7\frac{n^2\pi^2}{b^2}+D_{16}\frac{m^2\pi^2}{a^2}+D_6\right),\quad K_{34}=D_8\frac{n\pi}{b},$$
$$K_{41}=K_{14},\quad K_{42}=K_{24},\quad K_{43}=K_{34},\quad K_{44}=\left(D_{16}\frac{m^2\pi^2}{a^2}+D_{16}\frac{n^2\pi^2}{b^2}+D_9\right) \tag{B.1}$$

The elements of mass matrix $[M]$ are as under:

$$M_{11}=\left(I_2\frac{m^2\pi^2}{a^2}+I_2\frac{n^2\pi^2}{b^2}+I_1\right),\quad M_{12}=-I_3\frac{m\pi}{a},\quad M_{13}=-I_3\frac{n\pi}{b},\quad M_{14}=I_4,$$
$$M_{21}=M_{12},\quad M_{22}=I_5,\quad M_{23}=0.0,\quad M_{24}=0.0,\quad M_{31}=M_{13},\quad M_{32}=M_{23},$$
$$M_{33}=I_5,\quad M_{34}=0.0,\quad M_{41}=M_{14},\quad M_{42}=M_{24},\quad M_{43}=M_{34},\quad M_{44}=I_5$$

The vector $\{\Delta\}$ in Eq. (21) is defined as: $\{\Delta\}^{\mathrm{T}}=\{w_{mn}\ \phi_{mn}\ \psi_{mn}\ \xi_{mn}\}$..

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