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Analysis of Nonlinear Vibrations for Multi-walled Carbon Nanotubes Embedded in an Elastic Medium

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ABSTRACT

Nonlinear free vibration analysis of double-walled carbon nanotubes (DWCNTs) embedded in an elastic medium is studied in this paper based on classical (local) Euler-Bernoulli beam theory. Using the averaging method, the nonlinear free vibration responses of DWCNTs are obtained. The result is compared with the obtained results from the harmonic balance method for single-walled carbon nanotubes (SWCNTs) and DWCNTs. The effects of the surrounding elastic medium, van der waals (vdW) forces and aspect ratio of SWCNTs and DWCNTs on the vibration amplitude are discussed. The error percentage of the nonlinear free vibration frequencies between two theories decreases with increasing the spring constant of elastic medium. Results are also shown that if the value of the spring constant is lower than $10^7 N/m^3$ ($k < 10^7 N/m^3$), the nonlinear free vibration frequencies are increased. In this case, the effect of the spring constant on frequency responses is significant, while if the value of the spring constant is higher than $10^9 N/m^3$ ($k > 10^9 N/m^3$), the curve of frequency responses has a constant value near to 1 and therefore the effect of the spring constant on frequency responses is negligible.

Keywords: Nonlinear vibration; Nanotubes; Van der Waals; Euler-bernoulli beam; Natural frequency

1 INTRODUCTION

CARBON nanotubes (CNTs), which were discovered by Iijima [1], have widespread applications in different fields such as chemistry, physics, engineering, material science, and reinforced composite structures. The discovery of CNTs has been a very significant breakthrough that has accelerated further developments in the field of nanotechnology. They are among the most promising new materials and are expected to play an important role in the nanotechnology. Because of difficulties in experimental characterization of nanotubes, mechanical responses of CNTs are mainly investigated through numerical simulations using either atomistic or continuum models. Yoon et al. [2] investigated the resonant frequencies and the associated vibration modes of multi-walled carbon nanotubes (MWCNTs) embedded in an elastic medium based on the Winkler model. Fu et al. [3] studied the nonlinear free vibration of embedded MWCNTs considering intertube radial displacement and the related internal degrees of freedom by using the continuum mechanics. They obtained the amplitude frequency response curves for the nonlinear free vibration of SWCNTs and DWCNTs. Moreover, they investigated the effects of the surrounding elastic medium, the vdW forces and the aspect ratio of the MWCNTs on the amplitude frequency response

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characteristics. The small scale effect has a significant role on buckling analysis of CNTs and hence should be considered in the formulation. They also showed that the internal pressure increased the critical load, while the external pressure tended to decrease it. Using the Timoshenko beam model, Wang et al. [4] studied a free vibration analysis of MWCNTs. Moreover, they solved the Timoshenko governing equations for CNTs for different length-to-diameter ratio and boundary conditions using the differential quadrature method. Using the generalized shear deformation-beam theory, Aydogdu [5] presented the free vibration of simply supported MWCNTs. Unlike the Timoshenko beam theory, Aydogdu's theory satisfied the zero traction boundary conditions on the upper and lower surfaces of the structures. Therefore, there was no need to use a shear correction factor. Because of the importance of the shear deformation effect especially at higher modes, their predictions were slightly higher than those of the Timoshenko beam theory. Ke et al. [6] investigated nonlinear free vibration of DWCNTs embedded in an elastic medium based on Eringen's nonlocal elasticity theory and von- Karman geometric nonlinearity. They considered the effects of the transverse shear deformation and rotary inertia in Timoshenko beam theory. Moreover, using the Hamilton's principle, they derived the governing equations and boundary conditions and the differential quadrature (DQ) method is employed to discrete the nonlinear governing equations. Using the Euler-Bernoulli beam theory, Zhang et al. [7] developed transverse vibration of DWCNTs under compressive axial load. They obtained explicit expressions for natural frequencies and associated amplitude ratios of the inner to the outer tubes. Moreover, they showed that the effects of compressive axial load of the natural frequencies of DWCNTs are sensitive to the vibration modes and aspect ratios. Using the nonlinear Euler-Bernoulli microbeams theory, Xia et al. [8] investigated the static bending, post buckling and free vibration with an energy formulation. They considered the nonlinearity associated with the mean axial extension of the beam and the internal material length scale constant. Kuang et al. [9] presented the effect of the geometric nonlinearity and the nonlinearity of vdW force on the transverse vibration of the DWCNTs conveying fluid and the interaction between two types of nonlinearities. Using the Hamilton's principle, they deduced the nonlinear governing equations of the DWCNTs conveying fluid and discussed the effects of two types of nonlinearities on the coaxial and noncoaxial vibrations of the DWCNTs conveying fluid in numerical examples. Their results showed that the effect of geometric nonlinearity on the amplitude–frequency properties can be neglected if two types of nonlinearities are simultaneously considered. Natsuki et al. [10] used a theoretical analysis of the resonant vibration of DWCNTs embedded in an elastic medium based on Euler–Bernoulli beam model and Winkler spring model. Their results showed that the vibration modes of DWCNTs are quite different from those of SWCNTs and moreover found the resonant vibrations of DWCNTs to have inphase and anti-phase modes, in which the deflections of the inner and outer nanotubes occur in the same and opposite directions, respectively. It is seen from their results that for the vibration of DWCNTs with the same harmonic numbers, the resonant frequencies of anti-phase mode are larger than the ones of in-phase mode. Based on the Euler–Bernoulli beam model, Natsuki et al. [11] investigated to analyze the resonant vibration of DWCNTs with inner and outer nanotubes of different lengths. Also, they found that the resonant vibration is significantly affected by the vibrational modes of the DWCNTs, and by the lengths of the inner and outer nanotubes and for an inner or outer nanotube of constant length, the vibrational frequencies of the DWCNTs increase initially and then decrease as the length of another nanotube increases. Simsek [12] studied forced vibration of a simply supported SWCNT subjected to a moving harmonic load using nonlocal Euler–Bernoulli beam theory. They obtained the time-domain responses using both the modal analysis method and the direct integration method and also the effects of nonlocal parameter, aspect ratio, velocity and the excitation frequency of the moving load on the dynamic responses of SWCNT are discussed.

Ghorbanpour Arani et al. [13] studied the transverse vibrations of SWCNT and DWCNTs under axial load by applying the Euler–Bernoulli and Timoshenko beam models, as well as the Donnell shell model. They concluded that predictions from the Euler–Bernoulli beam model and the Donnell shell model have the lowest and highest accuracies, respectively. In order to predict the vibration behavior of the CNT more accurately, the current classical models were modified using the nonlocal theory. Moreover, they obtained the natural frequencies and amplitude coefficient for the simple supported boundary conditions.

To the best acknowledgement authors, the nonlinear free vibration analysis of embedded SWCNTs and DWCNTs using Euler-Bernoulli beam theory by averaging method has not been done in literatures.

Motivated by these considerations, the need for the development of a nonlinear free vibration analysis of SWCNTs and DWCNTs embedded in an elastic medium using averaging method is very much felt. Moreover, the effects of the surrounding elastic medium, van der waals (vdW) forces and aspect ratio of SWCNTs and DWCNTs on the vibration amplitude are discussed.

2 NONLINEAR FREE VIBRATION ANALYSIS FOR DWCNTs

The nonlinear free vibration of the SWCNTs, DWCNTs and MWCNTs were investigated by the averaging method [14]. The analysis of DWCNTs was conducted by the Euler-Bernoulli's beam model

2.1 Euler-Bernoulli-Beam model

As mentioned before, the Euler-Bernoulli' beam model was applied to the analysis the nonlinear free vibrations of CNTs [15]. It is assumed that the beam aspect ratio (diameter to the length of nanotubes) is smaller than one $(d/l << 1)$. For example, in this study, the aspect ratio of nanotubes is considered as $(0.02 < d/l < 0.1)$.

Using the Euler-Bernoulli beam model, we obtain a second degree equation of motion leads to a well-known Duffing equation solving by the averaging method.

2.2 Van der Waals force between the nanotubes

Fig. 1 illustrates a DWCNT embedded in an elastic medium. This figure shows a DWCNT of the outer radius r_2, the inner radius r_1, length *L*, and thickness *h*. Using the Leonard-Jones model, the vdW force between two atoms is estimated [16]. It is generally known that the interaction forces between the inner and outer tubes are equal in magnitude and are opposite in sign; therefore, this can be illustrated by the following [16]:

$$p_2^V(x,\theta)r_2 = -p_1^V(x,\theta)r_1 \tag{1}$$

where r_1 and r_2 is radii of the inner and outer tubes, p_1^V and p_2^V are the vdW pressures on the inner and outer tubes, respectively. Subscripts “1” and “2” refer to the inner and outer tubes, respectively. In analyzing of the extremely small lateral vibrations, the interaction pressure due to vdW forces between two adjacent tubes is expressed as $p_V(x,\theta)$ that is the linear function of space in that point[17].

$$P_V(x,\theta) = \left[\frac{\mathrm{d}g(\delta)}{\mathrm{d}\delta}\right]_{\delta=t} + c(\Delta w), \qquad c = \left.\frac{\mathrm{d}^2 g}{\mathrm{d}\delta^2}\right|_{\delta=t} \tag{2}$$

where $g(\delta)$ is a nonlinear function of the intertube spacing δ. In Eq. (2), (t) the inter layer distance is before the vibration and it is very close or equal to the thickness of SWCNTs. Δw is the difference of the radial displacement between two layers and *c* is the vdW interaction coefficient. Since the interlayer space is small (0.34 nm), thus $\mathrm{d}g(\delta)/\mathrm{d}\delta = 0$. The vdW interaction coefficient is assumed as follows [16]

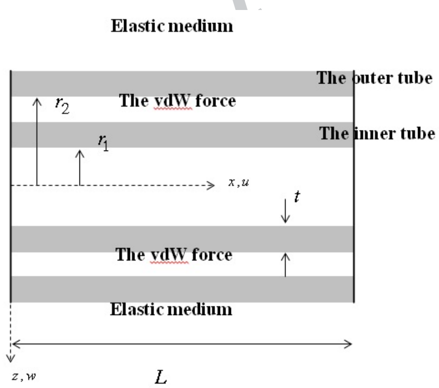

Fig. 1
A DWCNT embedded in an elastic medium.

$$c = \frac{320 \times 10^{-3}\ \text{j/m}^2}{0.16 d^2} = 9.918667 \times 10^{19}\ \text{N/m}^3, \qquad d = 1.42 \times 10^{-10}\ \text{m} \tag{3}$$

Therefore, the interaction pressure on the outer tubes after the vibration is calculated as follows:

$$P_2^V = P_0^V + c(w_1 - w_2) \tag{4}$$

where P_0^V is the vdW pressure before vibration. Using Eq. (1), the value of vdW pressure on the inner tube is obtained by the following equation:

$$P_1^V = -\frac{r_2}{r_1}\left[P_0^V + c(w_1 - w_2)\right] \tag{5}$$

2. 3 Pressure of elastic medium

The Winkler model is applied to measure the effects of the surrounding elastic medium on the external nanotubes in this model [18]. The initial pressure between the outer tube and the elastic medium is obtained as:

$$P_2^E = P_0^E \tag{6}$$

where, the P_0^E is as the linear function of axial strain[16]

$$P_0^E = c\varepsilon_{x0} \tag{7}$$

In Eq. (7), c is depended on the kind of external nanotubes material and the elastic medium. If the difference of nanotubes Poisson's ratio and the elastic medium is neglected, the axial strain will be uniform. In this case, there is no initial pressure between the external nanotube and the elastic medium.

$$P_0^E = 0 \tag{8}$$

After the vibration, the normal pressure in any point between the external nanotube and the elastic medium depends on the displacement of external nanotube in each point according to the Winkler model. Therefore:

$$P_2^E = P_0^E - kw_2 \tag{9}$$

k is the elastic medium constant based on the Winkler model [18]. The Whitney Riley's model was applied to determine the elastic constant. The elastic constant is obtained as [19, 20]

$$k = \frac{E_M}{(1+\mu_M)a}, \tag{10}$$

where k is the elastic medium constant, E_m and μ_m are the Young's modulus and Poisson's ratio in an elastic medium, respectively.

2.4 Free vibration analysis of DWCNTs

The couple equations for free vibration analysis of DWCNTs are expressed as [3]:

$$\begin{aligned}
&\frac{d^2W_1}{dt^2}+\left(\frac{\pi^4 EI_1}{l^4\rho S_1}+\frac{c}{\rho S_1}\right)W_1+\frac{\pi^4 E}{4l^4\rho}W_1^3-\frac{c}{\rho S_1}W_2=0,\\
&\frac{d^2W_2}{dt^2}+\left(\frac{\pi^4 EI_2}{l^4\rho S_2}+\frac{c}{\rho S_2}+\frac{k}{\rho S_2}\right)W_2+\frac{\pi^4 E}{4l^4\rho}W_2^3-\frac{c}{\rho S_2}W_1=0,
\end{aligned} \tag{11}$$

in which I_1 and I_2 are inertias of the inner and outer nanotubes, respectively, S_1 and S_2 are the cross-section of the inner and outer nanotubes, respectively, E is Young's modulus for CNTs and ρ is the density of CNTs. The dimensionless parameters are defined as

$$\begin{aligned}
&r=\sqrt{I_1/S_1},\qquad a_1=W_1/r,\qquad a_2=W_2/r,\qquad \omega_k=\sqrt{k/\rho S_1},\qquad \tau=\omega_t t,\\
&\omega_l=\pi^2/l^2\sqrt{EI_1/\rho S_1},\qquad \omega_c=\sqrt{c/\rho S_1},\qquad \omega_t^2=\omega_k^2+\omega_l^2
\end{aligned} \tag{12}$$

Substituting Eq. (12) into Eq. (11) yields:

$$\begin{aligned}
&\frac{d^2a_1}{d\tau^2}+\left[\left(\frac{\omega_l}{\omega_t}\right)^2+\left(\frac{\omega_c}{\omega_t}\right)^2\right]a_1+\frac{1}{4}\times\frac{\omega_l^2}{\omega_l^2+\omega_k^2}a_1^3-\left(\frac{\omega_c}{\omega_t}\right)^2a_2=0,\\
&\frac{d^2a_2}{d\tau^2}+\frac{S_1}{S_2}\left[\frac{I_2}{I_1}\times\left(\frac{\omega_l}{\omega_t}\right)^2+\left(\frac{\omega_c}{\omega_t}\right)^2+\left(\frac{\omega_k}{\omega_t}\right)^2\right]a_2+\frac{1}{4}\times\frac{\omega_l^2}{\omega_l^2+\omega_k^2}a_2^3-\frac{S_1}{S_2}\left(\frac{\omega_c}{\omega_t}\right)^2a_1=0,
\end{aligned} \tag{13}$$

Eq. (13) can be simplified to solve the problem by the averaging method.

$$\begin{aligned}
&\ddot a_1+\omega_{01}^2a_1=\varepsilon f_1(a_1,\dot a_1,\tau,\varepsilon)+\alpha_1a_2,\qquad \ddot a_2+\omega_{02}^2a_2=\varepsilon f_2(a_2,\dot a_2,\tau,\varepsilon)+\alpha_2a_1\ ,\\
&\omega_{01}^2\equiv\left(\frac{\omega_l}{\omega_t}\right)^2+\left(\frac{\omega_c}{\omega_t}\right)^2,\qquad \omega_{02}^2\equiv\frac{S_1}{S_2}\left[\frac{I_1}{I_2}\times\left(\frac{\omega_l}{\omega_t}\right)^2+\left(\frac{\omega_k}{\omega_t}\right)^2+\left(\frac{\omega_c}{\omega_t}\right)^2\right],\\
&\varepsilon\equiv\frac{1}{4}\times\frac{\omega_l^2}{\omega_l^2+\omega_k^2}<1,\qquad \alpha_1=\left(\frac{\omega_c}{\omega_t}\right)^2,\qquad \alpha_2=\frac{S_1}{S_2}\left(\frac{\omega_c}{\omega_t}\right)^2
\end{aligned} \tag{14}$$

If ε is equal to zero, thus Eq. (14) can be written as

$$\ddot a_1+\omega_{01}^2a_1=\alpha_1a_2,\qquad \ddot a_2+\omega_{02}^2a_2=\alpha_2a_1 \tag{15a}$$

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{Bmatrix}\ddot a_1\\\ddot a_2\end{Bmatrix}+\begin{bmatrix}\omega_{01}^2&-\alpha_1\\-\alpha_2&\omega_{02}^2\end{bmatrix}\begin{Bmatrix}a_1\\a_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \tag{15b}$$

The following harmonic motion equations can be written as

$$\begin{aligned}
&a_1(\tau)=A_1\cos(\omega\tau+\phi)\\
&a_2(\tau)=A_2\cos(\omega\tau+\phi)
\end{aligned} \tag{16}$$

where A_1 and A_2 are two real constants, ϕ is the phase angle. Substituting Eq. (16) into Eq. (15b) yields:

$$-\omega^2\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{Bmatrix}A_1\\A_2\end{Bmatrix}+\begin{bmatrix}\omega_{01}^2&-\alpha_1\\-\alpha_2&\omega_{02}^2\end{bmatrix}\begin{Bmatrix}A_1\\A_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \tag{17}$$

Therefore,

$$\begin{bmatrix} \omega_{01}^2-\omega^2 & -\alpha_1 \\ -\alpha_2 & \omega_{02}^2-\omega^2 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{18}$$

The values of natural frequencies of the DWCNTs can be obtained by solving Eq. (18)

$$(\omega^2)_{1,2}=\tfrac{1}{2}\left[(\omega_{01}^2+\omega_{02}^2)\pm\sqrt{(\omega_{01}^2-\omega_{02}^2)^2+4\alpha_1\alpha_2}\right] \tag{19}$$

If $\omega=\omega_1$, the amplitude ratios for first natural frequency is obtained as:

$$\left(\frac{A_2}{A_1}\right)_{\omega=\omega_1}=\frac{\omega_{01}^2-\omega_1^2}{\alpha_1} \tag{20}$$

Hence, as the frequency of system is ω_1 then the equations of motion can be simplified as**:**

$$a_1(\tau)=A_1\cos(\omega_1\tau+\phi_1),\qquad a_2(\tau)=\frac{\omega_{01}^2-\omega_1^2}{\alpha_1}A_1\cos(\omega_1\tau+\phi_1) \tag{21}$$

If $\omega=\omega_2$, the amplitude ratios for second natural frequency is expressed as:

$$\left(\frac{A_2}{A_1}\right)_{\omega=\omega_1}=\frac{\alpha_2}{\omega_{02}^2-\omega_2^2} \tag{22}$$

Therefore, when the frequency of the system is ω_2, the equations of motion can be simplified as:

$$a_1(\tau)=A_2\cos(\omega_2\tau+\phi_2),\qquad a_2(\tau)=\frac{\alpha_2}{\omega_{02}^2-\omega_2^2}A_2\cos(\omega_2\tau+\phi_2) \tag{23}$$

So the final answer can be obtained by:

$$\begin{aligned} a_1(\tau)&=A_1\cos(\omega_1\tau+\phi_1)+A_2\cos(\omega_2\tau+\phi_2) \\ a_2(\tau)&=\frac{\omega_{01}^2-\omega_1^2}{\alpha_1}A_1\cos(\omega_1\tau+\phi_1)+\frac{\alpha_2}{\omega_{02}^2-\omega_2^2}A_2\cos(\omega_2\tau+\phi_2) \end{aligned} \tag{24}$$

Therefore, if $\varepsilon=0$, then the values of the amplitude and phase angle become constant, then the solution of Eq. (14) expressed as Eq. (24), while if $\varepsilon\neq 0$, in this case, using the averaging method and with differentiation of $a_1(\tau)$ and $a_2(\tau)$ with respect to τ (Eq. (24)) , the following equations are obtained:

$$\begin{aligned} \dot{a}_1(\tau)&=-A_1\omega_1\sin(\omega_1\tau+\phi_1)-A_2\omega_2\sin(\omega_2\tau+\phi_2) \\ \dot{a}_2(\tau)&=-\frac{\omega_{01}^2-\omega_1^2}{\alpha_1}A_1\omega_1\sin(\omega_1\tau+\phi_1)-\frac{\alpha_2}{\omega_{02}^2-\omega_2^2}A_2\omega_2\sin(\omega_2\tau+\phi_2) \end{aligned} \tag{25}$$

υ_1 and υ_2 parameters are defined as:

$$\upsilon_1=\frac{\omega_{01}^2-\omega_1^2}{\alpha_1}=\frac{\alpha_2}{\omega_{02}^2-\omega_1^2},\qquad \upsilon_2=\frac{\alpha_2}{\omega_{02}^2-\omega_2^2}=\frac{\omega_{01}^2-\omega_2^2}{\alpha_1} \tag{26a}$$

Using Eqs. (24) and (26a), the following nonlinear equations are obtained as:

$$\begin{aligned}a_1(\tau)&=A_1(\tau)\cos(\omega_1\tau+\phi_1(\tau))+A_2(\tau)\cos(\omega_2\tau+\phi_2(\tau))\\ a_2(\tau)&=\upsilon_1A_1(\tau)\cos(\omega_1\tau+\phi_1(\tau))+\upsilon_2A_2(\tau)\cos(\omega_2\tau+\phi_2(\tau))\end{aligned} \tag{26b}$$

With differentiation of $a_1(\tau)$ and $a_2(\tau)$ with respect to τ, the following equation is obtained

$$\begin{aligned}\dot a_1(\tau)&=\dot A_1\cos(\omega_1\tau+\phi_1)+\dot A_2\cos(\omega_2\tau+\phi_2)-A_1\omega_1\sin(\omega_1\tau+\phi_1),\\ &\quad -A_2\omega_2\sin(\omega_2\tau+\phi_2)-A_1\dot\phi_1\sin(\omega_1\tau+\phi_1)-A_2\dot\phi_2\sin(\omega_2\tau+\phi_2)\\ \dot a_2(\tau)&=\upsilon_1\dot A_1\cos(\omega_1\tau+\phi_1)+\upsilon_2\dot A_2\cos(\omega_2\tau+\phi_2)-\upsilon_1A_1\omega_1\sin(\omega_1\tau+\phi_1),\\ &\quad -\upsilon_2A_2\omega_2\sin(\omega_2\tau+\phi_2)-\upsilon_1A_1\dot\phi_1\sin(\omega_1\tau+\phi_1)-\upsilon_2A_2\dot\phi_2\sin(\omega_2\tau+\phi_2)\end{aligned} \tag{27}$$

With differentiation of $\dot a_1(\tau)$ and $\dot a_2(\tau)$ with respect to τ yields the following equation

$$\begin{aligned}\ddot a_1(\tau)&\equiv-\dot A_1\omega_1\sin(\omega_1\tau+\phi_1)-\dot A_2\omega_2\sin(\omega_2\tau+\phi_2)-(\omega_1+\dot\phi_1)A_1\omega_1\cos(\omega_1\tau+\phi_1),\\ &\quad -(\omega_2+\dot\phi_2)A_2\omega_2\cos(\omega_2\tau+\phi_2),\\ \ddot a_2(\tau)&\equiv-\upsilon_1\dot A_1\omega_1\sin(\omega_1\tau+\phi_1)-\upsilon_2\dot A_2\omega_2\sin(\omega_2\tau+\phi_2)-(\omega_1+\dot\phi_1)\upsilon_1A_1\omega_1\cos(\omega_1\tau+\phi_1),\\ &\quad -(\omega_2+\dot\phi_2)\upsilon_2A_2\omega_2\cos(\omega_2\tau+\phi_2).\end{aligned} \tag{28}$$

Assuming $\psi_1=\omega_1\tau+\phi_1,\ \psi_2=\omega_2\tau+\phi_2$ and substituting Eq. (28) into Eq. (14), the following relations is obtained as

$$\begin{aligned}&-\dot A_1\omega_1\sin\psi_1-\dot A_2\omega_2\sin\psi_2-\dot\phi_1A_1\omega_1\cos\psi_1-\dot\phi_2A_2\omega_2\cos\psi_2=\varepsilon f_1(a_1,\dot a_1,\tau,\varepsilon)\\ &-\upsilon_1\dot A_1\omega_1\sin\psi_1-\upsilon_2\dot A_2\omega_2\sin\psi_2-\dot\phi_1\upsilon_1A_1\omega_1\cos\psi_1-\dot\phi_2\upsilon_2A_2\omega_2\cos\psi_2=\varepsilon f_2(a_2,\dot a_2,\tau,\varepsilon)\end{aligned} \tag{29}$$

Eqs. (26b) and (29) can be written as the following matrix form:

$$\begin{aligned}&\begin{bmatrix}\cos\psi_1 & -\sin\psi_1\\ -\omega_1\sin\psi_1 & -\omega_1\cos\psi_1\end{bmatrix}\begin{Bmatrix}\dot A_1\\ A_1\dot\phi_1\end{Bmatrix}+\begin{bmatrix}\cos\psi_2 & -\sin\psi_2\\ -\omega_2\sin\psi_2 & -\omega_2\cos\psi_2\end{bmatrix}\begin{Bmatrix}\dot A_2\\ A_2\dot\phi_2\end{Bmatrix}=\begin{Bmatrix}0\\ \varepsilon f_1(a_1,\dot a_1,\tau,\varepsilon)\end{Bmatrix}\\ &\begin{bmatrix}\upsilon_1\cos\psi_1 & -\upsilon_1\sin\psi_1\\ -\upsilon_1\omega_1\sin\psi_1 & -\upsilon_1\omega_1\cos\psi_1\end{bmatrix}\begin{Bmatrix}\dot A_1\\ A_1\dot\phi_1\end{Bmatrix}+\begin{bmatrix}\upsilon_2\cos\psi_2 & -\upsilon_2\sin\psi_2\\ -\upsilon_2\omega_2\sin\psi_2 & -\upsilon_2\omega_2\cos\psi_2\end{bmatrix}\begin{Bmatrix}\dot A_2\\ A_2\dot\phi_2\end{Bmatrix}=\begin{Bmatrix}0\\ \varepsilon f_2(a_2,\dot a_2,\tau,\varepsilon)\end{Bmatrix}\end{aligned} \tag{30}$$

where

$$\begin{aligned}\begin{Bmatrix}\dot A_1\\ A_1\dot\phi_1\end{Bmatrix}&=\frac{1}{\omega_1(\upsilon_2-\upsilon_1)}\begin{bmatrix}-\omega_1\cos\psi_1 & \sin\psi_1\\ \omega_1\sin\psi_1 & \cos\psi_1\end{bmatrix}\begin{Bmatrix}0\\ \varepsilon(f_2-\upsilon_2f_1)\end{Bmatrix}\\ \begin{Bmatrix}\dot A_2\\ A_2\dot\phi_2\end{Bmatrix}&=\frac{1}{\omega_2(\upsilon_1-\upsilon_2)}\begin{bmatrix}-\omega_2\cos\psi_2 & \sin\psi_2\\ \omega_2\sin\psi_2 & \cos\psi_2\end{bmatrix}\begin{Bmatrix}0\\ \varepsilon(f_2-\upsilon_1f_1)\end{Bmatrix}\end{aligned} \tag{31}$$

Extracting $\dot A_1,\dot A_2,\dot\phi_1,\dot\phi_2$ parameters of Eq. (31) and defining the *h* and *g* functions yield

$$\begin{cases} \dot{A}_1 = \dfrac{\sin\psi_1}{\omega_1(\upsilon_2-\upsilon_1)}\varepsilon(f_2-\upsilon_2 f_1) \equiv \varepsilon g_1(A_1,\phi_1,A_2,\phi_2,\tau,\varepsilon) = \varepsilon g_1(\psi_1,\psi_2,\varepsilon) \\ \dot{\phi}_1 = \dfrac{1}{A_1}\times\dfrac{\cos\psi_1}{\omega_1(\upsilon_2-\upsilon_1)}\varepsilon(f_2-\upsilon_2 f_1) \equiv \varepsilon h_1(A_1,\phi_1,A_2,\phi_2,\tau,\varepsilon) = \varepsilon h_1(\psi_1,\psi_2,\varepsilon) \end{cases} \tag{32}$$

and

$$\begin{cases} \dot{A}_2 = \dfrac{\sin\psi_2}{\omega_2(\upsilon_1-\upsilon_2)}\varepsilon(f_2-\upsilon_1 f_1) \equiv \varepsilon g_2(A_1,\phi_1,A_2,\phi_2,\tau,\varepsilon) = \varepsilon g_2(\psi_1,\psi_2,\varepsilon) \\ \dot{\phi}_2 = \dfrac{1}{A_2}\times\dfrac{\cos\psi_2}{\omega_2(\upsilon_1-\upsilon_2)}\varepsilon(f_2-\upsilon_1 f_1) \equiv \varepsilon h_2(A_1,\phi_1,A_2,\phi_2,\tau,\varepsilon) = \varepsilon h_2(\psi_1,\psi_2,\varepsilon) \end{cases} \tag{33}$$

Using the binary-integration of g_1, g_2, h_1, h_2 in the period of $\psi_1\left[0,2\pi\right]$, $\psi_2\left[0,2\pi\right]$ to obtain the average value.

$$\begin{aligned} g_{01} &\equiv avg\left\{g_1(\psi_1,\psi_2,\varepsilon)\right\} = a\left\{g_1\right\} = \frac{\int_0^{\bar{\psi}_1}\int_0^{\bar{\psi}_2} g_1(\psi_1,\psi_2)d\psi_1 d\psi_2}{\int_0^{\bar{\psi}_1}\int_0^{\bar{\psi}_2} d\psi_1 d\psi_2} \\ h_{01} &\equiv avg\left\{h_1(\psi_1,\psi_2,\varepsilon)\right\} = a\left\{h_1\right\} = \frac{\int_0^{\bar{\psi}_1}\int_0^{\bar{\psi}_2} h_1(\psi_1,\psi_2)d\psi_1 d\psi_2}{\int_0^{\bar{\psi}_1}\int_0^{\bar{\psi}_2} d\psi_1 d\psi_2} \end{aligned} \tag{34}$$

and

$$\begin{aligned} g_{02} &\equiv avg\left\{g_2(\psi_1,\psi_2,\varepsilon)\right\} = a\left\{g_2\right\} = \frac{\int_0^{\bar{\psi}_1}\int_0^{\bar{\psi}_2} g_2(\psi_1,\psi_2)d\psi_1 d\psi_2}{\int_0^{\bar{\psi}_1}\int_0^{\bar{\psi}_2} d\psi_1 d\psi_2} \\ h_{02} &\equiv avg\left\{h_2(\psi_1,\psi_2,\varepsilon)\right\} = a\left\{h_2\right\} = \frac{\int_0^{\bar{\psi}_1}\int_0^{\bar{\psi}_2} h_2(\psi_1,\psi_2)d\psi_1 d\psi_2}{\int_0^{\bar{\psi}_1}\int_0^{\bar{\psi}_2} d\psi_1 d\psi_2} \end{aligned} \tag{35}$$

Therefore,

$$\dot{A}_1 = \varepsilon a\left\{g_1\right\} = \varepsilon g_{01}, \qquad \dot{\phi}_1 = \varepsilon a\left\{h_1\right\} = \varepsilon h_{01} \tag{36}$$

and also

$$\dot{A}_2 = \varepsilon a\left\{g_2\right\} = \varepsilon g_{02}, \qquad \dot{\phi}_2 = \varepsilon a\left\{h_2\right\} = \varepsilon h_{02} \tag{37}$$

Choosing $f_1\left(a_1,\dot{a}_1,\tau,\varepsilon\right) = -a_1^3$, $f_2\left(a_2,\dot{a}_2,\tau,\varepsilon\right) = -a_2^3$ and substituting f_1 and f_2 into Eqs. (32) and (33), the following equations can be written as

$$g_1(\psi_1,\psi_2,\varepsilon) = \frac{\sin\psi_1}{\omega_1(\upsilon_2-\upsilon_1)}(-a_2^3+\upsilon_2 a_1^3), \qquad h_1(\psi_1,\psi_2,\varepsilon) = \frac{1}{A_1}\times\frac{\cos\psi_1}{\omega_1(\upsilon_2-\upsilon_1)}(-a_2^3+\upsilon_2 a_1^3) \tag{38}$$

and

$$g_2(\psi_1,\psi_2,\varepsilon)=\frac{\sin\psi_2}{\omega_2(\upsilon_1-\upsilon_2)}(-a_2^3+\upsilon_1 a_1^3), \qquad h_2(\psi_1,\psi_2,\varepsilon)=\frac{1}{A_2}\times\frac{\cos\psi_2}{\omega_2(\upsilon_{\cdot}-\upsilon_2)}(-a_2^3+\upsilon_1 a_1^3) \tag{39}$$

Substituting $a_1 = A_1\cos\psi_1$ and $a_1 = A_1\cos\psi_1$ into Eqs. (38) and (39) yield

$$\begin{aligned} g_1(\psi_1,\psi_2,\varepsilon)&=\frac{1}{\omega_1(\upsilon_2-\upsilon_1)}\left[-A_2^3\sin\psi_1(\frac{3}{4}\cos\psi_2+\frac{1}{4}\cos3\psi_2)+\upsilon_2 A_1^3(\frac{1}{4}\sin2\psi_1+\frac{1}{8}\sin4\psi_1)\right]\\ h_1(\psi_1,\psi_2,\varepsilon)&=\frac{1}{A_1}\times\frac{1}{\omega_1(\upsilon_2-\upsilon_1)}\left[-A_2^3\cos\psi_1(\frac{3}{4}\cos\psi_2+\frac{1}{4}\cos3\psi_2)+\upsilon_2 A_1^3(\frac{3}{8}+\frac{1}{2}\cos2\psi_1+\frac{1}{8}\cos4\psi_1)\right]\end{aligned} \tag{40}$$

and

$$\begin{aligned} g_2(\psi_1,\psi_2,\varepsilon)&=\frac{1}{\omega_2(\upsilon_1-\upsilon_2)}\left[-A_2^3(\frac{1}{4}\sin2\psi_2+\frac{1}{8}\sin4\psi_2)+\upsilon_1 A_1^3\sin\psi_2(\frac{3}{4}\cos\psi_1+\frac{1}{4}\cos3\psi_1)\right]\\ h_2(\psi_1,\psi_2,\varepsilon)&=\frac{1}{A_2}\times\frac{1}{\omega_2(\upsilon_1-\upsilon_2)}\left[-A_2^3(\frac{3}{8}+\frac{1}{2}\cos2\psi_2+\frac{1}{8}\cos4\psi_2)+\upsilon_1 A_1^3\cos\psi_2(\frac{3}{4}\cos\psi_1+\frac{1}{4}\cos3\psi_1)\right]\end{aligned} \tag{41}$$

Substituting Eqs. (40) and (41) into Eqs. (34) and (35), respectively and averaging in this interval $\{[0,2\pi],[0,2\pi]\}$, the following equations can be obtained as**:**

$$g_{01}=0, \qquad h_{01}=\frac{3}{8}\times\frac{\upsilon_2}{\omega_1(\upsilon_2-\upsilon_1)}A_1^2 \tag{42}$$

$$g_{02}=0, \qquad h_{02}=\frac{3}{8}\times\frac{1}{\omega_2(\upsilon_2-\upsilon_1)}A_2^2 \tag{43}$$

Substituting Eqs. (42) and (43) into Eqs. (38) and (39), respectively, yield:

$$\dot{A}_1=0, \qquad \dot{\phi}_1=\frac{3}{8}\times\frac{\upsilon_2}{\omega_1(\upsilon_2-\upsilon_1)}\varepsilon A_1^2 \tag{44}$$

$$\dot{A}_2=0, \qquad \dot{\phi}_2=\frac{3}{8}\times\frac{1}{\omega_2(\upsilon_2-\upsilon_1)}\varepsilon A_2^2 \tag{45}$$

Solving Eqs. (44) and (45), amplitude and phase angle for DWCNTs is obtained as

$$A_1(\tau)=A_1=\text{cte}, \qquad \phi_1(\tau)=\frac{3}{8}\times\frac{\upsilon_2}{\omega_1(\upsilon_2-\upsilon_1)}\varepsilon A_1^2\tau+\phi_{01} \tag{46}$$

$$A_2(\tau)=A_2=\text{cte}, \qquad \phi_2(\tau)=\frac{3}{8}\times\frac{1}{\omega_2(\upsilon_2-\upsilon_1)}\varepsilon A_2^2\tau+\phi_{02} \tag{47}$$

Substituting Eqs. (46) and (47) into Eq. (26b) yield the vibration response for DWCNTs

$$W_1(t) = A_1 r\cos[(\underbrace{\omega_1 + \frac{3}{8}\times\frac{\upsilon_2}{\omega_1(\upsilon_2-\upsilon_1)}\varepsilon A_1^2}_{\omega_I})\omega_t t + \phi_{01}] + A_2 r\cos[(\underbrace{\omega_2 + \frac{3}{8}\times\frac{1}{\omega_2(\upsilon_2-\upsilon_1)}\varepsilon A_2^2}_{\omega_{II}})\omega_t t + \phi_{02}]$$
$$W_2(t) = \upsilon_1 A_1 r\cos[(\underbrace{\omega_1 + \frac{3}{8}\times\frac{\upsilon_2}{\omega_1(\upsilon_2-\upsilon_1)}\varepsilon A_1^2}_{\omega_I})\omega_t t + \phi_{01}] + \upsilon_2 A_2 r\cos[(\underbrace{\omega_2 + \frac{3}{8}\times\frac{1}{\omega_2(\upsilon_2-\upsilon_1)}\varepsilon A_2^2}_{\omega_{II}})\omega_t t + \phi_{02}] \tag{48}$$

Eq. (48) shows that in nonlinear vibration, nanotubes vibration frequencies are not constant and depends on A_1, A_2, but in the linear vibration the vibration frequencies are constant and depend on the geometrical and physical data and the elastic medium of the carbon nanotubes (see Eq. (26)). Finally, substituting the values of υ_1, υ_2, α_1, α_2, ω_{01}, ω_{02} into Eqs. (19) and (48), respectively, yield vibration frequencies for DWCNTs:

$$\frac{\omega_I}{\omega_1\omega_t} = 1 + \frac{3}{8}\times\frac{\upsilon_2}{\omega_1^2(\upsilon_2-\upsilon_1)}\varepsilon A_1^2, \qquad \frac{\omega_{II}}{\omega_2\omega_t} = 1 + \frac{3}{8}\times\frac{1}{\omega_2^2(\upsilon_2-\upsilon_1)}\varepsilon A_2^2 \tag{49}$$

where $\omega_1\omega_t$ and $\omega_2\omega_t$ are the linear frequencies.

3 NUMERICAL RESULTS AND DISCUSSION

The following data for geometry and material properties are used for DWCNTS [3, 21, and 22]:

$$\begin{aligned} &E = 1.1\text{ Tpa}, \quad \rho = 1.3\times10^3\ \text{Kg/m}^3, \quad l = 45\text{ nm}, \quad h = 0.34\text{ nm}, \\ &S_1 = 2.115\times10^{-18}\,\text{m}^2, \quad S_2 = 2.841\times10^{-18}\,\text{m}^2, \quad I_1 = 1.067\times10^{-36}\,\text{m}^4, \\ &I_2 = 2.554\times10^{-36}\,\text{m}^4 \quad ,c = 0.3\times10^{12}\,\text{N/m}^2 \end{aligned} \tag{50}$$

Using Eq. (49) and defining $\min\ (\omega_1\omega_t, \omega_2\omega_t) = \omega_{lin}$, the nonlinear free vibration frequencies for DWCNTs is obtained as

$$\frac{\omega_I}{\omega_{lin}} = 1 + \frac{3}{8}\times\frac{\upsilon_2}{\omega_1^2(\upsilon_2-\upsilon_1)}\varepsilon A_1^2, \qquad \frac{\omega_{II}}{\omega_{lin}} = 1 + \frac{3}{8}\times\frac{1}{\omega_2^2(\upsilon_2-\upsilon_1)}\varepsilon A_2^2 \tag{51}$$

Substituting values of $I_1, I_2, S_1, S_2, \rho, \alpha$ into Eq. (22), the values of ω_{01}^2 and ω_{02}^2 are obtained in terms of *k*.

$$\begin{aligned} &\omega_{01}^2 = 1.014\times10^{22} + 1.091\times10^{26} \\ &\omega_{02}^2 = 1.806\times10^{22} + 0.812\times10^{26} + 2.708\times10^{14}\times k \end{aligned} \tag{52}$$

Using Eq. (19), the values of nonlinear natural frequencies for DWCNTs are obtained as follows:

$$\begin{aligned} (\omega^2)_{1,2} = \frac{1}{2}[&2.82\times10^{22} + 1.903\times10^{26} + 2.708\times10^{14}k \\ &\pm\sqrt{(0.279\times10^{26} - 0.792\times10^{22} - 2.708\times10^{14}k)^2 + 3.544\times10^{52}}\,] \end{aligned} \tag{53}$$

Substituting Eq. (53) into Eq. (51), Table 1. shows the nonlinear natural frequencies for the various values of *k* for the second layer of nanotubes.

Table 1
The nonlinear free frequencies of DWCNTs for the different values of spring constant k

k (N/m^2)	ω_1 (THz)	ω_2 (THz)	$\omega_{lin} = \min\{\omega_1, \omega_2\}$	$\omega_{II}/\omega_{lin} = func.(A_2)$
0	13.795	0.095	0.095	$1+1.455\times10^{17}A_2^2$
10^7	13.795	0.103	0.103	$1+0.855\times10^{17}A_2^2$
10^8	13.795	0.157	0.157	$1+0.368\times10^{17}A_2^2$
10^9	13.795	0.405	0.405	$1+0.055\times10^{17}A_2^2$

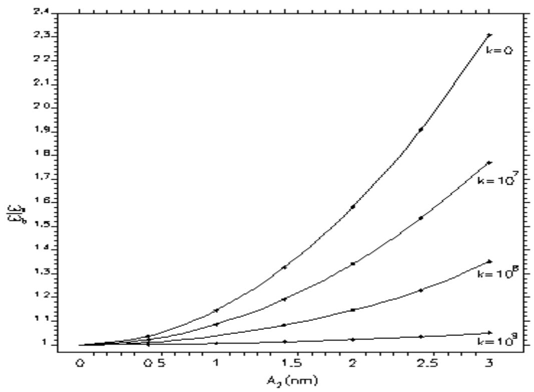

Fig. 2
The nonlinear free vibration frequency of DWCNTs for different values of elastic medium constant using averaging method.

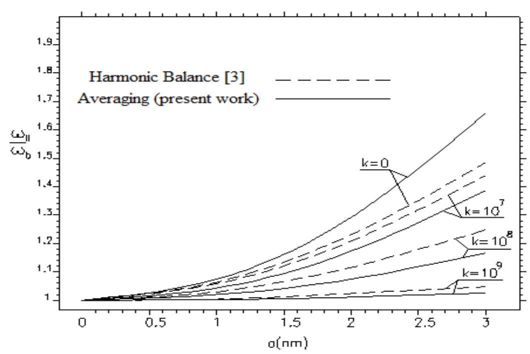

Fig. 3
The nonlinear free vibration frequencies of SWCNTs for different values of elastic medium constant using averaging and harmonic balance methods.

Fig. 2 indicates nonlinear free vibration frequency of DWCNTs versus the amplitude vibration for different values of elastic medium constant. It is seen that the values of nonlinear free vibration frequencies for DWCNTs decrease with increasing the value of elastic medium constant. If the value of the spring constant of elastic medium is higher than $10^9\,\mathrm{N/m}^3$ ($(k > 10^9\,\mathrm{N/m}^3)$), the curve of natural frequency response has a constant value near to 1 and therefore, the effect of the spring constant on frequency response is insignificant. In this case, the difference between the linear and nonlinear free frequency response is negligible. Fig. 3 illustrates the nonlinear natural frequency ratios versus the nondimentional amplitude $(a = A_2 / r)$ for different values of elastic medium constant k . In this study, using the averaging method, the nonlinear free vibration of amplitude and natural frequency response of SWCNTs are obtained. These results are compared with the obtained results from the harmonic balance method for SWCNTs [3].

Table 2. shows the error percentage of the nonlinear natural frequency ratios between the averaging and harmonic balance methods for SWCNTs. It is seen that the error percentage of the nonlinear free vibration frequencies between two theories decreases with increasing the elastic medium constant. Fig. 4 shows the nonlinear natural frequency ratios versus the non-dimensional amplitude $(a = A_2 / r)$ for different values of elastic medium constant k. In this study, using the averaging method, the nonlinear free vibration response of DWCNTs are obtained. These results are compared with the obtained results from the harmonic balance method for DWCNTs [3].

Table 2
Comparison of the nonlinear free vibration frequencies between the obtained results of averaging and harmonic balance methods for SWCNTs

k	0	10^7	10^8	10^9
Averaging method ω/ω_0 (present work)	1.829	1.717	1.332	1.050
Harmonic balance method ω/ω_0 [3]	1.66	1.583	1.285	1.047
Error	10.2% +	8.5%+	3.7%+	2.9%+

Table 3
Comparison of the nonlinear free vibration frequencies between the obtained results of averaging and harmonic balance methods

k	0	10^7	10^8	10^9
Averaging method ω/ω_0 (present work)	1.659	1.388	1.167	1.025
Harmonic balance method ω/ω_0 [3]	1.484	1.44	1.248	1.048
Error	11.7%+	-3.6%	-6.5%	2.1%-

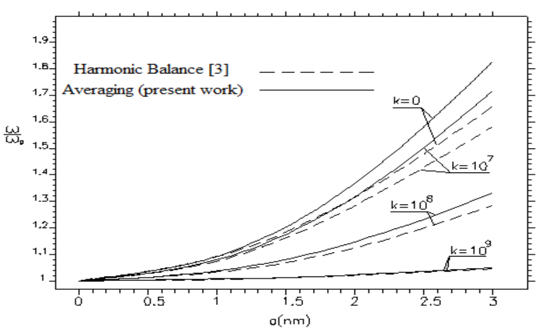

Fig.4
The nonlinear free vibration frequencies of DWCNTs for different values of elastic medium constant using averaging and harmonic balance methods.

The error percentage of the natural frequency ratios between the averaging and harmonic balance methods for DWCNTs is tabulated in Table 3. It is observed that the error percentage of the nonlinear free vibration frequencies between two theories decreases with increasing the elastic medium constant *k*.

4 CONCLUSIONS

Based on classical Euler-Bernoulli beam model, the nonlinear free vibration analysis of DWCNTs embedded in an elastic medium investigated in this study. Using the averaging method, the nonlinear free vibration responses of SWCNTs and DWCNTs obtained. Results compared with the obtained results from the harmonic balance method for SWCNTs and DWCNTs [3]. Moreover, it is seen from the results that the error percentage of the nonlinear free vibration frequencies between two theories decreases with increasing the spring constant of elastic medium. The error percentage of the nonlinear free vibration frequencies between two theories for DWCNTs, due to the vdW force between the inner and outer tubes for DWCNTs, is also lower than those SWCNTs.

The effects of the surrounding elastic medium, vdW forces and aspect ratio of the DWCNTs on the vibration amplitude discussed. It is seen from the obtained results that the spring constant of Winkler-type is noticeable on the frequency response of CNTs. If the value of the spring constant is lower than $10^7\,\mathrm{N/m^3}$ $(k<10^7\,\mathrm{N/m^3})$, the nonlinear free vibration frequencies are increased. Then, in this state, the effect of the spring constant on frequency response is significant. If the value of the spring constant is higher than $10^9\,\mathrm{N/m^3}$ $(k<10^9\,\mathrm{N/m^3})$, the curve of frequency response has a constant value near to 1 and in this case, the effect of the spring constant on the ratio the nonlinear vibration frequency to the linear vibration frequency is negligible. Thus, the nonlinear and linear free vibration solutions are near to each other for high values of spring constant.

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