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Wave Propagation and Fundamental Solution of Initially Stressed Thermoelastic Diffusion with Voids

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ABSTRACT

The present article deals with the study of propagation of plane waves in isotropic generalized thermoelastic diffusion with voids under initial stress. It is found that, for two dimensional model of isotropic generalized thermoelastic diffusion with voids under initial stress, there exists four coupled waves namely, P wave, Mass Diffusion (MD) wave, thermal (T) wave and Volume Fraction (VF) wave. The phase propagation velocities and attenuation quality factor of these plane waves are also computed and depicted graphically. In addition, the fundamental solution of system of differential equations in the theory of initially stressed thermoelastic diffusion with voids in case of steady oscillations in terms of elementary functions has been constructed. Some basic properties of the fundamental solution are established and some particular cases are also discussed.

Keywords: Plane waves; Fundamental solution; Initial stress; Thermoelastic diffusion with voids; Steady oscillations

1 INTRODUCTION

THERE are a number of theories which describe mechanical properties of porous materials, and one of them is a Biot consolidation theory of fluid-saturated porous solids [1,2]. These theories reduce to classical elasticity when the pore fluid is absent. Goodman and Cowin [3] established a continuum theory for granular materials, whose matrix material (or skeletal) is elastic and interstices are voids. They formulated this theory from the formal arguments of continuum mechanics and introduced the concept of distributed body, which represents a continuum model for granular materials (sand, grain, powder, etc.) as well as porous materials (rock, soil, sponge, pressed powder, cork, etc.). The basic concept underlying this theory is that the bulk density of the material is written as the product of two fields, the density field of the matrix material and the volume fraction field (the ratio of the volume occupied by grains to the bulk volume at a point of the material). This representation of the bulk density of the material introduces an additional kinematic variable in the theory. This idea of such representation of the bulk density was employed by Nunziato and Cowin [4] to develop a non-linear theory of elastic material with voids. Later on Cowin and Nunziato [5] developed a theory of linear elastic materials with voids for the mathematical study of the mechanical behaviour of porous solids. Iesan [6, 7] has developed a linear theory of thermoelastic material with voids by generalizing some ideas of the papers [8, 5, 9]. The theory of initially stressed thermoelastic material with voids is also given by Iesan [10].

During the last three decades, non-classical theories of thermoelasticity so called generalized thermoelasticity have been developed in order to remove the paradox of physically impossible phenomenon of infinite velocity of thermal signals in the conventional coupled thermoelasticity. Lord-Shulman theory [11] and Green-Lindsay theory

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[12] are important generalized theories of thermoelasticity that become center of interest of recent research in this area. The Lord and Shulman [11] theory of generalized thermoelasticity was further extended to homogeneous anisotropic heat conducting materials recommended by Dhaliwal and Sherief [13]. All these theories predict a finite speed of heat propagation. Nowacki [14-17] developed the theory of thermoelastic diffusion by using coupled thermoelastic model. Sherief and Saleh [18] investigated the problem of a thermoelastic half-space in the context of the theory of generalized thermoelastic diffusion with one relaxation time. Singh discussed the reflection phenomena of waves from free surface of a thermoelastic diffusion elastic solid with one relaxation time in [19] and with two relaxation times in [20]. Various authors [21-28] discussed different types of problems in thermoelastic diffusion. Aouadi [29] gives a theory of thermoelastic diffusion material with voids.

Initial stresses are developed in the medium due to many reasons, resulting from difference of temperature, process of quenching, shot pinning and cold working, slow process of creep, differential external of forces, gravity variations, etc. The Earth is supposed to be under high initial stresses. It is therefore of great interest to study the effect of these stresses on the propagation of stress waves. During the last five decades, considerable attenuation has been directed towards this phenomenon. Biot[30] depicted that the acoustic propagation under initial stresses would be fundamentally different from that under stress free state. Hetnarski [31, 32] was the first to study the fundamental solutions in the classical theory of coupled thermoelasticity. Iesan[33] presented the fundamental solution in the theory of thermoelasticity without energy dissipation. The fundamental solutions in the micro continuum fields theories were constructed by Svanadze [34-38]. The information related to fundamental solutions of differential equations is contained in the books of Hörmander [39, 40].

In this article, two dimensional wave propagation in isotropic generalized thermoelastic diffusion with voids under initially stress has been investigated. The phase propagation velocity and attenuation quality factor of plane waves have been computed and presented graphically. The fundamental solution of system of equations in the case of steady oscillations has also been considered in terms of elementary functions.

2 BASIC EQUATIONS

Let $x=(x_1,x_2,x_3)$ be the point of the Euclidean three-dimensional space E^3, $|x|=(x_1{}^2+x_2{}^2+x_3{}^2)^{1/2}$, $D_x=(\partial/\partial x_1,\partial/\partial x_2,\partial/\partial x_3)$ and let t denote the time variable. Following Lord and Shulman [11], Magana and Quintanilla [41], Aouadi [42], Iesan [10], the basic equations for homogeneous initially stressed generalized thermoelastic diffusion with voids material are:

Constitutive relations

$$t_{ij}=(c_{ijlm}+t^0_{jm}\,\delta_{il})u_{l,m}+D_{ijl}\,\phi_{,l}+B_{ij}\,\phi-\beta_{ij}(T+\tau_1\dot{T})-\gamma_{ij}(C+\tau^1\dot{C}), \tag{1}$$

$$h_i=A_{ij}\,\phi_{,j}+D_{lmi}\,u_{l,m}+f_i\,\phi-a_iT, \tag{2}$$

$$g=-\omega_0\,\dot{\phi}-\zeta\,\phi-B_{ij}\,u_{i,j}-f_i\,\phi_{,i}+b_1^*(T+\tau_1\dot{T})+b_2^*(C+\tau^1\dot{C}), \tag{3}$$

$$\rho T_0\eta=\rho C^*(T+\tau_t\dot{T})+T_0\beta_{ij}u_{i,j}+aT_0(C+\tau^c\dot{C})+b_1^*T_0\phi, \tag{4}$$

$$P=-\gamma_{ij}u_{i,j}-b_2^*\phi-a(T+\tau_1\dot{T})+d(C+\tau^1\dot{C}), \tag{5}$$

$$\rho T_0\dot{\eta}=-q_{i,i}\,, \tag{6}$$

$$\dot{C}=-\eta_{i,i}, \tag{7}$$

where $\phi(=\nu-\nu_0)$ is the volume fraction field and ν_0 is the matrix volume fraction at the reference state. T is the temperature measured from the absolute temperature $T_0\,(T_0\neq 0)$. We assume that T_0 and ν_0 are constants.

The equation of motion in absence of body force is

$$\rho\,\ddot{u}_i=t_{ij,j}\,, \tag{8}$$

The equation of volume fraction is

$$\rho\;\chi\,\ddot{\phi}=h_{i,i}\;+g+\rho\,l, \tag{9}$$

The equation of heat conduction is

$$q_i + m_0\tau_0\dot{q}_i = -k_{ij}T_{,j}\,, \tag{10}$$

The equation of chemical potential is

$$\eta_i + m_0\tau^0\dot{\eta}_i = -\alpha^*_{ij}P_{,j}\,, \tag{11}$$

where $c_{ijlm}, D_{ijl}, A_{ij}, B_{ij}, \omega_0, \zeta, f_i, \beta_{ij}, d, a, b_1^*, b_2^*, a_i, k_{ij}, \gamma_{ij}$ are the constitutive coefficients, ρ is the density, t_{ij} is the stress tensor, t^0_{jm} is the initial stress parameter, q_i is the heat flux, C is the concentration, η_i is the mass diffusion vector, P is the chemical potential per unit mass, C^* is the specific heat, η is the specific entropy, h_i is the equilibrated stress vector, χ is the equilibrated inertia, g is the intrinsic equilibrated body force and l is the extrinsic equilibrated body force. If the material symmetry is of a type that posses a center of symmetry then D_{ijk}, a_i and f_i are identically zero.

The general system of equations for anisotropic materials in absence of body force and extrinsic equilibrated body force are obtained by the substituting Eqs. (1)-(7) into Eqs. (8)-(11),

$$[(c_{ijlm} + t^0_{jm}\,\delta_{il})\,u_{l,m} + B_{ij}\,\phi - \beta_{ij}(T + \tau_1\dot{T}) - \gamma_{ij}(C + \tau^1\dot{C})]_{,j} = \rho\,\ddot{u}_i\,, \tag{12}$$

$$A_{ij}\phi_{,ij} - \omega_0\,\dot{\phi} - \zeta\,\phi - B_{ij}\,u_{i,j} + b_1^*(T + \tau_1\dot{T}) + b_2^*(C + \tau^1\dot{C}) = \rho\chi\ddot{\phi}, \tag{13}$$

$$(1 + \tau_0 m_0\frac{\partial}{\partial t})[\rho C^*(\dot{T} + \tau_t\ddot{T}) + T_0(b_1^*\dot{\phi} + \beta_{ij}\dot{u}_{i,j}) + aT_0(\dot{C} + \tau^c\ddot{C})] = k_{ij}T_{,ij}\,, \tag{14}$$

$$\alpha^*_{ij}[-\gamma_{ij}u_{i,j} - b_2^*\phi - a(T + \tau_1\dot{T}) + d(C + \tau^1\dot{C})]_{,ij} = (1 + m_0\tau^0\frac{\partial}{\partial t})\dot{C}, \tag{15}$$

Here, τ^0, τ^1 are diffusion relaxation times and τ_0, τ_1 are thermal relaxation times. For Lord and Shulman (LS) theory $\tau_t = \tau^c = \tau_1 = \tau^1 = 0,\ m_0 = 1,\ m^0 = 0$ and for Green and Lindsay (GL) theory $\tau_t = \tau_0,\ \tau^c = \tau^0,\ m_0 = 0,\ m^0 = 1.$ The thermal relaxation times τ_0 and τ_1 satisfy the inequality $\tau_1 \geq \tau > 0$ for GL-theory only. However, it has been proved by Sturnin [43] that the inequality is not mandatory for τ_0 and τ_1 to follow. In the above equations, a superposed dot denotes the derivative with respect to time.

In case of isotropic medium, we have

$$\begin{aligned} &c_{ijlm} = \lambda\delta_{ij}\delta_{lm} + \mu(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}),\ \ \beta_{ij} = \beta\delta_{ij},\ \ \gamma_{ij} = \gamma\delta_{ij},\ \ B_{ij} = B\delta_{ij},\ \ A_{ij} = A\delta_{ij}, \\ &k_{ij} = k\delta_{ij},\ \ \alpha^*_{ij} = \alpha^*\delta_{ij},\ \ t^0_{jm} = t^0\delta_{jm} \end{aligned} \tag{16}$$

The values of the coefficients from Eq. (16) put in Eqs. (12)-(15), we get

$$(\mu + t^0)\Delta\boldsymbol{u} + (\lambda + \mu)grad\,div\boldsymbol{u} + B\,grad\,\phi - \beta\ (1 + \tau_1\frac{\partial}{\partial t})\,gard\,T - \gamma\ (1 + \tau^1\frac{\partial}{\partial t})\,gard\,C = \rho\,\ddot{\boldsymbol{u}}, \tag{17}$$

$$(A\Delta - \zeta)\phi - \omega_0\dot{\phi} - B\,div\,\boldsymbol{u} + b_1^*(1 + \tau_1\frac{\partial}{\partial t})T + b_2^*(1 + \tau^1\frac{\partial}{\partial t})C = \rho\chi\ddot{\phi}, \tag{18}$$

$$(1 + \tau_0\frac{\partial}{\partial t})\rho C^*\dot{T} + \tau_{m0}\,\tau_c^{m0}aT_0\dot{C} + T_0\tau_{m0}(b_1^*\dot{\phi} + \beta\ \ div\,\dot{\boldsymbol{u}}) = k\Delta T\,, \tag{19}$$

$$\alpha^*\,\Delta[-\gamma\ \ div\,\boldsymbol{u} - b_2^*\phi - a(1 + \tau_1\frac{\partial}{\partial t})T + d(1 + \tau^1\frac{\partial}{\partial t})C] = \tau^{m0}\,\dot{C}, \tag{20}$$

and

$$\tau_{m0} = 1 + m_0\tau_0 \frac{\partial}{\partial t}, \qquad \tau^{m0} = 1 + m_0\tau^0 \frac{\partial}{\partial t}, \qquad \tau_c^{m0} = 1 + m^0\tau^0 \frac{\partial}{\partial t} \tag{21}$$

where $\beta_1 = (3\lambda + 2\mu)\alpha_t$ and $\gamma_1 = (3\lambda + 2\mu)\alpha_c$, λ, μ are Lame's constants, α_t is the coefficient of linear thermal expansion and α_t is the coefficient of linear diffusion expansion. $\boldsymbol{u} = (u_1, u_2, u_3)$ is the displacement vector and Δ is the Laplacian operator.

3 FORMULATION OF THE PROBLEM

We consider the medium of isotropic generalized thermoelastic diffusion with voids under initial stress. The origin of the Cartesian coordinate system (x_1, x_2, x_3) is taken at any point and x_3-axis taking vertically downward into the medium. We consider plane waves in the x_1x_3 – plane with wave front parallel to the x_2-axis. For two dimensional problem, we have

$$u = (u_1, 0, u_3) \tag{22}$$

We define the dimensionless quantities:

$$\begin{gathered} x_i' = \frac{\omega_1^* x_i}{c_1},\ t' = \omega_1^*,\ u_i' = \frac{\omega_1^* u_i}{c_1},\ \phi' = \frac{\omega_1^{*2}\chi\phi}{c_1^2},\ T' = \frac{\beta\ T}{\rho c_1^2},\ C' = \frac{\gamma C}{\rho c_1^2}, \\ \tau_0' = \omega_1^*\tau_0,\ \tau^{0'} = \omega_1^*\tau^0,\ \tau_1' = \omega_1^*\tau_1,\ \tau^{1'} = \omega_1^*\tau^1,\ \omega_1^* = \frac{C^*(\lambda + 2\mu)}{k},\ c_1^2 = \frac{\lambda + 2\mu}{\rho} \end{gathered} \tag{23}$$

Here, ω_1^* and c_1 are the characteristic frequency and longitudinal wave velocity in the medium respectively. Upon introducing the quantities (23) in the Eqs. (17)-(20) with the aid of (22) and after suppressing the primes, we obtain

$$\delta_1\Delta\boldsymbol{u} + \delta_2\, grad\, div\,\boldsymbol{u} + \delta_3\, grad\,\phi - \left(1 + \tau_1\frac{\partial}{\partial t}\right) grad\, T - \left(1 + \tau^1\frac{\partial}{\partial t}\right) grad\, C = \ddot{\boldsymbol{u}} \tag{24}$$

$$(\delta_4\Delta + \delta_5 + \delta_6\frac{\partial}{\partial t})\phi + \delta_7\, div\,\boldsymbol{u} + \delta_8\left(1 + \tau_1\frac{\partial}{\partial t}\right)T + \delta_9(1 + \tau^1\frac{\partial}{\partial t})C = \ddot{\phi} \tag{25}$$

$$\tau_{m0}(\delta_{10}\, div\,\dot{\boldsymbol{u}} + \delta_{11}\dot{\phi}) + \delta_{12}\left(1 + \tau_0\frac{\partial}{\partial t}\right)\dot{T} + \delta_{13}\tau_{m0}\,\tau^{m0}\dot{C} = K\Delta T \tag{26}$$

$$\Delta\left[\delta_{14}div\,\boldsymbol{u} + \delta_{15}\,\phi + \delta_{16}\left(1 + \tau_1\frac{\partial}{\partial t}\right)T + \delta_{17}\left(1 + \tau^1\frac{\partial}{\partial t}\right)C\right] = -\tau^{m0}\,\dot{C} \tag{27}$$

Here, $\Delta = \partial/\partial x_1^2 + \partial/\partial x_3^2$, $div\,\boldsymbol{u} = \partial u_1/\partial x_1 + \partial u_3/\partial x_3$ and value of all deltas are

$$\delta_1=\frac{t^0+\mu}{\lambda+2\mu},\ \delta_2=\frac{\lambda+\mu}{\lambda+2\mu},\ \delta_3=\frac{B}{\chi\rho\omega_1^{*2}},\ \delta_4=\frac{A}{\chi(\lambda+2\mu)},\ \delta_5=-\frac{\zeta}{\rho\chi\omega_1^*},\ \delta_6=-\frac{\omega_0}{\rho\chi\omega_1^*},$$
$$\delta_7=-\frac{B}{\lambda+2\mu},\ \delta_8=\frac{b_1^*}{\beta_1},\ \delta_9=\frac{b_2^*}{\gamma_1},\ \delta_{10}=\frac{T_0\beta^2}{\rho\omega_1^*},\ \delta_{11}=\frac{T_0b_1^*c_1^2\beta}{\rho\chi\omega_1^{*3}},\ \delta_{12}=\frac{\rho C^*c_1^2}{\omega_1^*},\tag{28}$$
$$\delta_{13}=\frac{aT_0c_1^2\beta}{\gamma\omega_1^*},\ \delta_{14}=\frac{\alpha^*\gamma^2\omega_1^*}{\rho c_1^4},\ \delta_{15}=\frac{\alpha^*\gamma b_2^*}{\rho c_1^2\chi\omega_1^*},\ \delta_{16}=\frac{\alpha^*a\gamma\omega_1^*}{\beta c_1^2},\ \delta_{17}=-\frac{d\alpha^*\omega_1^*}{c_1^2},\ \delta_{18}=1+\frac{t^0}{\lambda+2\mu}$$

We introduce the potential functions Φ and ϕ through the relations

$$u_1=\frac{\partial\Phi}{\partial x_1}-\frac{\partial\varphi}{\partial x_3},\qquad u_3=\frac{\partial\Phi}{\partial x_3}+\frac{\partial\varphi}{\partial x_1}\tag{29}$$

Substituting Eq. (29) in the Eqs. (24)-(27), we obtain

$$\delta_{18}\Delta\Phi+\delta_3\phi-\left(1+\tau_1\frac{\partial}{\partial t}\right)T-\left(1+\tau^1\frac{\partial}{\partial t}\right)C=\ddot{\Phi},\tag{30}$$

$$\Delta\varphi-\frac{1}{\delta_1}\ddot{\varphi}=0,\tag{31}$$

$$\left(\delta_4\Delta+\delta_5+\delta_6\frac{\partial}{\partial t}\right)\phi+\delta_7\Delta\Phi+\delta_8\left(1+\tau_1\frac{\partial}{\partial t}\right)T+\delta_9\left(1+\tau^1\frac{\partial}{\partial t}\right)C=\ddot{\phi},\tag{32}$$

$$\tau_{m0}(\delta_{10}\Delta\dot{\Phi}+\delta_{11}\dot{\phi})+\delta_{12}\left(1+\tau_0\frac{\partial}{\partial t}\right)\dot{T}+\delta_{13}\tau_{m0}\tau_c^{m0}\dot{C}=k\Delta T,\tag{33}$$

$$\Delta[\delta_{14}\Delta\Phi+\delta_{15}\phi+\delta_{16}\left(1+\tau_1\frac{\partial}{\partial t}\right)T+\delta_{17}\left(1+\tau^1\frac{\partial}{\partial t}\right)C]=-\tau^{m0}\dot{C}\tag{34}$$

The Eq. (31) corresponds to transverse wave that decouples from rest of the motion, and is not affected by the thermal, voids and diffusion parameters.

4 PLANE WAVE PROPAGATION

For plane harmonic waves, we assume the solution of the form

$$(\Phi,\phi,T,C)=(\overline{\Phi},\overline{\phi},\overline{T},\overline{C})\,e^{i(\xi(n_1x_1+n_3x_3)-\omega t)}\tag{35}$$

where ω is the angular frequency and ξ is the complex wave number and $\vec{n}=(n_1,0,n_3)$ is the unit propagation vector. $\overline{\Phi},\overline{\phi},\overline{T}$ and $\overline{C}$ are the undetermined amplitude vectors that are independent of time t and coordinates $x_m(m=1,3)$. The unit propagation vector n_1 and n_2 onto the x_1x_3-plane have the property $n_1^2+n_3^2=1$.

Using Eq. (35) in the Eqs. (30), (32)-(34), we obtain

$$[\omega^2-(\delta_{18}\xi^2)]\overline{\Phi}+\delta_3\overline{\phi}-\tau_{22}\overline{T}-\tau_{33}\overline{C}=0,\tag{36}$$
$$-\delta_7\xi^2\overline{\Phi}+(\delta_5-\xi^2\delta_4-i\omega\delta_6+\omega^2)\overline{\phi}+\delta_8\tau_{22}\overline{T}+\delta_9\tau_{33}\overline{C}=0,\tag{37}$$
$$-\xi^2\tau_{44}\delta_{10}\overline{\Phi}+\delta_{11}\tau_{44}\overline{\phi}+(k\xi^2+\delta_{12}\tau_{55})\overline{T}+\delta_{13}\tau_{66}\overline{C}=0,\tag{38}$$
$$-\xi^4\delta_{14}\overline{\Phi}+\xi^2\delta_{15}\overline{\phi}+\xi^2\tau_{22}\delta_{16}\overline{T}+(\delta_{17}\tau_{33}\xi^2-\tau_{77})\overline{C}=0,\tag{39}$$

where

$$\tau_{22} = 1 - \mathrm{i}\omega\tau_1, \qquad \tau_{33} = 1 - \mathrm{i}\omega\tau^1, \qquad \tau_{44} = -\mathrm{i}\omega(1 - i\omega\tau_0 m_0), \qquad \tau_{55} = -\mathrm{i}\omega(1 - i\omega\tau_0),$$
$$\tau_{66} = \tau_{44}(1 - \mathrm{i}\omega\tau^0 m^0), \qquad \tau_{77} = -\mathrm{i}\omega(1 - \mathrm{i}\omega\tau^0 m_0) \tag{40}$$

The system of the Eqs. (36)-(39) has a non-trivial solution if the determinant of the coefficients $\left[\bar{\Phi}, \bar{\phi}, \bar{T}, \bar{C}\right]^{\mathrm{tr}}$ vanishes i.e.

$$\begin{vmatrix} \omega^2 - \delta_{18}\xi^2 & \delta_3 & -\tau_{22} & -\tau_{33} \\ -\delta_7\xi^2 & (\delta_5 - \xi^2\delta_4 - i\omega\delta_6 + \omega^2) & \delta_8\tau_{22} & \delta_9\tau_{33} \\ -\tau_{44}\xi^2\delta_{10} & \delta_{11}\tau_{44} & \delta_{12}\tau_{55} + K\xi^2 & \delta_{13}\tau_{66} \\ -\xi^4\delta_{14} & \xi^2\delta_{15} & \xi^2\tau_{22}\delta_{16} & \delta_{17}\tau_{33}\xi^2 - \tau_{77} \end{vmatrix} = 0 \tag{41}$$

Solving the determinant of Eq. (41), a quartic equation in c^2 is obtained that can be written as,

$$F_1^* c^8 + F_2^* c^6 + F_3^* c^4 + F_4^* c^2 + F_5^* = 0 \tag{42}$$

where

$$F_1^* = -\omega^2(\delta_5 - i\omega\delta_6 + \omega^2)\tau_{77}\tau_{55}\delta_{12} + \omega^2\tau_{77}\tau_{22}\tau_{44}\delta_{11}\delta_8,$$
$$F_2^* = \omega^2(\omega^2(\delta_5 - i\omega\delta_6 + \omega^2)(\tau_{33}\tau_{55}\delta_{12}\delta_{17} - k\tau_{77} - \tau_{22}\tau_{66}\delta_{13}\delta_{16}) + (\delta_5 - i\omega\delta_6 + \omega^2)\delta_8\delta_{12}\tau_{55}\tau_{77}$$
$$- \omega^2\delta_8\tau_{22}(\delta_{11}\delta_{17}\tau_{33}\tau_{44} - \delta_{15}\delta_{13}\tau_{66}) - \delta_8\delta_{11}\delta_{18}\tau_{22}\tau_{44}\tau_{77} + \omega^2\delta_9\tau_{33}(\delta_{11}\delta_{16}\tau_{22}\tau_{44} - \delta_{12}\delta_{15}\tau_{55})$$
$$+ \delta_3(\delta_{12}\tau_{55}\tau_{77} + \delta_{10}\delta_{13}\tau_{22}\tau_{66}) + \delta_{10}\delta_3\delta_8\tau_{44}\tau_{22}\tau_{77} - \delta_7\delta_{11}\tau_{44}\tau_{22}\tau_{77} + \delta_{10}\tau_{44}\tau_{22}\tau_{77}(\delta_5 - i\omega\delta_6 + \omega^2)),$$
$$F_3^* = \omega^4(-\omega^2\delta_4(\tau_{33}\tau_{55}\delta_{12}\delta_{17} - k\tau_{77} - \tau_{22}\tau_{66}\delta_{13}\delta_{16}) + \omega^2(\delta_5 - i\omega\delta_6 + \omega^2)\delta_{17}k\tau_{33} - \delta_{18}(\delta_5 - i\omega\delta_6 + \omega^2)$$
$$(\tau_{33}\tau_{55}\delta_{12}\delta_{17} - k\tau_{77} - \tau_{22}\tau_{66}\delta_{13}\delta_{16}) - \tau_{55}\tau_{77}\delta_{12}\delta_{14}\delta_{18} - k\tau_{33}\delta_9\delta_{15} + \delta_3\delta_7(\delta_{12}\tau_{55}\tau_{77} + \delta_{10}\delta_{13}\tau_{22}\tau_{66})$$
$$+ \tau_{22}\delta_3\delta_8(\delta_{13}\delta_{14}\tau_{66} - \delta_{10}\delta_{17}\tau_{33}\tau_{44}) - \tau_{33}\delta_3\delta_9(\delta_{12}\delta_{14}\tau_{55} - \delta_{10}\delta_{10}\tau_{22}\tau_{44}) + \tau_{22}\delta_7(-\delta_{13}\delta_{15}\tau_{66} + \delta_{11}\delta_{17}\tau_{33}\tau_{44})$$
$$- \tau_{22}(\delta_5 - i\omega\delta_6 + \omega^2)(\delta_{13}\delta_{14}\tau_{66} + \delta_{10}\delta_{17}\tau_{33}\tau_{44}) - \tau_{22}\tau_{44}\tau_{77}\delta_4\delta_{10} - \tau_{22}\tau_{33}\delta_9(\tau_{44}\delta_{11}\delta_{14} - \tau_{44}\delta_{15}\delta_{10})$$
$$- \tau_{33}\delta_7(\tau_{44}\delta_{11}\delta_{16} - \tau_{55}\delta_{15}\delta_{12}) - \tau_{33}(\delta_5 - i\omega\delta_6 + \omega^2)(\delta_{12}\delta_{14}\tau_{55} - \delta_{10}\delta_{16}\tau_{22}\tau_{44}) + \tau_{22}\tau_{33}\delta_8(\tau_{44}\delta_{11}\delta_{14} - \tau_{44}\delta_{15}\delta$$
$$F_4^* = \omega^6(-\delta_{18}(\delta_5 - i\omega\delta_6 + \omega^2)\delta_{17}k\tau_{33} + \tau_{33}\delta_9\delta_{15}\delta_{18}k - \omega^2\tau_{33}\delta_4\delta_{17}k + \tau_{33}\delta_3\delta_7\delta_{17}k - \tau_{33}\delta_3\delta_9\delta_{14}k$$
$$+ \tau_{22}\delta_4(\delta_{13}\delta_{14}\tau_{66} + \delta_{10}\delta_{17}\tau_{33}\tau_{44}) + \tau_{33}\delta_7\delta_{15}k - (\delta_5 - i\omega\delta_6 + \omega^2)\delta_{14}k\tau_{33} + \tau_{33}\delta_4(\delta_{12}\delta_{14}\tau_{55} - \delta_{10}\delta_{16}\tau_{22}\tau_{44})),$$
$$F_5^* = \omega^8 k\delta_4\delta_{17}\delta_{18}\tau_{33}, \qquad \omega = \xi c$$

Eq. (42) is quartic in c^2, therefore the roots of this equation gives four values of c^2. Each value of c^2 corresponds to a velocity of propagation of four possible waves. The waves with velocity c_j (j=1, 2, 3, 4) corresponds to four type of waves. The complex coefficients $F_1^* - F_5^*$ in Eq. (42) implies that four roots of this equation may be complex. The complex velocity of wave 'j', i.e., $c_j = (c_R + ic_I)$, j=1,…4, define the phase propagation velocity $V_j = (c_R^2 + c_I^2)/c_R$ and the attenuation quality factor $Q_j^{-1} = -2c_I/c_R$ for the corresponding wave [44]. Therefore, the four waves in such a medium are attenuating waves. Let we name these four waves corresponding to descending order of their phase velocities, namely a P wave, a Mass Diffusion(MD) wave, a Thermal(T) wave and Volume Fraction(VF) wave.

5 STEADY OSCILLATIONS

Now we consider the case of steady oscillations. We assume the displacement vector, volume fraction,temperature change and concentration functions as

$$(u(x,t),\phi(x,t),\, T(x,t),\, C(x,t)) = (u,\phi,T,C)\,e^{-i\omega t} \tag{43}$$

Using Eq. (43) into Eqs. (17)-(20), we obtain the system of equations of steady oscillations as

$$\delta_1^* \Delta \boldsymbol{u} + \delta_2^* \operatorname{grad}\operatorname{div}\boldsymbol{u} + B \operatorname{grad}\phi - \beta\tau_{22} \operatorname{grad} \mathrm{T} - \gamma\tau_{33} \operatorname{grad} \mathrm{C} + \rho\omega^2 \boldsymbol{u} = 0 \tag{44}$$

$$-B \operatorname{div}\boldsymbol{u} + (\delta_3^* + A\Delta)\phi + b_1^*\tau_{22}\,\mathrm{T} + b_2^*\tau_{33}\,\mathrm{C} = 0 \tag{45}$$

$$\delta_4^* \operatorname{div}\boldsymbol{u} + \delta_5^*\,\phi - (\delta_6^* + K\Delta)\mathrm{T} + \delta_7^*\,\mathrm{C} = 0 \tag{46}$$

$$\delta_8^* \Delta \operatorname{div}\boldsymbol{u} - \delta_9^*\,\Delta\,\phi + \delta_{10}^*\,\Delta\,\mathrm{T} + (\delta_{11}^*\Delta - \tau_{77})\mathrm{C} = 0 \tag{47}$$

where

$$\begin{gathered}\delta_1^* = \mu + t^0,\ \ \delta_2^* = \mu + \lambda,\ \ \delta_3^* = \rho\chi\omega^2 + i\omega\omega_o - \varsigma,\ \ \delta_4^* = \beta T_0\tau_{44},\ \ \delta_5^* = b_1^* T_0 \tau_{44},\\ \delta_6^* = -\rho C^* \tau_{55},\ \ \delta_7^* = aT_0\tau_{66},\ \ \delta_8^* = -\alpha^*\gamma,\ \ \delta_9^* = \alpha^* b_2^*\ ,\ \ \delta_{10}^* = -\alpha^* a\tau_{22},\ \ \delta_{11}^* = \alpha^* d\tau_{33}\end{gathered} \tag{48}$$

We introduce the matrix differential operator

$$F(D_x) = \|F_{mn}(D_x)\|_{6\times 6}$$

where

$$F_{mn}(D_x) = [\delta_1^*\Delta + \rho\omega^2]\delta_{mn} + \delta_2^* \frac{\partial^2}{\partial x_m \partial x_n},\qquad F_{m4}(D_x) = B\frac{\partial}{\partial x_m},\qquad F_{m5}(D_x) = -\beta_1\tau_{22}\frac{\partial}{\partial x_m},$$

$$F_{m6}(D_x) = -\gamma_1\tau_{33}\frac{\partial}{\partial x_m},\qquad F_{4n}(D_x) = -B\frac{\partial}{\partial x_n},\qquad F_{44}(D_x) = \delta_3^* + A\Delta,\qquad F_{45} = b_1^*\tau_{22},$$

$$F_{46} = b_2^*\tau_{33},\qquad F_{5n}(D_x) = \delta_4^*\frac{\partial}{\partial x_n},\qquad F_{54}(D_x) = \delta_5^*,\qquad F_{55}(D_x) = -(\delta_6^* + K\Delta),$$

$$F_{56}(D_x) = \delta_7^*,\qquad F_{6n}(D_x) = \delta_8^*\Delta\frac{\partial}{\partial x_n},\qquad F_{64}(D_x) = -\delta_9^*\Delta,\qquad F_{65}(D_x) = \delta_{10}^*\Delta,$$

$$F_{66}(D_x) = \delta_{11}^*\Delta - \tau_{77},\qquad m,n = 1,2,3$$

and δ_{mn} is Kronecker delta. The system of Eqs. (44)-(47) can be written as

$$F(D_x)U(x) = 0$$

where U=(u,φ,T,C) is a six-component vector function on E^3.

Definition: The fundamental solution of the system of Eqs. (44) - (47) (the fundamental matrix of operator F) is the matrix G(x)=II G_{mn}(x) II_{6x6} satisfying condition[40]

$$F(D_x)G(x) = \delta(x)\ I(x) \tag{49}$$

where δ is the Dirac delta and $I = \|\delta_{mn}\|_{6x6}$ is the unit matrix and $x\ \varepsilon\ E^3$. Now we construct $G(x)$ in terms of elementary functions.

6 FUNDAMENTAL SOLUTION OF SYSTEM OF EQUATIONS OF STEADY OSCILLATIONS

We consider the system of equations

$$\delta_1^*\Delta \boldsymbol{u}+\delta_2^* \operatorname{grad}\operatorname{div}\boldsymbol{u}-\boldsymbol{B}\operatorname{grad}\phi+\delta_4{}^*\operatorname{grad}\mathrm{T}+\delta_8{}^*\Delta\operatorname{grad}\mathrm{C}+\rho\,\omega^2\boldsymbol{u}=\boldsymbol{H} \quad (50)$$

$$B\operatorname{div}\boldsymbol{u}+(\delta_3^*+A\Delta)\phi+\delta_5{}^*\mathrm{T}-\delta_9{}^*\Delta\mathrm{C}=Z \quad (51)$$

$$-\beta_1\tau_{22}\operatorname{div}\boldsymbol{u}+b_1^*\tau_{22}\phi-(\delta_6^*+K\Delta)\mathrm{T}+\delta_{10}{}^*\Delta\mathrm{C}=L \quad (52)$$

$$-\gamma_1\tau_{33}\Delta\operatorname{div}\boldsymbol{u}+b_2^*\tau_{33}\Delta\phi+\delta_7^*\Delta\mathrm{T}+(\delta_{11}^*\Delta-\tau_{77})\mathrm{C}=M \quad (53)$$

where H is three-component vector function on E^3; Z, L and M are scalar functions on E^3. The system of Eqs. (50)-(53) may be written in the form

$$F^{tr}(D_x)U(x)=Q(x) \quad (54)$$

where F^{tr} is the transpose of matrix F, Q=(H, Z, L, M) and $x\,\varepsilon\ E^3$. Applying the operator div to Eq. (50), we obtain

$$\begin{aligned}
&(\delta_9{}^{**}\Delta+\rho\omega^2)div\,\boldsymbol{u}-B\Delta\phi+\delta_4{}^*\Delta T+\delta_8{}^*\Delta C=div\,\mathbf{H}\\
&B\operatorname{div}\boldsymbol{u}+(\delta_3^*+A\Delta)\phi+\delta_5{}^*\mathrm{T}-\delta_9{}^*\Delta\mathrm{C}=Z\\
&-\beta_1\tau_{22}\operatorname{div}\boldsymbol{u}+b_1^*\tau_{22}\phi-(\delta_6^*+K\Delta)\mathrm{T}+\delta_{10}{}^*\Delta\mathrm{C}=L\\
&-\gamma_1\tau_{33}\Delta\operatorname{div}\boldsymbol{u}+b_2^*\tau_{33}\Delta\phi+\delta_7^*\Delta\mathrm{T}+(\delta_{11}^*\Delta-\tau_{77})\mathrm{C}=M
\end{aligned} \quad (55)$$

where

$$\delta_9{}^{**}=\lambda+2\mu+t^0$$

The system of Eqs. (55) may be written in the form

$$\boldsymbol{N}(\Delta)\boldsymbol{S}=\boldsymbol{Q} \quad (56)$$

where S=div $\boldsymbol{u}$, ϕ, T, C), Q=(d_1, d_2, d_3, d_4)=(div H, ϕ, T, C) and

$$N(\Delta)=\left\|N_{mn}(\Delta)\right\|_{4\times4}=\begin{Vmatrix}
\delta_9{}^{**}\Delta+\rho\omega^2 & -B\Delta & \delta_4{}^*\Delta & \delta_8{}^*\Delta\\
B & \delta_3{}^*+A\Delta & \delta_5{}^* & -\delta_9{}^*\\
-\beta\tau_{22} & b_1^*\tau_{22} & -(\delta_6{}^*+K\Delta) & \delta_{10}^*\\
-\gamma\tau_{33}\Delta & b_2^*\tau_{33}\Delta & \delta_7^*\Delta & (\delta_{11}{}^*\Delta-\tau_{77})
\end{Vmatrix}_{4x4} \quad (57)$$

The system (56) can be also written as

$$\Gamma(\Delta)S=\psi \quad (58)$$

where

$$\psi=(\psi_1,\psi_2,\psi_3,\psi_4)\,,\quad \psi_n=\frac{1}{AK\delta_{11}{}^*\delta_9{}^{**}}\sum_{m=1}^{4}N_{mn}{}^*d_m,\quad \Gamma(\Delta)=\frac{1}{AK\delta_{11}{}^*\delta_9{}^{**}}\det N(\Delta),\qquad n=1,2,3,4 \quad (59)$$

and $\boldsymbol{N}^*_{mn}$ is the cofactor of the elements N_{mn} of the matrix $\boldsymbol{N}$. From Eqs. (57) and (59), we see that

$$\Gamma(\Delta)=\prod_{m=1}^{4}(\Delta+\lambda_m{}^2) \tag{60}$$

where $\lambda^2_{m,}$ m=1, 2, 3, 4 are the roots of the equation Γ(-k)=0 (with respect to k). Applying the operator $\Gamma(\Delta)$ to the Eq. (50), we get

$$\Gamma(\Delta)(\Delta+\lambda_5^2)u=\psi' \tag{61}$$

where $\lambda_5{}^2=\dfrac{\rho\omega^2}{\delta_1{}^*}$ and

$$\psi'=\frac{1}{\delta_1{}^*}\{\Gamma(\Delta)H-grad[\delta_2{}^*\psi_1-B\psi_2+\delta_4{}^*\psi_3+\delta_8{}^*\Delta\psi_4\} \tag{62}$$

From Eqs. (58) and (61), we obtain

$$\Theta(\Delta)U(x)=\hat{\psi}(x) \tag{63}$$

where

$$\hat{\psi}=(\psi',\psi_2,\psi_3,\psi_4) \text{ and } \Theta(\Delta)=\left\|\Theta_{qn}(\Delta)\right\|_{6x6}$$
$$\Theta_{mn}(\Delta)=\Gamma(\Delta)(\Delta+\lambda_5{}^2),\qquad \Theta_{qn}(\Delta)=0,\qquad \Theta_{44}(\Delta)=\Theta_{55}(\Delta)=\Theta_{66}(\Delta)=\Gamma(\Delta),$$
$$m=1,2,3,\qquad q,n=1,2,3,4,5,6,\qquad q\neq n$$

The Eqs. (59) and (62) can be rewritten in the form

$$\begin{aligned}
\psi'&=[\frac{1}{\delta_1^*}\Gamma(\Delta)\ \bullet\ +q_{11}(\Delta)\text{grad div}]H+q_{21}(\Delta)\text{grad}\,Z+q_{31}(\Delta)\text{grad}\,L+q_{41}(\Delta)\text{grad}\,M,\\
\psi_2&=q_{12}(\Delta)\text{div}\,H+q_{22}(\Delta)Z+q_{32}(\Delta)L+q_{42}(\Delta)M,\\
\psi_3&=q_{13}(\Delta)\text{div}H+q_{23}(\Delta)Z+q_{33}(\Delta)L+q_{43}(\Delta)M,\\
\psi_4&=q_{14}(\Delta)\text{div}\,H+q_{24}(\Delta)Z+q_{34}(\Delta)L+q_{44}(\Delta)M,
\end{aligned} \tag{64}$$

where $\bullet=\left\|\delta_{mn}\right\|_{3\times3}$ is the unit matrix. In Eq. (64), we have used the following notations:

$$q_{m1}(\Delta)=\frac{1}{AK\delta_1\delta_{11}^*\delta_9^{**}}[\delta_2^*N_{m1}^*-BN_{m2}^*-\delta_4N_{m3}^*+\delta_8{}^*\Delta N_{m4}^*],\qquad q_{m2}(\Delta)=\frac{1}{AK\delta_{11}^*\delta_9^{**}}N_{m2}^*,$$
$$q_{m3}(\Delta)=\frac{1}{AK\delta_{11}^*\delta_9^{**}}N_{m3}^*,\qquad q_{m4}(\Delta)=\frac{1}{AK\delta_{11}^*\delta_9^{**}}N_{m4}^*,\qquad m=1,2,3,4$$

Now from Eq. (64), we have

$$\hat{\psi}(x)=R^{tr}(D_x)Q(x) \tag{65}$$

where

$$R=\left\|R_{mn}\right\|_{6\times 6}=\left\|\begin{matrix} R^{(1)} & R^{(2)} \\ R^{(3)} & R^{(4)} \end{matrix}\right\|_{6\times 6}$$

$$R^{(r)}=\left\|R_{mn}^{r}\right\|_{3\times 3}, \qquad R^{(4)}=\left\|R_{mn}^{4}\right\|_{3\times 3},$$

$$R_{mn}^{(1)}(D_x)=\frac{1}{\delta_1^*}\Gamma(\Delta)\delta_{mn}+q_{11}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n}, \tag{66}$$

$$R_{mn}^{(2)}(D_x)=q_{1n}(\Delta)\frac{\partial}{\partial x_n}, \qquad R_{mn}^{(3)}(D_x)=q_{n1}(\Delta)\frac{\partial}{\partial x_n}, \qquad R_{mn}^{(4)}(D_x)=q_{m+1,n+1}(\Delta), \qquad r=1,2,3$$

From Eqs. (54), (63) and (65), we obtain

$$\Theta U=R^{tr}F^{tr}U$$

It implies that

$$R^{tr}F^{tr}=\Theta$$
$$F(D_x)R(D_x)=\Theta(\Delta) \tag{67}$$

We assume that

$$\lambda^2{}_m \neq \lambda^2{}_n \neq 0, \qquad m,n=1,2,3,4,5, \qquad m\neq n$$

Let

$$\boldsymbol{Y}(x)=\left\|Y_{rs}(x)\right\|_{6\times 6}, \qquad Y_{mn}(x)=\sum_{n=1}^{5} r_{1n}\varsigma_n(x), \qquad Y_{44}(x)=Y_{55}(x)=Y_{66}(x)=\sum_{n=1}^{4} r_{2n}\varsigma_n(x), \qquad \boldsymbol{Y}_{vw}(x)=0,$$
$$m=1,2,3, \qquad v,w=1,2,3,4,5,6, \qquad v\neq w$$

where

$$\varsigma_n(x)=-\frac{1}{4\pi|x|}\exp(i\lambda_n|x|), \qquad r_{1n}=\prod_{m=1,m\neq n}^{5}(\lambda_m^2-\lambda_n^2)^{-1}, \qquad n=1,2,3,4,5$$
$$r_{2v}=\prod_{m=1,m\neq v}^{4}(\lambda_m^2-\lambda_n^2)^{-1}, \qquad v=1,2,3,4$$

We will prove the following Lemma:
Lemma: The matrix **Y** defined above is the fundamental matrix of operator $\Theta(\Delta)$, that is

$$\Theta(\Delta)Y(x)=\delta(x)I(x) \tag{68}$$

Proof: To prove the lemma, it is sufficient to prove that

$$\Gamma(\Delta)(\Delta+\lambda_5^2)Y_{11}(x)=\delta(x), \qquad \Gamma(\Delta)Y_{44}(x)=\delta(x) \tag{69}$$

Consider

$$r_{11}+r_{12}+r_{13}+r_{14}+r_{15}=\frac{z_1-z_2+z_3-z_4+z_5}{z_6}, \tag{70}$$

where

$$z_1 = (\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_4^2)(\lambda_3^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2),$$
$$z_2 = (\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_4^2)(\lambda_1^2-\lambda_5^2)(\lambda_3^2-\lambda_4^2)(\lambda_3^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2),$$
$$z_3 = (\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_4^2)(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_4^2)(\lambda_2^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2),$$
$$z_4 = (\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_5^2),$$
$$z_5 = (\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_4^2)(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)(\lambda_3^2-\lambda_4^2),$$
$$z_6 = (\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_1^2-\lambda_4^2)(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_4^2)(\lambda_3^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2)$$

On simplifying the right hand side of above Eq.(70) , we obtain

$$r_{11}+r_{12}+r_{13}+r_{14}+r_{15}=0 \tag{71}$$

Similarly, we find that

$$r_{12}(\lambda_1^2-\lambda_2^2)+r_{13}(\lambda_1^2-\lambda_3^2)+r_{14}(\lambda_1^2-\lambda_4^2)+r_{15}(\lambda_1^2-\lambda_5^2)=0$$
$$r_{13}(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)+r_{14}(\lambda_1^2-\lambda_4^2)(\lambda_2^2-\lambda_4^2)+r_{15}(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_5^2)=0$$
$$r_{14}(\lambda_1^2-\lambda_4^2)(\lambda_2^2-\lambda^2{}_4)(\lambda_3^2-\lambda_4^2)+r_{15}(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_5^2)=0$$
$$r_{15}(\lambda_1^2-\lambda_5^2)(\lambda_2^2-\lambda_5^2)(\lambda_3^2-\lambda_5^2)(\lambda_4^2-\lambda_5^2)=1$$
$$(\Delta+\lambda^2{}_m)\varsigma_n(x)=\delta(x)+(\lambda_m^2-\lambda_n^2)\varsigma_n(x), \qquad m,n=1,2,3,4,5 \tag{72}$$

Now consider

$$\Gamma(\Delta)(\Delta+\lambda_5^2)Y_{11}(x)=(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{n=1}^{5}r_{1n}\,\varsigma_n(x)$$
$$=(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{n=1}^{5}r_{1n}[\delta(x)+(\lambda_1^2-\lambda_n^2)\varsigma_n(x)]$$
$$=(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)[\delta(x)\sum_{n=1}^{5}r_{1n}+\sum_{n=2}^{5}r_{1n}(\lambda_1^2-\lambda_n^2)\varsigma_n(x)]$$

Using Eq. (71) in the above relation, we obtain

$$\Gamma(\Delta)(\Delta+\lambda_5^2)Y_{11}(x)=(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda^2{}_5)\sum_{n=2}^{5}r_{1n}(\lambda_1^2-\lambda_n^2)\varsigma_n(x)$$
$$=(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{n=2}^{5}r_{1n}(\lambda_1^2-\lambda_n^2)[\delta(x)+(\lambda_2^2-\lambda_n^2)\varsigma_n(x)]$$
$$=(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{n=3}^{5}r_{1n}(\lambda_1^2-\lambda_n^2)(\lambda_2^2-\lambda_n^2)\varsigma_n(x)$$
$$=(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{n=3}^{5}r_{1n}(\lambda_1^2-\lambda_n^2)(\lambda_2^2-\lambda_n^2)[\delta(x)+(\lambda_3^2-\lambda_n^2)\varsigma_n(x)]$$
$$=(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{n=4}^{5}r_{1n}(\lambda_1^2-\lambda_n^2)(\lambda_2^2-\lambda_n^2)(\lambda_3^2-\lambda_n^2)\varsigma_n(x)$$
$$=(\Delta+\lambda_5^2)\sum_{n=4}^{5}r_{1n}(\lambda_1^2-\lambda_n^2)(\lambda_2^2-\lambda_n^2)(\lambda_3^2-\lambda_n^2)[\delta(x)+(\lambda_4^2-\lambda_n^2)\varsigma_n(x)]$$
$$=(\Delta+\lambda_5^2)\varsigma_5(x)=\delta(x)$$

Similarly, the Eq. (69) can be proved. We introduce the matrix

$$G(x) = R(D_x)Y(X) \tag{73}$$

From Eqs. (67), (68) and (73), we obtain

$$F(D_x)G(x) = F(D_x)R(D_x)Y(x) = \Theta(\Delta)Y\left(x\right) = \delta(x)I(x)$$

Hence, $G(x)$ is a solution to Eq. (49).Therefore, we have proved the following theorem:
Theorem: The matrix ***G(x)*** defined by Eq. (73) is the fundamental solution of system of Eqs .(44)-(47).

7 BASIC PROPERTIES OF THE MATRIX *G(x)*

Property 1. Each column of the matrix $G(x)$ is the solution of the system of Eqs. (44)-(47) at every point $x \;\varepsilon\; E^3$ except the origin.
Property 2. The matrix $G(x)$ can be written in the form

$$\begin{aligned} G &= \left\| G_{mn} \right\|_{6\times 6} = \left\| \begin{matrix} G^{(1)} & G^{(2)} \\ G^{(3)} & G^{(4)} \end{matrix} \right\|_{6\times 6} \\ G^{(m)}(x) &= R^{(m)}(D_x)Y_{11}(x) \quad , \qquad m = 1,3 \\ G^{(2)}(x) &= R^{(2)}(D_x)Y_{44}(x) \\ G^{(4)}(x) &= R^{(4)}(D_x)Y_{44}(x) \end{aligned} \tag{74}$$

8 PARTICULAR CASES

1. If we neglect the voids effect in the Eq. (73), we obtain the fundamental solution of initially stressed generalized thermoelastic diffusion material.
2. Further in the absence of voids, diffusion and initial stress effects in the basic Eqs. (73), we obtain the similar results for fundamental solution as obtained by Iesan [33] in case of CT theory (i.e. taking all thermal relaxation times are zero).

9 NUMERICAL RESULTS AND DISCUSSION

With the view of illustrating theoretical results obtained in the preceding sections and compare these in the context of two theories of thermoelasticity for the medium of initially stressed thermoelastic diffusion with voids. For numerical computations we take the values of relevant parameters for copper material, the physical data is given below [18]:

λ=7.76 × 10^{10} Kg m^{-1} s^{-2}, μ=3.86 × 10^{10} Kg m^{-1} s^{-2}, T_0=0.293 × 10^3 K, C^*=0.3831× 10^3 J/kg K,
α_t=1.78 × 10^{-5} K^{-1}, α_c=1.98 × 10^{-4} m^3 Kg^{-1}, α^*=0.85 × 10^{-8} m^{-3} Kg s, a=0.0012 × 10^4 $m^2s^{-2}K^{-1}$, d=003 × 10^5 Kg^{-1} m^5 s^{-2}, ρ=8.954 × 10^3 Kg/m^3, k=386 ×10^3 W m^{-1} K^{-1}.

The voids and initial stress parameters are

t^0=0.5 × 10^{10} Kg $m^{-1}s^{-2}$, χ=1.75× 10^{-15} m^2, B=1.13 × 10^{10} Kg $m^{-1}s^{-2}$, ω_0 =2.687 Kg $m^{-1}s^{-1}$,
A=3.688 ×10^{-5} Kg m s^{-2}, ς =1.475 ×10^{10} Kg m^{-1} s^{-2}, b_1^*=20 ×10^5 Kg $m^{-1}s^{-2}K^{-1}$, b_2^*=2.9 ×10^6 m^2s^{-2},

τ_0=0.005 s, τ^0=0.006 s, τ_1=0.007 s, τ^1=0.008 s.

The software MATLAB 7.0.4 has been used to determine the values of phase propagation velocity and attenuation quality factor. The variations of phase propagation velocity and attenuation quality factor with respect to frequency have been shown in Figs.1-4 and 5-8 respectively. In all the Figures, two curves for Lord and Shulman(LS)-theory with and without initial stress cases of isotropic thermoelastic diffusion with voids are presented by ITVDIS(LS) and ITVD(LS) respectively, and two curves for Green and Lindsay(GL)-theory with and without initial stress cases of isotropic thermoelastic diffusion with voids are presented by ITVDIS(GL) and ITVD(GL) respectively.

From Fig.1, it is noticed that the values of phase velocity V_1 increase for both theories (LS-theory and GL-theory) as frequency ω increases. Figs. 2, and 4 show that, the values of phase velocities V_2 and V_4 oscillate for lower value of frequency ω and increase smoothly for higher value of frequency ω for both theories. Fig. 3, on the other hand, shows that the values of phase velocity V_3, increase as frequency ω increases for both theories. On comparing the theories, we find that the values of V_1, V_2, V_3 and V_4 are more for LS-theory in comparison to GL-theory and due to the effect of initial stress the values of phase velocities V_1, V_2, V_3 and V_4 are little more.

Figs. 5-8, show that the variation of attenuation quality factor with respect to frequency ω. From Fig. 5, the values of attenuation quality factor Q_1^{-1} are more for lower value of frequency ω and small for higher value of frequency ω for both theories (LS-theory and GL-theory).

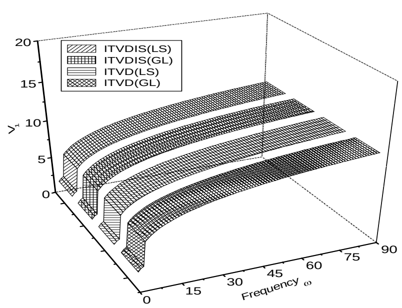

Fig. 1
Variation of phase propagation velocity V_1 w.r.t frequency ω.

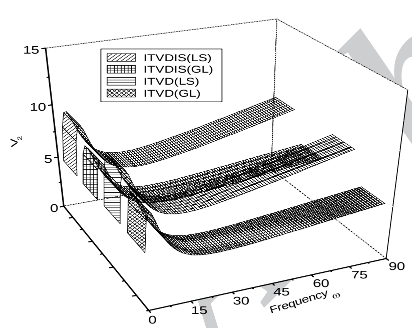

Fig. 2
Variation of phase propagation velocity V_2 w.r.t frequency ω.

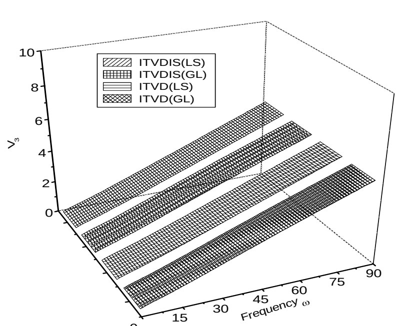

Fig. 3
Variation of phase propagation velocity V_3 w.r.t frequency ω.

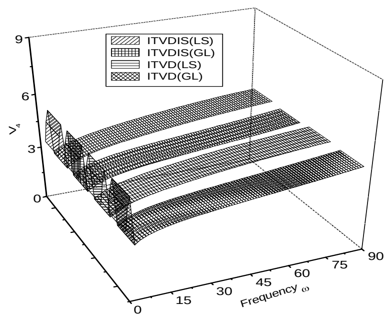

Fig. 4
Variation of phase propagation velocity V_4 w.r.t frequency ω.

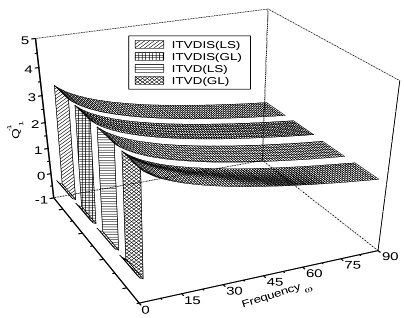

Fig. 5
Variation of attenuation quality factor $Q^{-1}{}_1$ w.r.t frequency ω.

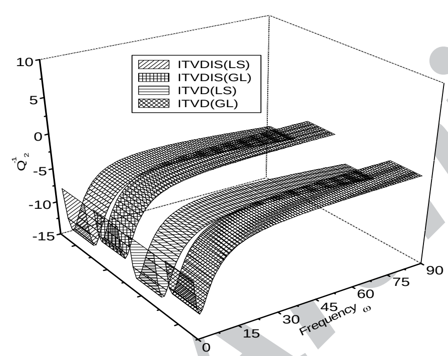

Fig. 6
Variation of attenuation quality factor $Q^{-1}{}_2$ w.r.t frequency ω.

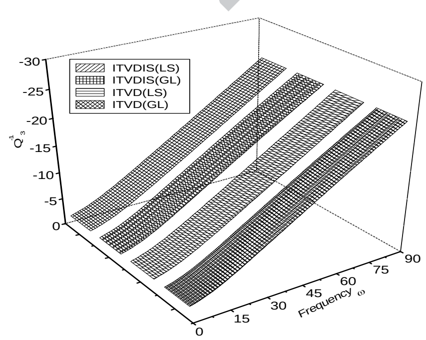

Fig. 7
Variation of attenuation quality factor $Q^{-1}{}_3$ w.r.t frequency ω.

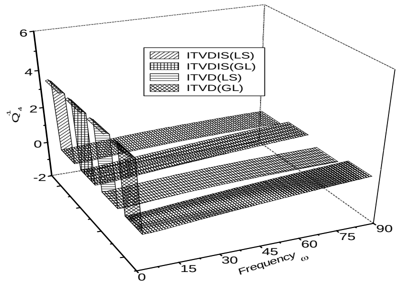

Fig. 8
Variation of attenuation quality factor $Q^{-1}{}_4$ w.r.t frequency *ω*.

Figs. 6, 8, show that the value of attenuation quality factors Q_1^{-1} and Q_4^{-1} decrease for lower value of frequency *ω* and increase for higher value of frequency *ω* for both theories. Fig. 7, on the other hand, depicts that the value of attenuation quality factor Q_3^{-1} decrease with respect to frequency *ω* increase for both theories. On comparing the theories, we find that the values of attenuation quality factors Q_1^{-1}, Q_2^{-1}, Q_3^{-1} and Q_4^{-1} are little more for GL-theory in comparison to LS-theory and due to the effect of initial stress the values of attenuation quality factors Q_1^{-1}, Q_2^{-1}, Q_3^{-1} and Q_4^{-1} are less.

10 CONCLUSIONS

The propagation of plane waves in homogeneous isotropic generalized thermoelastic diffusion with voids under initial stress has been studied. For two dimensional model of initially stressed isotropic generalized thermoelastic diffusion with voids, there exists four coupled waves namely, P wave, Mass Diffusion(MD) wave, Thermal(T) wave and Volume Fraction(VF) wave and one transverse wave is decoupled from rest of the motion which is not affected by the thermal, diffusion and voids parameters. The phase propagation velocities and attenuation quality factors of these plane waves are also computed and presented graphically with respect to frequency. The fundamental solution of system of equations in the generalized theories of thermoelastic diffusion with voids under initial stress in case of steady oscillations in terms of elementary functions has also been constructed. Some special cases are also discussed.

This type of study is useful due to its applications in geophysics and electronic industry. Study of phenomenon of diffusion is used to improve the conditions of oil extractions (seeking ways of more efficiently recovering oil from oil deposits). Also the investigation of thermal, diffusion and initial stress effects on elastic wave propagation plays an important role in understanding many seismological processes.

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