Journal of Solid Mechanics Vol. 3, No. 4 (2011) pp. 332-345

Theoretical Formulations for Finite Element Models of Functionally Graded Beams with Piezoelectric Layers

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Received 25 October 2011; accepted 30 November 2011

ABSTRACT

In this paper an overview of functionally graded materials and constitutive relations of electro elasticity for three-dimensional deformable solids is presented, and governing equations of the Bernoulli–Euler and Timoshenko beam theories which account for through-thickness power-law variation of a two-constituent material and piezoelectric layers are developed using the principle of virtual displacements. The formulation is based on a power-law variation of the material in the core with piezoelectric layers at the top and bottom. Virtual work statements of the two theories are also developed and their finite element models are presented. The theoretical formulations and finite element models presented herein can be used in the analysis of piezolaminated and adaptive structures such as beams and plates.
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Keywords: Bending; Bernoulli–Euler beam theory; Functionally graded material; Piezoelectricity; PZT; Timoshenko beam theory

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1 INTRODUCTION

1.1 Functionally graded materials

FUNCTIONALLY graded materials (FGM) are a class of composites that have a gradual variation of material properties from one surface to another [1–3]. These novel materials were proposed as thermal barrier materials for applications in space planes, space structures, nuclear reactors, turbine rotors, flywheels, gears, and so on. These materials are isotropic and nonhomogeneous. In general, all the multi-phase materials, in which the material properties are varied gradually in a predetermined manner, fall into the category of functionally gradient materials.

Two-constituent FGMs are usually made of a mixture of ceramic and metals for use in thermal environments. The ceramic constituent of the material provides the high temperature resistance due to its low thermal conductivity. The ductile metal constituent, on the other hand, prevents fracture due to high temperature gradient in a very short period of time. The gradation in properties of the material reduces thermal stresses, residual stresses, and stress concentration factors. The gradual variation results in a very efficient material tailored to suit the needs.

Noda [4] presented an extensive review that covers a wide range of topics from thermo- elastic to thermo-inelastic problems, where he discussed the importance of temperature dependent properties on thermo elastic problems. He further presented analytical methods to handle transient heat conduction problems and indicates the necessity for the optimization of FGM properties. Zhang et al. [5] modeled an isotropic ceramic/metal laminated beam subjected to an abrupt heating condition and demonstrated the influence of thermo- mechanical coupling on thermal shock response. Tanigawa [6] compiled a comprehensive review on the thermo elastic analyses of functionally graded materials. In this review, he discussed closed-form solutions for some simple geometries. Sankar

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and Tzeng [7] studied thermal stresses in beams. These solutions, however, were restricted to steady-state conditions and linear analyses. Praveen and Reddy [8-10] considered the von Kármán geometric nonlinearity in their bending and transient analysis of functionally graded plates. The von Kármán strains are computed using the assumption that the strains are small enough to neglect the difference between various measures of stress and strain but the rotations are moderately large. The book by Shen [11] provides an excellent treatment of the subject.

1.2 Hyperelastic materials

1.2.1 Isothermal elasticity

As a precursor to electroelasticity, we begin with linear elastic materials. Materials for which the constitutive behavior is only a function of the current state of deformation are known as elastic. When the work done by stresses in deforming an elastic material is independent of the path (i.e., depends only the initial and final states), then the material is termed hyperelastic (see Bonet and Wood [12] and Reddy [13]). For such materials, there exists a strain energy density function of strain, $U_0 = U_0(\varepsilon)$ such that stress σ is derivable from:

$$\sigma = \frac{\partial U_0}{\partial \varepsilon} \qquad \text{or} \qquad \sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}} \tag{1}$$

For infinitesimal deformations considered herein, we do not distinguish between various stress and strain measures, although σ should be thought of as the second Piola-Kirchhoff stress tensor and ε as the Green-Lagrange strain tensor. In some books (see Reddy [13]), a hyperelastic material is defined to be one for which Eq. (1) holds. For linear elastic materials under isothermal conditions, the strain energy density function can be expressed as

$$U_0 = \frac{1}{2}\varepsilon : C : \varepsilon \qquad \text{or} \qquad U_0 = \frac{1}{2} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \tag{2}$$

where : denotes the double-dot product (see Reddy [13]) and we have

$$\sigma = \frac{\partial U_0}{\partial \varepsilon} = \mathrm{C} : \varepsilon \qquad \text{or} \qquad \sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}} = C_{ijkl}\,\varepsilon_{kl} \tag{3}$$

which is known the generalized Hooke's law. The total virtual work done for an elastic body is

$$\delta W = \delta U + \delta V = \int_V \delta U_0 \,\mathrm{d}v = \int_V \delta u \cdot f \,\mathrm{d}v - \int_{S_1} \delta u \cdot t \,\mathrm{d}s \tag{4}$$

where f is the body force vector, t is the traction vector on the surface S_1 of the volume V occupied by the elastic body, δu is the virtual displacement vector, and

$$\delta U_0 = \frac{\partial U_0}{\partial \varepsilon} : \delta\varepsilon = \sigma : \delta\varepsilon = \sigma_{ij}\delta\varepsilon_{ij} \tag{5}$$

1.2.2 Thermoelasticity

If thermal effects are to be included, the strain energy density function δU_0 is replaced with the Helmholtz free-energy function Ψ_0 (see Reddy [13] and Gurtin et al. [14]) $\Psi_0 = \Psi_0(\varepsilon, \theta)$) such that

$$\sigma = \frac{\partial \Psi_0}{\partial \varepsilon} \qquad \text{or} \qquad \sigma_{ij} = \frac{\partial \Psi_0}{\partial \varepsilon_{ij}} \qquad \text{and} \qquad \eta = -\frac{\partial \Psi_0}{\partial \theta} \tag{6}$$

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where θ is the absolute temperature and η is the entropy. Using thermodynamic considerations [13-18], Ψ_0 can be expressed as

$$\Psi_0(\varepsilon,\theta) = U_0 - \beta : \varepsilon\theta - \frac{\rho c_v}{2\theta_0}\theta^2 \tag{7}$$

where β is the symmetric second-order tensor of material coefficients, $\beta = C:\alpha$, α is the symmetric second-order tensor of thermal coefficients of expansion, θ_0 is a reference temperature, ρ is the material density, and c_v is the specific heat at constant temperature. The constitutive relations become

$$\sigma = \frac{\partial \Psi_0}{\partial \varepsilon} = \frac{\partial U_0}{\partial \varepsilon} - \beta\theta \qquad \text{or} \qquad \sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \beta_{ij}\theta \tag{8}$$

$$\eta = -\frac{\partial \Psi_0}{\partial \theta} = \beta : \varepsilon + \rho c_v \frac{\theta}{\theta_0} \qquad \text{or} \qquad \eta = \beta_{ij}\varepsilon_{ij} + \rho c_v \frac{\theta}{\theta_0} \tag{9}$$

The principle of virtual displacements for a thermoelastic body with known temperature field $\theta = \theta(X)$ (hence, $\delta\theta = 0$) takes the form

$$\begin{aligned} \delta W &= \int_V \delta\Psi_0 \mathrm{d}v - \int_V \delta u \cdot f \ \mathrm{d}v - \int_{S_1} \delta u \cdot t \ \mathrm{ds} \\ &= \int_V [\sigma : \delta\varepsilon - \delta u \cdot f] \ \mathrm{d}v - \int_{S_1} \delta u \cdot t \ \mathrm{ds} \\ &= \int_V [(C_{ijkl}\varepsilon_{kl} - \beta_{ij}\theta)\delta\varepsilon_{ij} - \delta u_i f_i] \ \mathrm{d}v - \int_{S_1} \delta u \cdot t \ \mathrm{ds} \end{aligned} \tag{10}$$

where we have used the fact

$$\delta\Psi_0 = \frac{\partial \Psi_0}{\partial \varepsilon} : \delta\varepsilon + \frac{\partial \Psi_0}{\partial \theta} : \delta\theta = \sigma : \delta\varepsilon = \sigma_{ij}\delta\varepsilon_{ij} \tag{11}$$

1.3 Piezoelectric materials

1.3.1 Constitutive models

Electro elasticity is the phenomena caused by interactions between electric and mechanical fields, and the piezoelectric effect is one such phenomenon, which is concerned with the effect of the electric charge on deformation [15–18]. A laminated or functionally graded beam structure with surface mounted piezoelectric layers receives actuation through an applied electric field, and the piezoelectric layers send electric signals that are used to measure the motion or deformation of the laminate. In these problems, the electric charge that is applied to actuate a structure is treated as an additional body force in the problem, much the same way a temperature field induces a body force through thermal strains.

The coupling between the mechanical, thermal, and electrical fields can be established using thermodynamic principles and Maxwell's relations. Analogous to the strain energy density function U_0 for isothermal elasticity and the Helmholtz free-energy function Ψ_0 for hermoelasticity, we assume the existence of a potential function $\Phi_0 = \Phi_0(\varepsilon, E, \theta)$, called the Gibbs free energy function or enthalpy function, such that

$$\sigma = \frac{\partial \Phi_0}{\partial \varepsilon} \qquad \left(\sigma_{ij} = \frac{\partial \Phi_0}{\partial \varepsilon_{ij}}\right), \qquad \eta = \frac{\partial \Phi_0}{\partial \theta}, \qquad D = \frac{\partial \Phi_0}{\partial E} \qquad \left(D_i = \frac{\partial \Phi_0}{\partial E_i}\right) \tag{12}$$

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where D is the electric displacement vector and E is the electric field vector. The specific form of Φ_0 is given by (see [15–18])

$$\begin{aligned}\Phi_0(\varepsilon,E,\theta) &= \Psi_0(\varepsilon,\theta) - H_0(\varepsilon,E,\theta)\\ &= U_0(\varepsilon) - \beta:\varepsilon\theta - \frac{\rho c_v}{2\theta_0}\theta^2 - \varepsilon:e\cdot E - \frac{1}{2}E\cdot\epsilon\cdot E - p\cdot e\,\theta\\ &= \frac{1}{2}C_{ijk\ell}\varepsilon_{ij}\,\varepsilon_{k\ell} - \beta_{ij}\,\varepsilon_{ij}\,\theta - \frac{\rho c_v}{2\theta_0}\theta^2 - e_{ijk}\,\varepsilon_{ij}E_k - \frac{1}{2}\epsilon_{k\ell}\,E_kE_\ell - p_kE_k\theta\end{aligned} \tag{13}$$

Here e denotes the third-order tensor of piezoelectric moduli, is the second-order tensor of dielectric constants (also known as the permittivity tensor), and p is the vector of pyroelectric constants. Then we have

$$\begin{aligned}\sigma &= \frac{\partial\Phi_0}{\partial\varepsilon} = \frac{\partial U_0}{\partial\varepsilon} - \beta\theta - e\cdot E = C:\varepsilon - e\cdot E - \beta\theta \quad (\sigma_{ij} = C_{ijk\ell}\,\varepsilon_{k\ell} - e_{ijk}\,E_k - \beta_{ij}\,\theta)\\ D &= -\frac{\partial\Phi_0}{\partial E} = \varepsilon:e + \epsilon\cdot E + p\theta \quad (D_k = \varepsilon_{ij}\,e_{ijk} + \epsilon_{k\ell}\,E_\ell + p_k\,\theta)\\ \eta &= -\frac{\partial\Phi_0}{\partial\theta} = \beta:\varepsilon + p\cdot E + \rho c_v\frac{\theta}{\theta_0} \quad \left(\eta = \beta_{ij}\,\varepsilon_{ij} + p_k\,E_k + \rho c_v\frac{\theta}{\theta_0}\right)\end{aligned} \tag{14}$$

1.3.2 Maxwell's equations

In addition to obeying the conservation principles of continuum mechanics, a deformable piezoelectric medium must satisfy Maxwell's equations governing the electric displacement vector D and electric field E in the absence of a magnetic field (see Reddy and Gartling [19])

$$-\nabla\cdot D + \rho_c = 0, \qquad \nabla\times E = 0 \tag{15}$$

where ρ_c is the distributed charge density (which can be assumed to be zero). It is often assumed that the electric field E is derivable from an electric scalar potential function φ such that

$$E = -\nabla\varphi \tag{16}$$

Consequently, the second equation in (15) is trivially satisfied (i.e., the curl of the gradient of any function is zero).

1.3.3 Virtual work statements

The principle of virtual displacements for a deformable piezoelectric medium with known temperature field $\theta(X)$ has the form

$$\begin{aligned}\delta W &= \int_V \delta\Phi_0\,\mathrm{d}v - \int_V \delta u\cdot f\,\mathrm{d}v - \int_{S_1}\delta u\cdot t\,\mathrm{d}s\\ &= \int_V\left[(C:\varepsilon - e\cdot E - \beta\theta):\delta\varepsilon - (\varepsilon:e + \epsilon\cdot E + p\theta)\cdot\delta E - \delta u\cdot f\right]\mathrm{d}v - \int_{S_1}\delta u\cdot t\,\mathrm{d}s\\ &= \int_V\left[(C_{ijkl}\varepsilon_{kl} - e_{ijk}E_k - \beta_{ij}\theta)\delta\varepsilon_{ij} - (\varepsilon_{ij}\,e_{ijk} + \epsilon_{k\ell}\,E_\ell + p_k\,\theta)\delta E_k - \delta u_i\,f_i\right]\mathrm{d}v - \int_{S_1}\delta u_i\,t_i\,\mathrm{d}s\end{aligned} \tag{17}$$

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where

$$\delta\Phi_0 = \frac{\partial\Phi_0}{\partial\varepsilon} : \delta\varepsilon + \frac{\partial\Phi_0}{\partial\theta}\delta\theta + \frac{\partial\Phi_0}{\partial E}\cdot\delta E \tag{18}$$

1.3.4 Condensed notation for various tensors of mechanics

Although most equations in mechanics are derived using vector and tensor notation, they are expressed in terms of condensed notation for solution purposes. For example, stresses and strains, which are second-order tensors, are expressed as column vectors using the notation (see Reddy [18])

$$\begin{aligned} &\sigma_{11} = \sigma_1,\ \sigma_{22} = \sigma_2,\ \sigma_{33} = \sigma_3,\ \sigma_{23} = \sigma_{32} = \sigma_4,\quad \sigma_{13} = \sigma_{31} = \sigma_5,\quad \sigma_{12} = \sigma_{21} = \sigma_6 \\ &\varepsilon_{11} = \varepsilon_1,\quad \varepsilon_{22} = \varepsilon_2,\quad \varepsilon_{33} = \varepsilon_3,\ 2\varepsilon_{23} = 2\varepsilon_{32} = \varepsilon_4,\ 2\varepsilon_{13} = 2\varepsilon_{31} = \varepsilon_5,\ 2\varepsilon_{12} = 2\varepsilon_{21} = \varepsilon_6 \end{aligned} \tag{19}$$

This notation for stresses and strains requires the need to use condensed notation for constitutive tensors of various orders. The constitutive relations are listed below for Lead zirconate titanate (PZT), a polarized ferroelectric ceramic. These are listed under the assumption of orthotropy (in the material coordinate system) and plane stress state:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 & 0 \\ 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 - \alpha_1\theta \\ \varepsilon_2 - \alpha_2\theta \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & e_{24} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} \tag{20}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} + \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix}\theta \tag{21}$$

The condensed elastic stiffness coefficients c_{ij} can be expressed in terms of the engineering constants

$$c_{11} = \frac{E_1^m}{1-\nu_{12}\nu_{21}},\ c_{12} = \frac{\nu_{12}E_2^m}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1^m}{1-\nu_{12}\nu_{21}},\quad c_{22} = \frac{E_2^m}{1-\nu_{12}\nu_{21}},\quad c_{44} = G_{23},\quad c_{55} = G_{31},\quad c_{66} = G_{12} \tag{22}$$

where E_1^m and E_2^m are Young's moduli in 1,2, and 3 material coordinate directions, respectively, v_{ij} is Poisson's ratio, defined as the ratio of transverse strain in the *j*th direction to the axial strain in the *i*th direction when stressed in the *i*th direction, and G_{23}, G_{13}, G_{12}, are the shear moduli in the 2-3, 1-3, 1-2 planes, respectively.

1.4 Present study

To the best of the authors' knowledge, no work has been reported till date which concerns the thermo-mechanical analysis of functionally graded beams with temperature-dependent properties and surface mounted piezoelectric material layers. This very fact motivates the investigation of the present study. In this study, through-thickness functionally graded beams with temperature-dependent material properties and surface mounted piezoelectric layers are considered. The beam is subjected to transverse force and thermal loading. This work aims to investigate the

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effects of the thermal field, the material grading index and the geometry of the beam on the displacement and stress fields under various boundary conditions. The material properties modeled as nonlinear functions of the temperature, which is determined as a function of the thickness coordinate, consistent with the power-law distribution of the constituent materials. Complete theoretical developments for functionally graded beams according to the Bernoulli-Euler beam theory and Timoshenko beam theory are presented and associated finite element models are developed.

2 THICKNESS PROFILE AND TEMPERATURE FILD

2.1 Through Thickness Variation of Material Properties

Consider a beam of length L, width b, and height (or thickness) h. The x-coordinate is taken along the length of the beam and is assumed to pass through the geometric centroid of the cross sections, the z-coordinate is taken transverse (i.e., along the height of the beam), and the y-coordinate is taken along the width of the beam, as shown in Fig 1. In addition, the piezoelectric layers are placed at the top and bottom of the beam and the thickness of each of these layers is taken to be H. Generally the variation of a typical material property of the material in the FGM beam along the thickness coordinate z is assumed to be represented by the simple power-law as (see [8-11]).

$$P(z,T) = \left[P_c(T) - P_m(T)\right] f(z) + P_m(T), \qquad f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n \tag{23}$$

where P_c and P_m are the material properties of the ceramic and metal faces of the beam respectively, n is the volume fraction exponent (power-law index), and T is the temperature above the room temperature, T_0. Note that when n=0, we obtain the single material beam (with property P_c). Fig 2. shows the variation of the volume fraction, $f(z)$, through the beam thickness for various values of the power-law index, n. Note that the volume fraction $f(z)$ of the ceramic material decreases with increasing values of n.

Since FGMs are generally used in high temperature environment, the material properties are temperature-dependent and they can be expressed as

$$P_\alpha(T) = c_0(c_{-1}T^{-1} + 1 + c_1T + c_2T^2 + c_3T^3), \qquad \alpha = c \text{ or } m \tag{24}$$

where c_0 is a constant appearing in the cubic fit of the material property with temperature; and c_{-1}, c_1, c_2, and c_3 coefficients of T^{-1}, T, T^2, and T^3, obtained after factoring out c_0 from the cubic fit of the property. The material properties were expressed in this way, so that the higher-order effects of the temperature on the material properties would be readily discernible.

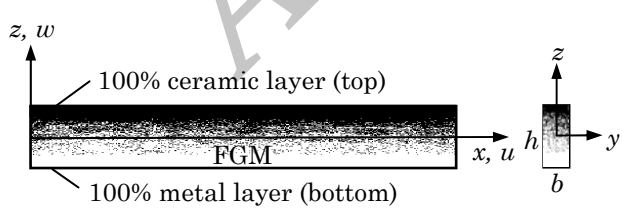

Fig. 1
Geometry of a through-thickness functionally graded beam.

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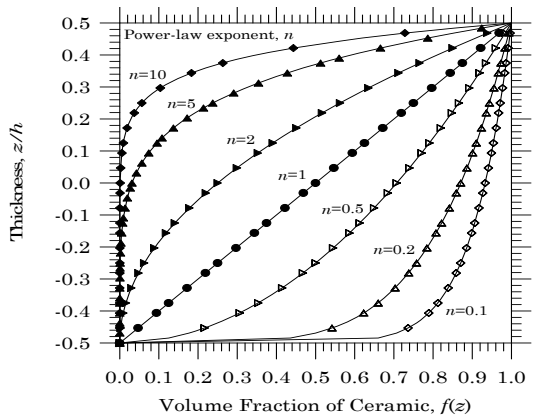

Fig. 2
Volume fraction of ceramic material, $f(z)$, through the beam thickness for various values of power-law index, n.

Table 1
Material properties of Zirconia (c-1=0).

Property	c_0	$c_1\times10^4$	$c_2\times10^8$	$c_3\times 0^{10}$
ρ, Density (kg/m3)	5700	0	0	0
k, Conductivity (W/m K)	1.7	1.276	664.85	0
α, Coefficient of thermal expansion (K)	2.7657×10^{-6}	14.9	0.0001	-0.06775
υ, Poisson's ratio	0.2882	1.13345	0	0
cv, Specific heat (J/kg K)	487.34279	3.04908	-6.037232	0
E, Young's modulus (Pa)	244.266×10^9	-13.707	121.393	-3.681378

Table 2
Material properties of Ti6AlV (c-1=0).

Property	c_0	$c_1\times10^4$	$c_2\times10^8$	$c_3\times 0^{10}$
ρ, Density (kg/m3)	4429	0	0	0
k, Conductivity (W/m K)	1.20947	139.375	0	0
α, Coefficient of thermal expansion (K)	7.57876×10^{-6}	6.5	31.467	0
υ, Poisson's ratio	0.28838235	1.12136	0	0
cv, Specific heat (J/kg K)	625.2969	-4.22387	71.786536	0
E, Young's modulus (Pa)	122.557 ×109	-4.58635	0	-3.681378

For the analysis with constant properties, the material properties were all evaluated at 25.15ºC. The values of each of the coefficients appearing in section (2.2) are listed for the metal and ceramic in Tables 1 and 2, respectively. The modulus of elasticity, conductivity, and the coefficient of thermal expansion are considered vary according to Eqs. (23) and (24).

2.2 Temperature distribution through beam thickness

The temperature is assumed to be uniform along the length of the beam but vary through the thickness of the beam. The temperature field θ through the beam thickness is determined by solving the one-dimensional heat conduction problem, with specified temperature boundary conditions at the top and bottom of the beam.

The energy equation for the temperature variation through the thickness is governed by

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left[k(z,T)\frac{\mathrm{d}T}{\mathrm{d}z}\right]=0, \qquad -\frac{h}{2}\le z\le\frac{h}{2} \tag{25}$$

$$T\left(-\frac{h}{2}\right)=T_m, \qquad T\left(\frac{h}{2}\right)=T_c \tag{26}$$

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where $k(z,T)$ is assumed to vary according to Eqs. (23) and (24), while density ρ and specific heat c_v are assumed to be constants. Fig. 3 shows linear finite element discretization through the beam thickness. The values of the parameters c_0, c_1, etc. of Eq. (24) for temperature dependent thermal conductivities of Titanium and Zirconia are taken from Tables 1 and 2, respectively. The calculated temperature profiles through the beam thickness are shown in Fig 4. For n=0, n=0.2, n=1, and n=5 for T_c=1000ºC and T_m=25ºC for temperature-independent (linear) and temperature-dependent (nonlinear) thermal conductivity. The heat flux through the thickness of the beam are shown in Fig 5 for n=0, n=0.2, n=5.

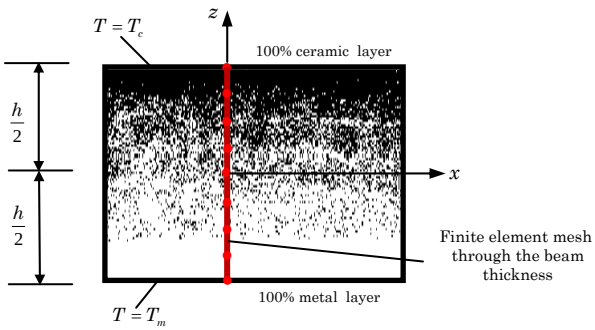

Fig. 3
Finite element discretization through the thickness of the beam.

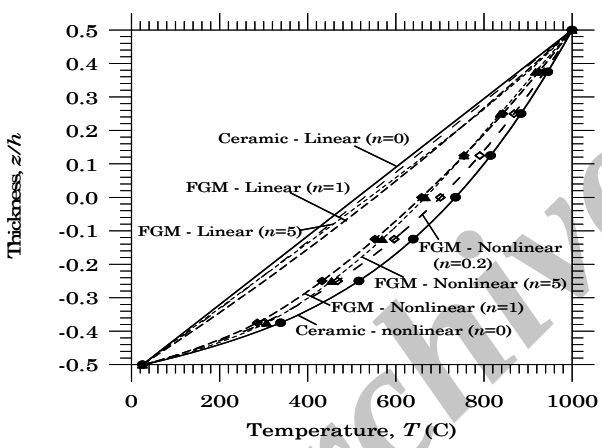

Fig. 4
Temperature distribution through the thickness of the beam.

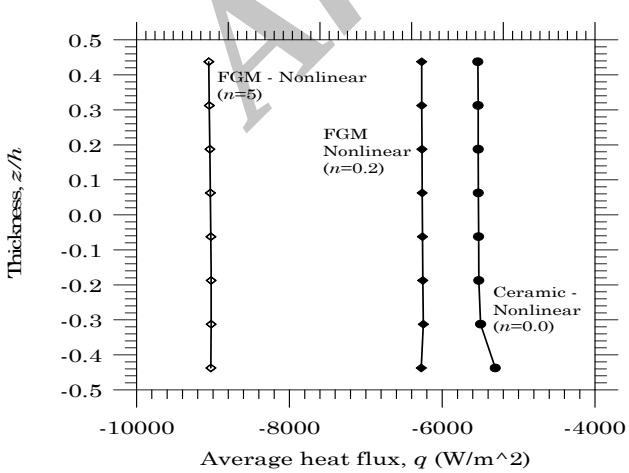

Fig. 5
Heat through the thickness of the beam.

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3 BEAM THEORIES

3.1 Preliminary comments

Equations governing the bending of beams can be formulated using (1) the Bernoulli–Euler beam theory (i.e., the theory in which transverse shear strain is assumed to be zero) and (2) the Timoshenko beam theory. Considering bending in the *xz*-plane, all displacements are assumed to be only functions of the *x* and *z* coordinates. Further, it is assumed that the displacement u_2 is identically zero. We suppose that the beam consists of a FGM material layer and layers made of piezoelectric material with no pyroelectric effect. Since bending is caused by, in addition to a transversely applied mechanical load $q(x)$, an electric field applied in the *z*-direction (i.e., thickness polarization), we have $\mathbf{E} = E_3(z)\hat{\mathbf{e}}_z$. Suppose that the electric scalar potential is a quadratic function of *z*

$$\varphi(z) = \varphi_0 + z\varphi_1 + \frac{z^2}{2}\varphi_2 \tag{27}$$

where $\varphi_0, \varphi_1,$ and φ_2 are to be determined in terms of electric potential. Suppose that the electrical boundary conditions are

$$\varphi(h/2) = V, \qquad \varphi(-h/2) = -V \tag{28}$$

Then

$$\varphi_0 = -\frac{h^2}{8}\varphi_2, \qquad \varphi_1 = 2\frac{V}{h} \tag{29}$$

Then Eq. (16) takes the form

$$E_3 = -\frac{d\varphi}{dz} = -(\varphi_1 + z\varphi_2) \equiv E_3^{(0)} + zE_3^{(1)}, \qquad \varphi_1 = -E_3^{(0)},\ \varphi_2 = -E_3^{(1)} \tag{30}$$

The constitutive relations for the one-dimensional case at hand become

$$\begin{aligned}
&\sigma_{xx}(x,z) = E^m(z,\theta)\left[\varepsilon_{xx}(x,z) - \alpha\theta(z)\right] - e_{31}E_3(z)\\
&\sigma_{xz}(x,z) = 2G(z,\theta)\varepsilon_{xz}(x,z)\\
&D_1 = 0\\
&D_2 = 2e_{15}\varepsilon_{xz}\\
&D_3 = e_{31}\varepsilon_{xx} + \epsilon_{33}E_3
\end{aligned} \tag{31}$$

where E^m is Young's modulus, $G = E^m/[2(1+\nu)]$, α is the coefficient of thermal expansion and is the temperature change from the room temperature, $\theta = T - T_0$. The temperature θ is assumed to vary only through the thickness.

3.2 Bernoulli-Euler beam theory

The total displacements (u_1, u_3) along the coordinate directions (x, z), as implied by the Bernoulli-Euler hypothesis are

$$u_1(x,z) = u(x) - z\frac{\mathrm{d}w}{\mathrm{d}x}, \qquad u_3(x,z) = w(x) \tag{32}$$

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where w denotes the transverse displacement of a point on the mid-plane of the beam. The only nonzero linear strains are

$$\varepsilon_{xx}(x,z)=\varepsilon_{xx}^{(0)}+z\varepsilon_{xx}^{(1)};\qquad \varepsilon_{xx}^{(0)}=\frac{\mathrm{d}u}{\mathrm{d}x},\ \varepsilon_{xx}^{(1)}=-\frac{\mathrm{d}^2w}{\mathrm{d}x^2} \tag{33}$$

The principle of virtual displacements (17) for the Bernoulli-Euler beam theory takes the form

$$\begin{aligned}0&=\int_0^L\int_A\left\{\left[E^m(\varepsilon_{xx}-\alpha\theta)-e_{31}E_3\right]\delta\varepsilon_{xx}-(e_{31}\varepsilon_{xx}+\epsilon_{33}E_3)\delta E_3\right\}\mathrm{d}A\mathrm{d}x-\int_0^L\delta wq\mathrm{d}x\\&=\int_0^L\left[N_{xx}\delta\varepsilon_{xx}^{(0)}+M_{xx}\delta\varepsilon_{xx}^{(1)}-(N_{xx}\delta E_3^{(0)}+M_{xx}\delta E_3^{(1)})-\delta wq\right]\mathrm{d}x\end{aligned} \tag{34}$$

where

$$\begin{aligned}N_{xx}&=\int_A\left[E^m(\varepsilon_{xx}-\alpha\theta)-e_{31}E_3\right]\mathrm{d}A=A_{xx}\varepsilon_{xx}^{(0)}+B_{xx}\varepsilon_{xx}^{(1)}-A_{31}E_3^{(0)}-N_{xx}^T\\M_{xx}&=\int_A z\left[E^m(\varepsilon_{xx}-\alpha\theta)-e_{31}E_3\right]\mathrm{d}A=B_{xx}\varepsilon_{xx}^{(0)}+D_{xx}\varepsilon_{xx}^{(1)}-D_{31}E_3^{(1)}-M_{xx}^T\end{aligned} \tag{35}$$

$$\begin{aligned}N_{xx}&=\int_A(e_{31}\varepsilon_{xx}+\epsilon_{33}E_3)\ \mathrm{d}A=A_{31}\varepsilon_{xx}^{(0)}+A_{33}E_3^{(0)}\\M_{xx}&=\int_A z(e_{31}\varepsilon_{xx}+\epsilon_{33}E_3)\ \mathrm{d}A=D_{31}\varepsilon_{xx}^{(1)}+D_{33}E_3^{(1)}\end{aligned} \tag{36}$$

$$N_{xx}^T=\int_A E^m(z,T)\alpha(z,T)\theta\ \mathrm{d}A,\ \ M_{xx}^T=\int_A zE^m(z,T)\alpha(z,T)\theta\ \mathrm{d}A \tag{37}$$

and

$$\begin{aligned}(A_{xx},B_{xx},D_{xx})&=\int_A(1,z,z^2)E(z,T)\ \mathrm{d}A\\(A_{31},D_{31})&=\int_A(1,z^2)e_{31}\ \mathrm{d}A=e_{31}(A,I)\\(A_{33},D_{33})&=\int_A(1,z^2)\ \epsilon_{33}\ \mathrm{d}A=\epsilon_{33}(A,I)\end{aligned} \tag{38}$$

3.3 Timoshenko beam theory

The Timoshenko beam theory (TBT) (see Reddy [20, 21]) is based on the displacement field of

$$u_1(x,z)=u(x)+z\phi_x(x),\qquad u_3(x,z)=w(x) \tag{39}$$

where ϕ_x denotes the rotation of the cross section about the y-axis. In the Timoshenko beam theory the normality assumption of the Bernoulli-Euler beam theory is relaxed and a constant state of transverse shear strain (and thus constant shear stress computed from the constitutive equation) with respect to the thickness coordinate z is included. The Timoshenko beam theory requires shear correction factors to compensate for the error due to this constant shear stress assumption. The shear correction factors depend not only on the material and geometric parameters but also on the load and boundary conditions.

The linear strains and stresses of the Timoshenko beam theory are

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$$\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)}, \qquad \varepsilon_{xx}^{(0)} = \frac{\mathrm{d}u}{\mathrm{d}x}, \qquad \varepsilon_{xx}^{(1)} = \frac{\mathrm{d}\phi_x}{\mathrm{d}x}, \qquad \gamma_{xz} = \phi_x + \frac{\mathrm{d}w}{\mathrm{d}x} \tag{40}$$

The virtual work statement takes the form

$$\begin{aligned} 0 &= \int_0^L \int_A \left\{ \left[E^m(\varepsilon_{xx} - \alpha\theta) - e_{31}E_3 \right] \delta\varepsilon_{xx} + G\gamma_{xz}\delta\gamma_{xz} - (e_{31}\varepsilon_{xx} + \epsilon_{33}E_3)\delta E_3 \right\} \mathrm{d}A\mathrm{d}x - \int_0^L \delta w q \mathrm{d}x \\ &= \int_0^L \left[N_{xx}\delta\varepsilon_{xx}^{(0)} + M_{xx}\delta\varepsilon_{xx}^{(1)} + Q_x\,\delta\gamma_{xz} - N_{xx}\delta E_3^{(0)} - M_{xx}\delta E_3^{(1)} - \delta w q \right] \mathrm{d}x \end{aligned} \tag{41}$$

where

$$Q_x = \int_A G(z,T)\gamma_{xy}\,\mathrm{d}A = \gamma_{xy}\int_A G(z,T)\mathrm{d}A \equiv S_{xz}\left(\phi_x + \frac{\mathrm{d}w}{\mathrm{d}x}\right) \tag{42}$$

and all other variables appearing in Eq. (41) are defined by Eqs. (35)-(38) with $\varepsilon_{xx}^{(0)}$ and $\varepsilon_{xx}^{(1)}$ defined by Eq. (40).

4 FINITE ELEMENT MODELS

4.1 Bernoulli-Euler beam theory

In the finite element method, the domain of the beam is discretized into a set of finite elements, the domain of a typical element being $\Omega^e = (x_a, x_b)$. Equations relating the nodal displacements to nodal forces are derived using the virtual work statements. As per Eq. (28), the virtual work statement in Eq. (32) is modified to

$$\begin{aligned} 0 = \int_{x_a}^{x_b} \Big[& \frac{\mathrm{d}\delta u}{\mathrm{d}x}\left(A_{xx}\frac{\mathrm{d}u}{\mathrm{d}x} - B_{xx}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - A_{31}E_3^{(0)} - N_{xx}^T \right) - \delta E_3^{(0)}\left(A_{31}\frac{\mathrm{d}u}{\mathrm{d}x} + A_{33}E_3^{(0)} \right) \\ & - \frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2}\left(B_{xx}\frac{\mathrm{d}u}{\mathrm{d}x} - D_{xx}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - D_{31}E_3^{(1)} - M_{xx}^T \right) - \delta E_3^{(1)}\left(-D_{31}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + D_{33}E_3^{(1)} \right) - \delta w q \Big] \mathrm{d}x \\ & - \sum_{j=1}^{6} Q_j\,\delta\Delta_j \end{aligned} \tag{43}$$

where Q_j are the generalized forces associated with the generalized displacements [see Figs. 6(a) and 6(b)]. The virtual work statement in Eq. (43) forms the basis of the Bernoulli-Euler beam finite element model (see Reddy [21]).

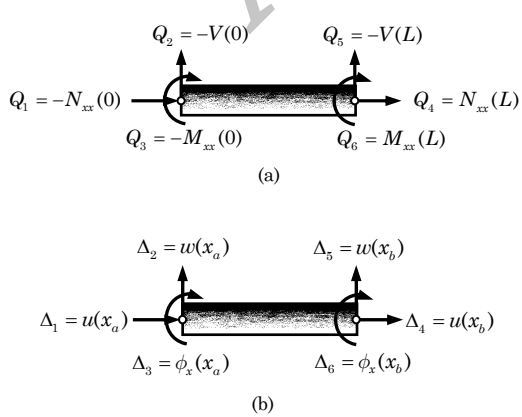

Fig. 6
Beam finite element. (a) Generalized displacements. (b) Generalized forces.

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We assume linear approximation of the axial displacement u and Hermite cubic approximation of the transverse displacement w over a typical beam element, and represent $E_3^{(0)}$ and $E_3^{(1)}$ as constant in the element

$$u(x) \approx \Delta_1\psi_1(x) + \Delta_4\psi_2(x), \qquad w(x) \approx \Delta_2\phi_1(x) + \Delta_3\phi_2(x) + \Delta_5\phi_3(x) + \Delta_6\phi_4(x) \tag{44}$$

where $\psi_j(x)$ are the linear polynomials, $\phi_j(x)$ are the Hermite cubic polynomials, and Δ_i are the nodal values

$$\Delta_1 = u(x_a),\ \Delta_2 = w(x_a),\ \Delta_3 = \theta_x(x_a),\ \Delta_4 = u(x_b),\ \Delta_5 = w(x_b),\ \Delta_6 = \theta_x(x_b) \tag{45}$$

where $\phi_x = -(\mathrm{d}w/\mathrm{d}x)$ Substitution of Eq. (44) for u, w, $\delta u = \psi_i$, and $\delta w = \phi_i$ into the virtual work statements in Eq. (44), we obtain the finite element equations

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} & 0 \\ \mathbf{K}^{21} & \mathbf{K}^{22} & 0 & \mathbf{K}^{24} \\ \mathbf{K}^{31} & 0 & K^{33} & 0 \\ 0 & \mathbf{K}^{42} & 0 & K^{44} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \bar{\mathbf{\Delta}} \\ E_3^{(0)} \\ E_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ 0 \\ 0 \end{Bmatrix} \tag{46}$$

where

$$\mathbf{u} = \{\Delta_1\ \Delta_4\}^{\mathrm{T}}, \qquad \bar{\mathbf{\Delta}} = \{\Delta_2\ \Delta_3\ \Delta_5\ \Delta_6\}^{\mathrm{T}} \tag{47}$$

And the symmetric stiffness coefficients $(K_{ij}^{\alpha\beta} = K_{ji}^{\beta\alpha}\ (\alpha, \beta = 1,2,3,4))$ are

$$\begin{aligned}
&K_{ij}^{11} = \int_{x_a}^{x_b} A_{xx} \frac{\mathrm{d}\psi_i}{\mathrm{d}x}\frac{\mathrm{d}\psi_j}{\mathrm{d}x}\mathrm{d}x, \qquad F_i^1 = \int_{x_a}^{x_b} \frac{\mathrm{d}\psi_i}{\mathrm{d}x} N_{xx}^T\,\mathrm{d}x + \psi_i(x_a)Q_1 + \psi_i(x_b)Q_4 \\
&K_{ij}^{12} = -\int_{x_a}^{x_b} B_{xx} \frac{\mathrm{d}\psi_i}{\mathrm{d}x}\frac{\mathrm{d}^2\phi_j}{\mathrm{d}x^2}\mathrm{d}x, \qquad K_{i1}^{13} = \int_{x_a}^{x_b} A_{31}\frac{\mathrm{d}\psi_i}{\mathrm{d}x}\mathrm{d}x \\
&K_{ij}^{22} = \int_{x_a}^{x_b} D_{xx} \frac{\mathrm{d}^2\phi_i}{\mathrm{d}x^2}\frac{\mathrm{d}^2\phi_j}{\mathrm{d}x^2}\mathrm{d}x, \qquad K_{i1}^{24} = \int_{x_a}^{x_b} D_{31}\frac{\mathrm{d}^2\phi_i}{\mathrm{d}x^2}\,\mathrm{d}x \\
&F_i^2 = \int_{x_a}^{x_b} \left(\frac{\mathrm{d}^2\phi_i}{\mathrm{d}x^2} M_{xx}^T + \phi_i q\right)\mathrm{d}x + \phi_i(x_a)Q_2 + \left(-\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\right)_{x_a} Q_3 + \phi_i(x_b)Q_5 + \left(-\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\right)_{x_b} Q_6 \\
&K_{11}^{33} = -\int_{x_a}^{x_b} A_{33}\,\mathrm{d}x, \qquad K_{11}^{44} = -\int_{x_a}^{x_b} D_{33}\,\mathrm{d}x
\end{aligned} \tag{48}$$

Clearly, there is a coupling between the axial displacement u and the transverse displacement w due to the extensional-bending coefficient B.

4.2 Timoshenko beam theory

The virtual work statement for the Timoshenko beam element is given by

$$\begin{aligned}
0 = \int_0^L \Bigg[&\frac{\mathrm{d}\delta u}{\mathrm{d}x}\left(A_{xx}\frac{\mathrm{d}u}{\mathrm{d}x} + B_{xx}\frac{\mathrm{d}\phi_x}{\mathrm{d}x} - A_{31}E_3^{(0)} - N_{xx}^T\right) + \frac{\mathrm{d}\delta\phi_x}{\mathrm{d}x}\left(B_{xx}\frac{\mathrm{d}u}{\mathrm{d}x} + D_{xx}\frac{\mathrm{d}\phi_x}{\mathrm{d}x} - D_{31}E_3^{(1)} - M_{xx}^T\right) \\
&+ S_{xz}\left(\delta\phi_x + \frac{\mathrm{d}\delta w}{\mathrm{d}x}\right)\left(\phi_x + \frac{\mathrm{d}w}{\mathrm{d}x}\right) - \delta E_3^{(0)}\left(A_{31}\frac{\mathrm{d}u}{\mathrm{d}x} + A_{33}E_3^{(0)}\right) - \delta E_3^{(1)}\left(D_{31}\frac{\mathrm{d}\phi_x}{\mathrm{d}x} + D_{33}E_3^{(1)}\right) - \delta w q\Bigg]\mathrm{d}x - \sum_{i=1}^{6} Q_i\,\delta\Delta_i
\end{aligned} \tag{49}$$

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where

$$S_{xz} = K_s \int_A G(x,z)\mathrm{d}A \tag{50}$$

We assume Lagrange approximation of the all field variables (u, w, ϕ_x) independently

$$u(x) \approx \sum_{j=1}^{m} u_j \psi_j^{(1)}(x), \qquad w(x) \approx \sum_{j=1}^{n} w_j \psi_j^{(2)}(x), \qquad \phi_x \approx \sum_{j=1}^{p} S_j \psi_j^{(3)}(x) \tag{51}$$

where $\psi_j^{(\alpha)}(x)$ are the Lagrange polynomials of different order used for the three variables. Substitution of Eq. (51) for (u, w, ϕ_x), $\delta u = \psi_i^{(1)}$, $\delta w = \psi_i^{(2)}$, and $\delta\phi_x = \psi_i^{(3)}$ and constant values of $E_3^{(0)}$ and $E_3^{(1)}$ into the virtual work statement in Eq. (49), we obtain the finite element equations

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{0} & \mathbf{K}^{13} & \mathbf{K}^{14} & \mathbf{K}^{15} \\ \mathbf{0} & \mathbf{K}^{22} & \mathbf{K}^{23} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} & \mathbf{K}^{34} & \mathbf{K}^{35} \\ \mathbf{K}^{41} & \mathbf{K}^{42} & \mathbf{K}^{43} & K^{44} & 0 \\ \mathbf{K}^{51} & \mathbf{K}^{52} & \mathbf{K}^{53} & 0 & K^{55} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{w} \\ \mathbf{s} \\ E_3^{(0)} \\ E_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \mathbf{F}^3 \\ 0 \\ 0 \end{Bmatrix} \tag{52}$$

where

$$\begin{aligned}
&K_{ij}^{11} = \int_{x_a}^{x_b} A_{xx} \frac{\mathrm{d}\psi_i^{(1)}}{\mathrm{d}x} \frac{\mathrm{d}\psi_j^{(1)}}{\mathrm{d}x} \mathrm{d}x, \qquad F_i^1 = \int_{x_a}^{x_b} \frac{\mathrm{d}\psi_i^{(1)}}{\mathrm{d}x} N_{xx}^T \mathrm{d}x + \psi_i^{(1)}(x_a)Q_1 + \psi_i^{(1)}(x_b)Q_4 \\
&K_{ij}^{13} = \int_{x_a}^{x_b} B_{xx} \frac{\mathrm{d}\psi_i^{(1)}}{\mathrm{d}x} \frac{\mathrm{d}\psi_j^{(3)}}{\mathrm{d}x} \mathrm{d}x, \qquad K_{i1}^{14} = -\int_{x_a}^{x_b} A_{31} \frac{\mathrm{d}\psi_i^{(1)}}{\mathrm{d}x} \mathrm{d}x \\
&K_{ij}^{22} = \int_{x_a}^{x_b} S_{xz} \frac{\mathrm{d}\psi_i^{(2)}}{\mathrm{d}x} \frac{\mathrm{d}\psi_j^{(2)}}{\mathrm{d}x} \mathrm{d}x, \quad K_{ij}^{23} = \int_{x_a}^{x_b} S_{xz} \frac{\mathrm{d}\psi^{(2)}}{\mathrm{d}x} \psi_j^{(3)} \mathrm{d}x \\
&F_i^2 = \int_{x_a}^{x_b} \frac{\mathrm{d}\psi_i^{(2)}}{\mathrm{d}x} \frac{\mathrm{d}w}{\mathrm{d}x} N_{xx}^T \mathrm{d}x + \psi_i^{(2)}(x_a)Q_2 + \psi_i^{(2)}(x_b)Q_5 \\
&K_{ij}^{33} = \int_{x_a}^{x_b} \left(S_{xz} \psi_i^{(3)} \psi_j^{(3)} + D_{xx} \frac{\mathrm{d}\psi_i^{(3)}}{\mathrm{d}x} \frac{\mathrm{d}\psi_j^{(3)}}{\mathrm{d}x} \right) \mathrm{d}x, \qquad K_{i1}^{34} = -\int_{x_a}^{x_b} D_{31} \frac{\mathrm{d}\psi_i^{(3)}}{\mathrm{d}x} \mathrm{d}x \\
&K_{i1}^{35} = -\int_{x_a}^{x_b} D_{31} \frac{\mathrm{d}\psi_i^{(3)}}{\mathrm{d}x} \mathrm{d}x, \qquad F_i^3 = \int_{x_a}^{x_b} \frac{\mathrm{d}\psi_i^{(3)}}{\mathrm{d}x} M_{xx}^T \mathrm{d}x + \psi_i^{(3)}(x_a)Q_3 + \psi_i^{(3)}(x_b)Q_6 \\
&K_{11}^{44} = -\int_{x_a}^{x_b} A_{33} \mathrm{d}x, \qquad K_{11}^{45} = -\int_{x_a}^{x_b} A_{33} \mathrm{d}x, \qquad K_{11}^{55} = -\int_{x_a}^{x_b} D_{33} \mathrm{d}x
\end{aligned} \tag{53}$$

5 SUMMARY

The constitutive equations of electroelasticity for three-dimensional deformable solids are presented, and governing equations of the Bernoulli–Euler and Timoshenko beam theories that account for through-thickness power-law variation of a two-constituent material and piezoelectric layers are derived using the principle of virtual displacements. Virtual work statements of the two theories in terms of the generalized displacements are developed and their finite element models are formulated. The theoretical formulations and finite element models presented herein should help in the analysis of piezolaminated and adaptive structures such as beams and plates.

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ACKNOWLEDGEMENT

The research reported herein was carried out under National Science Foundation research grant CMMI-1030836.

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