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A Zigzag Theory with Local Shear Correction Factors for Semi-Analytical Bending Modal Analysis of Functionally Graded Viscoelastic Circular Sandwich Plates

M. Shariyat*, M.M. Alipour

Faculty of Mechanical Engineering, K.N. Toosi University of Technology, Tehran 19991-43344, Iran

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ABSTRACT

Free bending vibration analysis of the functionally graded viscoelastic circular sandwich plates is accomplished in the present paper, for the first time. Furthermore, local shear corrections factors are presented that may consider simultaneous effects of the gradual variations of the material properties and the viscoelastic behaviors of the materials, for the first time. Moreover, in contrast to the available works, a global-local zigzag theory rather than an equivalent single-layer theory is employed in the analysis. Another novelty is solving the resulted governing equations by a power series that may cover several boundary conditions. To extract more general conclusions, sandwich plates with both symmetric and asymmetric (with a bending-extension coupling) layups are considered. Results are validated by comparing some of them with results of the three-dimensional theory of elasticity, even for the thick plates. Influences of various geometric and material properties parameters on free vibration of the circular sandwich plates are evaluated in detail in the results section.

Keywords: Free bending vibration; global-local zigzag theory; functionally graded viscoelastic circular sandwich plate; semi-analytical solution

1 INTRODUCTION

SANDWICH plates, frequently employed in a three-layer construction, are extensively used in the engineering structures, especially for light structures with high cross section rigidities. In some applications, e.g. in circumstances where the upper or lower surface of the entire plate is likely to be subjected to concentrated impact loads or thermal shocks, these layers can be fabricated from functionally graded materials (FGMs) to meet the strength and toughness requirements [1]. The researches performed so far have mainly focused on sandwich viscoelastic circular plates with isotropic/composite face sheets and elastic/viscoelastic cores [2-5].

Vibration of circular plates exhibiting viscoelastic behaviors has been given less attention. Bailey and Chen [6] investigated vibration of the viscoelastic circular plates using the classical bending theory. Roy and Ganesan [7] developed a formulation for vibration and damping analyses of circular plates with constrained damping layer treatments. Yu and Huang [8] employed the classical shell theory to derive equations of motion of a three-layered circular plate with a thin viscoelastic layer. Natural frequencies and modal loss factors of a three-layered annular plate with a viscoelastic core were studied by Wang and Chen [9] using the complex modulus concept. Vibration and stability of a rotating sandwich plate with two face layers of polar orthotropic materials and a viscoelastic core layer were analyzed by Chen and Chen [10]. Shariyat and Alipour [11] and Alipour and Shariyat [12] presented

* Corresponding author. Tel.: +98 912 272 7199.
E-mail address: shariyat@kntu.ac.ir and m_shariyat@yahoo.com (M.Shariyat).

analytical solutions for vibration and buckling of heterogeneous single layer viscoelastic plates resting on elastic foundations.

Due to the simplicity in programming and economic computational run times, using the first-order global, local or global-local theories has been popular in engineering analyses. The main disadvantage of these models is that distributions of the transverse shear and normal stresses and in turn, the relevant strain energies are somewhat erroneous, especially for thicker structures. Some correction factors have mainly been proposed for the Timoshenko beams [13-15]. Using an energy equivalence principle, a shear correction factor was derived by Prabhu and Davalos [14] for laminated rectangular beams with arbitrary lay-up configurations. Stephen [13] was the first to incorporate the dependence on the aspect ratio of the cross-section into the shear correction factor.

Mindlin [16] derived two different shear coefficients; one was dependent on Poisson's ratio and the second was a constant. Stephen [17] by matching long wavelength phase velocity predictions between the second mode of Mindlin finite plate theory and the exact Rayleigh-Lamb frequency equation for flexural waves, came to a conclusion that the best correction factor is due to Mindlin (that includes Poisson's ratio effect). Andrew [18] derived a shear coefficient for a plate of infinite spatial extent. Liu and Soh [19] proposed two methods for determination of the shear correction and proposed the 0.8 shear correction factor for the homogeneous elastic materials. Kirakosyan [20] introduced extended concept of two correction coefficients for an orthotropic plate subjected to tangential shear at the top and bottom surfaces using the energy equivalence principle. Two shear correction expressions that account for the influence of the transverse normal stress component were derived by Batista [21] where the first was a slightly modified Mindlin factor. Birman and Bert [22] proposed using a correction factor equal to unity, for the two-skin and multi-skin sandwich structures. Sometimes, the shear correction factor concept may be used to impose the continuity condition of the transverse stresses at the layer interfaces. Huang [23] used concept of the shear correction factor in Reddy's third-order shell theory to accommodate effect of the continuity conditions of the interlaminar transverse shear stresses, based on the shear strain energy equivalence.

In the above mentioned researches, material properties were constant within each layer. To extend Mindlin's correction factor to the FGM plates, Efraim and Eisenberger [24] replaced Poisson's ratio with the average Poisson ratio of the mixture (using the volume fraction of each material in the entire cross-section). Extending an approach previously used for the composite beams [14,25] and plates [26,27], Nguyen et al. [28] proposed transverse shear factors for the FGM plates using the shear energy equivalence method. Then, the obtained shear correction factors were extended to three-layered sandwich plates having functionally graded face sheets.

The above brief literature survey reveals that almost only the global theories have been used for analysis of the circular sandwich plates so far. On the other hand, the traditional zigzag theories have not been employed for the circular plates. Recently, Shariyat [29-36] proposed global-local theories to extend ideas of the zigzag theories for bending, vibration, buckling, and postbuckling analyses of composite, FGM, and viscoelastic plates/shells under thermo-mechanical loads.

In the present paper, a power series solution and a new zigzag theory with local correction factors are developed for free damped bending vibration analysis of the three-layer circular functionally graded viscoelastic sandwich plates. Different novelties are included in the present analysis. The presented solution covers some boundary conditions and in contrast to some of the available zigzag theories, a priori satisfies continuity conditions of the transverse shear stress components.

2 DESCRIPTION OF THE GLOBAL-LOCAL DISPLACEMENT FIELD FOR DEVELOPMENT OF THE ZIGZAG SANDWICH PLATE THEORY

Let us consider the three-layered sandwich plate shown in Fig. 1. Generally, each layer of the sandwich plate may be an FGM viscoelastic layer. As shown in Fig. 1, the *z* coordinate is perpendicular to the reference plane of the plate (e.g., the mid-surface of the plate) and is positive upward and h_1, h_3, and h_2 are thicknesses of the top and bottom face sheets and thickness of the core, respectively.

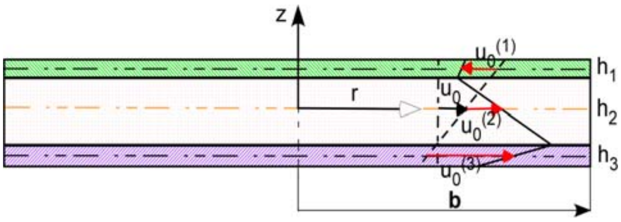

Fig.1
Geometric parameters of the considered functionally graded viscoelastic circular sandwich plate.

The displacement field within each individual layer may be considered to be composed of a local field that is superimposed on the global one. Consequently, the displacement field is affected by the local and global stiffness properties of the entire sandwich plate. Therefore, using linear global and local displacement fields and imposing the kinematic continuity conditions at the interfaces between layers, the displacement field of the entire sandwich plate may be described as follows [1]:

$$\begin{cases} u_1 = u_0 + z\psi_r + u_0^{(1)} + \xi^{(1)}\psi_r^{(1)} = u_0 + z\psi_r + \left(z - \frac{h_2}{2}\right)\psi_r^{(1)} + \frac{h_2}{2}\psi_r^{(2)}, & \frac{h_2}{2} \le z \le \frac{h_2}{2} + h_1 \\ u_2 = u_0 + z\psi_r + u_0^{(2)} + \xi^{(2)}\psi_r^{(2)} = u_0 + z\psi_r + z\psi_r^{(2)}, & -\frac{h_2}{2} \le z \le \frac{h_2}{2} \\ u_3 = u_0 + z\psi_r + u_0^{(3)} + \xi^{(3)}\psi_r^{(3)} = u_0 + z\psi_r + \left(z + \frac{h_2}{2}\right)\psi_r^{(3)} - \frac{h_2}{2}\psi_r^{(2)}, & \frac{-h_2}{2} \le z \le -\frac{h_2}{2} \\ w = w_0, & \frac{-h_2}{2} - h_3 \le z \le \frac{h_2}{2} + h_1 \end{cases} \tag{1}$$

where u_0 and w_0 are the radial and transverse displacement components of the reference layer (e.g. the mid-surface) of the circular plate, respectively. ψ_r is the global rotation of the normal to the reference surface. $u_0^{(i)}$ and $\psi_r^{(i)}$ are the local mid-surface radial displacement and rotation of the *i*th layer of the plate, respectively. The displacement components u_0 and w_0 can incorporate the extensions-bending couplings caused by the material heterogeneity. Eq. (1) implies that the defined radial displacement field leads to an asymptotic curve for a third-order curve. However, since the traction continuity conditions at the interfaces between layers are incorporated implicitly in the present theory, the resulted field may be more accurate than that of the third-order global (equivalent single layer) theories.

For small deflections, the strain-displacement relations may be written as [37]:

$$\varepsilon_r = u,_r \quad \varepsilon_\theta = \frac{u}{r} \quad \varepsilon_{rz} = u,_z + w,_r \tag{2}$$

where the symbol "," stands for the partial derivative. On the other hand, if order of the transverse normal strain is ignorable in comparison with that of the in-plane strains, Hooke's generalized stress-strain law may be expressed as [37]:

$$\sigma_r = \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta), \quad \sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r), \quad \sigma_{rz} = \frac{E}{2(1+\nu)}\varepsilon_{rz} \tag{3}$$

Assuming that each layer of the plate is made from a mixture of ceramic and metal materials and following a common procedure, one may express variations of a representative material property p within each layer as [38]:

$$p = p_c V_c + p_m V_m \tag{4}$$

in which the subscripts c and m refer to ceramic and metal, respectively (the same relation is valid for a material mixture made of any other two constituent materials). V_c and V_m are the volume fractions of the ceramic and metal, respectively and are related as follows:

$$V_c + V_m = 1 \tag{5}$$

The metal volume fraction is assumed to follow a power–law distribution [38]:

$$V_m = \left(\frac{1}{2} - \frac{z}{h}\right)^g \tag{6}$$

where g is the positive definite volume fraction index. From Eqs. (4-6), the layerwise variations of the moduli of elasticity may be expressed as:

$$E_1 = \left[(E_{t1} - E_{b1})\left(\frac{z}{h_1} - \frac{h_2}{2h_1}\right)^{g_1} + E_{b1}\right] \qquad \frac{h_2}{2} \le z \le \frac{h_2}{2} + h_1 \tag{7}$$

$$E_2(z) = [E_2'(z) + iE_2''(z)] = E_2'(z)\left(1 + i\eta_2\right) \qquad -\frac{h_2}{2} \le z \le \frac{h_2}{2} \tag{8}$$

$$E_3 = \left[(E_{t3} - E_{b3})\left(\frac{z}{h_3} + \frac{h_2}{2h_3} + 1\right)^{g_3} + E_{b3}\right] \qquad -h_3 - \frac{h_2}{2} \le z \le -\frac{h_2}{2} \tag{9}$$

where the subscripts t and b respectively denote the top and bottom layers of the corresponding layer and η is the loss factor of the viscoelastic material. Generally, all the layers may be made of viscoelastic functionally graded materials. In this regard, elastic moduli E_t and E_b appeared in Eqs. (7) and (9) may have real and imaginary components, i.e. [39]:

$$E_t = E_t'(1 + i\eta_t), \quad E_b = E_b'(1 + i\eta_b) \tag{10}$$

where E' is the real component. On the other hand, the transverse coordinate dependency of the elastic modulus of the mid-layer is chosen to be general, e.g.

$$E_2'(z) = \left[(E_{b2} - E_{t2})\left(\frac{1}{2} - \frac{z}{h_2}\right)^{g_2} + E_{t2}\right] \qquad -\frac{h_2}{2} \le z \le \frac{h_2}{2} \tag{11}$$

3 DERIVATION OF THE LOCAL SHEAR CORRECTION FACTORS

Two correction procedures are commonly used. One procedure is using an iterative or successive approximation method which begins with estimating the in-plane stress components based on the initial assumptions and revising the results by employing the transverse stress components predicted by the three-dimensional theory of elasticity and the resulted displacement field as a new staring point and so on. A somewhat similar but single-stage procedure is introducing the shear correction factor concept in the total potential energy or the governing equations of the structure. However, the resulting shear correction factor has to lead to results that are consistent with the normal and shear traction boundary conditions of the top, bottom, and interfaces between layers (in case of a multilayered or sandwich plate/shell).

The available correction factors have been derived based on the equivalent single layer theories employing the $\psi_r = \psi_r^{(1)} = \psi_r^{(2)} = \psi_r^{(3)}$ assumption. Generally, rotations of the layers are not identical and in some cases, even sign of the rotation of the core may be opposite to those of the face sheets.

Based on the three-dimensional theory of elasticity, the equilibrium equations in terms of the stress components of a specific layer may be expressed as [40]:

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad x \leftrightarrow y \\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0
\end{aligned} \tag{12}$$

where σ_x, σ_y, and σ_z are the normal and $\tau_{xz}, \tau_{yz}, \tau_{xy}$ are the shear stress components.

The symbol $x \leftrightarrow y$ means that an additional equation may be written by interchanging x and y. Hence, for a three-layered sandwich plate, Eq. (12) must hold for each individual layer:

$$\begin{cases}
\dfrac{\partial \sigma_x^{(1)}}{\partial x} + \dfrac{\partial \tau_{xy}^{(1)}}{\partial y} + \dfrac{\partial \tau_{xz}^{(1)}}{\partial z} = 0 \qquad x \leftrightarrow y \\
\dfrac{\partial \tau_{xz}^{(1)}}{\partial x} + \dfrac{\partial \tau_{yz}^{(1)}}{\partial y} + \dfrac{\partial \sigma_z^{(1)}}{\partial z} = 0
\end{cases} \qquad \frac{h_2}{2} \le z \le \frac{h_2}{2} + h_1 \tag{13}$$

By integration of the elasticity equilibrium Eq. (13) across the plate thickness and using the following boundary condition

$$\tau_{xz}^{(1)}\Big|_{\left(x,y,\frac{h_2}{2}+h_1\right)} = \tau_{xz}^{(3)}\Big|_{\left(x,y,-\frac{h_2}{2}-h_3\right)} = 0 \quad x \leftrightarrow y \tag{14}$$

One arrives at

$$\begin{cases}
\dfrac{\partial M_x^{(1)}}{\partial x} + \dfrac{\partial M_{xy}^{(1)}}{\partial y} - \dfrac{h_2}{2}\tau_{xz}^{(1)}\Big|_{\left(z=\frac{h_2}{2}\right)} - Q_x^{(1)} = 0 \qquad x \leftrightarrow y \\
Q_{x,x}^{(1)} + Q_{y,y}^{(1)} = 0
\end{cases} \qquad \frac{h_2}{2} \le z \le \frac{h_2}{2} + h_1 \tag{15}$$

where M_x, M_y are the bending and M_{xy} is the twisting moments per unit length and Q_x , Q_y are the transverse shear forces per unit length [37]:

$$\begin{aligned}
Q_x^{(1)} &= \kappa \int_{h_2/2}^{h_2/2+h_1} \tau_{xz}^{(1)} dz \\
Q_y^{(1)} &= \kappa \int_{h_2/2}^{h_2/2+h_1} \tau_{yz}^{(1)} dz
\end{aligned} \tag{16}$$

where κ denotes the transverse shear correction factor which is usually introduced in the first-order shear-deformation plate/shell theories (FSDT) in order to correct the transverse shear rigidities of the plate. Due to the local variations of the rotations, the available shear correction factors that have been proposed based on definition of a single rotation for the entire thickness of the plate, cannot be used. Therefore, each layer should have a distinct local shear correction factor.

Since the through-thickness distributions of the in-plane stress components are linear in bending, one may write:

$$\begin{cases} M_x^{(1)} = \int_{h_2/2}^{h_2/2+h_1} \sigma_x^{(1)} z dz \;\rightarrow\; \sigma_x^{(1)} = \dfrac{z}{C_1} M_x^{(1)} \qquad x \leftrightarrow y \\ M_{xy}^{(1)} = \int_{h_2/2}^{h_2/2+h_1} \tau_{xy}^{(1)} z dz \;\rightarrow\; \tau_{xy}^{(1)} = \dfrac{z}{C_1} M_{xy}^{(1)} \end{cases} , \; C_1 = \frac{1}{3}\left[\left(\frac{h_2}{2}+h_1\right)^3 - \left(\frac{h_2}{2}\right)^3\right]$$

$$\begin{cases} M_x^{(2)} = \int_{-h_2/2}^{h_2/2} \sigma_x^{(2)} z dz \;\rightarrow\; \sigma_x^{(2)} = \dfrac{z}{C_2} M_x^{(2)} \qquad x \leftrightarrow y \\ M_{xy}^{(2)} = \int_{-h_2/2}^{h_2/2} \tau_{xy}^{(2)} z dz \;\rightarrow\; \tau_{xy}^{(2)} = \dfrac{z}{C_2} M_{xy}^{(2)} \end{cases} , \; C_2 = \frac{h_2^3}{12} \tag{17}$$

$$\begin{cases} M_x^{(3)} = \int_{-h_2/2-h_3}^{-h_2/2} \sigma_x^{(3)} z dz \;\rightarrow\; \sigma_x^{(3)} = \dfrac{z}{C_3} M_x^{(3)} \qquad x \leftrightarrow y \\ M_{xy}^{(3)} = \int_{-h_2/2-h_3}^{-h_2/2} \tau_{xy}^{(3)} z dz \;\rightarrow\; \tau_{xy}^{(3)} = \dfrac{z}{C_3} M_{xy}^{(3)} \end{cases} , \; C_3 = \frac{1}{3}\left[\left(\frac{h_2}{2}+h_3\right)^3 - \left(\frac{h_2}{2}\right)^3\right]$$

Substituting Eq. (17) into Eq. (13) and using Eq. (15) lead to the following results:

$$\frac{\partial \tau_{xz}^{(1)}}{\partial z} = -\frac{z}{C_1}\left(\frac{h_2}{2}\tau_{xz\left(z=\frac{h_2}{2}\right)}^{(1)} + Q_x^{(1)}\right) \qquad x \leftrightarrow y$$

$$\frac{\partial \tau_{xz}^{(2)}}{\partial z} = \frac{z}{C_2}\left[\frac{h_2}{2}\left(\tau_{xz\left(z=\frac{h_2}{2}\right)}^{(2)} - \tau_{xz\left(z=-\frac{h_2}{2}\right)}^{(2)}\right) - Q_x^{(2)}\right] \qquad x \leftrightarrow y \tag{18}$$

$$\frac{\partial \tau_{xz}^{(3)}}{\partial z} = \frac{z}{C_3}\left(-\frac{h_2}{2}\tau_{xz\left(z=-\frac{h_2}{2}\right)}^{(3)} - Q_x^{(3)}\right) \qquad x \leftrightarrow y$$

Integrating Eq. (18) across the plate thickness and using the boundary condition appearing in Eq. (14):

$$\tau_{xz}^{(1)} = \frac{1}{2C_1}\left[\left(\frac{h_2}{2}+h_1\right)^2 - z^2\right]\left(\frac{h_2}{2}\tau_{xz\left(z=\frac{h_2}{2}\right)}^{(1)} + Q_x^{(1)}\right)$$

$$\tau_{xz}^{(2)} = \frac{1}{2C_2}\left[z^2 - \left(\frac{h_2}{2}\right)^2\right]\left[\frac{h_2}{2}\left(\tau_{xz\left(z=\frac{h_2}{2}\right)}^{(2)} - \tau_{xz\left(z=-\frac{h_2}{2}\right)}^{(2)}\right) - Q_x^{(2)}\right] + \tau_{xz\left(z=\frac{h_2}{2}\right)}^{(1)} \tag{19}$$

$$\tau_{xz}^{(3)} = \frac{1}{2C_3}\left[\left(\frac{h_2}{2}+h_3\right)^2 - z^2\right]\left(\frac{h_2}{2}\tau_{xz\left(z=-\frac{h_2}{2}\right)}^{(3)} + Q_x^{(3)}\right)$$

The transverse shear stresses at the interfaces between the layers are obtained by substituting $z = h_2/2$ and $z = -h_2/2$ into first and third equations of Eq. (19) , respectively:

$$\tau_{xz\left(z=\frac{h_2}{2}\right)} = \frac{2h_1(h_1+h_2)}{4C_1 - h_1h_2(h_1+h_2)}Q_x^{(1)} \qquad \tau_{xz\left(z=-\frac{h_2}{2}\right)} = \frac{2h_3(h_3+h_2)}{4C_3 - h_2h_3(h_3+h_2)}Q_x^{(3)} \tag{20}$$

Finally, by substituting Eq. (20) into Eq. (19), variations of the transverse shear stress within each layer are obtained as follow:

$$
\begin{aligned}
\tau_{xz}^{(1)} &= \frac{1}{2C_1}\left[\left(\frac{h_2}{2}+h_1\right)^2 - z^2\right]\left(\frac{h_2}{2}\frac{2h_1\left(h_1+h_2\right)}{4C_1-h_1h_2(h_1+h_2)}+1\right)Q_x^{(1)} \\
\tau_{xz}^{(2)} &= \frac{1}{2C_2}\left[z^2-\left(\frac{h_2}{2}\right)^2\right]\left[\frac{h_2}{2}\left(\frac{2h_1\left(h_1+h_2\right)}{4C_1-h_1h_2(h_1+h_2)}Q_x^{(1)} - \frac{2h_3(h_3+h_2)}{4C_3-h_2h_3(h_3+h_2)}Q_x^{(3)}\right)-Q_x^{(2)}\right]+\frac{2h_1\left(h_1+h_2\right)}{4C_1-h_1h_2(h_1+h_2)}Q_x^{(1)} \\
\tau_{xz}^{(3)} &= \frac{1}{2C_3}\left[\left(\frac{h_2}{2}+h_3\right)^2 - z^2\right]\left[\frac{h_2}{2}\frac{2h_3(h_3+h_2)}{4C_3-h_2h_3(h_3+h_2)}+1\right]Q_x^{(3)}
\end{aligned} \tag{21}
$$

The accurate strain energy of the transverse shear stress may be derived based on the transverse shear stresses determined based on the theory of elasticity:

$$
U_s^{(1)} = \frac{1}{2}\int_{h_2/2}^{h_2/2+h_1}\frac{1}{G_1(z)}\left(\tau_{xz}^{(1)}\right)^2 dz \qquad U_s^{(2)} = \frac{1}{2}\int_{-h_2/2}^{h_2/2}\frac{1}{G_2(z)}\left(\tau_{xz}^{(2)}\right)^2 dz \qquad U_s^{(3)} = \frac{1}{2}\int_{-h_2/2-h_3}^{-h_2/2}\frac{1}{G_2(z)}\left(\tau_{xz}^{(3)}\right)^2 dz \tag{22}
$$

On the other hand, the strain energies determined based on the constitutive equations may be corrected as follows:

$$
\tilde{U}_s^{(1)} = \frac{1}{2}\int_{h_2/2}^{h_2/2+h_1}\frac{\left(Q_x^{(1)}\right)^2}{\kappa^{(1)}G_1(z)h_1^2}dz,\quad \tilde{U}_s^{(2)} = \frac{1}{2}\int_{-h_2/2}^{h_2/2}\frac{\left(Q_x^{(2)}\right)^2}{\kappa^{(2)}G_2(z)h_2^2}dz,\quad \tilde{U}_s^{(3)} = \frac{1}{2}\int_{-h_2/2-h_3}^{-h_2/2}\frac{\left(Q_x^{(3)}\right)^2}{\kappa^{(3)}G_3(z)h_3^2}dz \tag{23}
$$

Equating the corresponding strain energies of the transverse shear stresses appeared in Eqs. (22) and (23), leads to the following set of the local correction factors for the sandwich plate:

$$
\begin{aligned}
\kappa^{(1)} &= \frac{\displaystyle\int_{h_2/2}^{h_2/2+h_1}\frac{\left(Q_x^{(1)}\right)^2}{G_1(z)h_1^2}dz}{\displaystyle\int_{h_2/2}^{h_2/2+h_1}\frac{\left(\tau_{xz}^{(1)}\right)^2}{G_1(z)}dz} = \frac{\displaystyle\int_{h_2/2}^{h_2/2+h_1}\frac{\left(Q_x^{(1)}\right)^2}{G_1(z)h_1^2}dz}{\displaystyle\int_{h_2/2}^{h_2/2+h_1}\frac{\left\{\frac{1}{2C_1}\left[\left(\frac{h_2}{2}+h_1\right)^2-z^2\right]\left[\frac{h_2}{2}\frac{2h_1\left(h_1+h_2\right)}{4C_1-h_1h_2(h_1+h_2)}+1\right]Q_x^{(1)}\right\}^2}{G_1(z)}dz} \\
\kappa^{(2)} &= \frac{\displaystyle\int_{-h_2/2}^{h_2/2}\frac{\left(Q_x^{(2)}\right)^2}{G_2(z)h_2^2}dz}{\displaystyle\int_{-h_2/2}^{h_2/2}\frac{\left(\tau_{xz}^{(2)}\right)^2}{G_2(z)}dz} = \int_{-h_2/2}^{h_2/2}\frac{\left(Q_x^{(2)}\right)^2}{G_2(z)h_2^2}dz \Bigg/ \int_{-h_2/2}^{h_2/2}\frac{1}{G_2(z)}\left\{\frac{1}{2C_2}\left[z^2-\left(\frac{h_2}{2}\right)^2\right]dz\right. \\
&\quad \left.\left[\frac{h_2}{2}\left(\frac{2h_1\left(h_1+h_2\right)}{4C_1-h_1h_2(h_1+h_2)}Q_x^{(1)}-\frac{2h_3(h_3+h_2)}{4C_3-h_2h_3(h_3+h_2)}Q_x^{(3)}\right)-Q_x^{(2)}\right]+\frac{2h_1\left(h_1+h_2\right)}{4C_1-h_1h_2(h_1+h_2)}Q_x^{(1)}\right\}^2 \\
\kappa^{(3)} &= \frac{\displaystyle\int_{-h_2/2-h_3}^{-h_2/2}\frac{\left(Q_x^{(3)}\right)^2}{G_3(z)h_3^2}dz}{\displaystyle\int_{-h_2/2-h_3}^{-h_2/2}\frac{\left(\tau_{xz}^{(3)}\right)^2}{G_3(z)}dz} = \frac{\displaystyle\int_{-h_2/2-h_3}^{-h_2/2}\frac{\left(Q_x^{(3)}\right)^2}{G_3(z)h_3^2}dz}{\displaystyle\int_{-h_2/2-h_3}^{-h_2/2}\frac{\left\{\frac{1}{2C_3}\left[\left(\frac{h_2}{2}+h_3\right)^2-z^2\right]\left[\frac{h_2}{2}\frac{2h_3(h_3+h_2)}{4C_3-h_2h_3(h_3+h_2)}+1\right]Q_x^{(3)}\right\}^2}{G_3(z)}dz}
\end{aligned} \tag{24}
$$

For the special case of a sandwich plate whose material properties are constant within each layer, Eq. (24) leads to the following local shear correction factors:

$$\kappa^{(1)} = \frac{5}{6}\frac{(3h_2+4h_1)^2}{(3h_2+4h_1)^2+h_2(h_2+h_1)}, \qquad \kappa^{(2)} = \frac{5\left(Q_x^{(2)}\right)^2 h_1^2 h_3^2}{6}\frac{\mathrm{X}_1}{\mathrm{X}_2}, \qquad \kappa^{(3)} = \frac{5}{6}\frac{(3h_2+4h_3)^2}{(3h_2+4h_3)^2+h_2(h_2+h_3)} \tag{25}$$

where

$$\begin{aligned}
X_1 &= 81h_2^4 + 216h_2^3(h_3+h_1) + 144h_2^2\left(h_3^2+h_1^2\right) + 576h_2^2h_3h_1 + 256h_1^2h_3^2 + 384h_3h_2h_1(h_3+h_1) \\
X_2 &= Q_x^{(1)}Q_x^{(3)}h_1h_2h_3\Big[588h_3h_2^3h_1 + 336h_2^2h_1h_3(h_1+h_3) + 252h_2^4(h_1+h_3) + 192h_2h_3^2h_1^2 \\
&\quad +144\,h_2^3\left(h_3^2+h_1^2\right) + 108h_2^5\Big] + Q_x^{(1)}Q_x^{(2)}h_1h_2h_3\Big(252h_3h_2^3h_1 + 672h_3^2h_2^2h_1 + 144h_3h_2^2h_1^2 + 384h_3^2h_2h_1^2 \\
&\quad +448h_2h_3^3h_1 + 256\,h_3^3h_1^2 + 108h_3h_2^4 + 288\,h_3^2h_2^3 + 192\,h_3^3h_2^2\Big) + Q_x^{(3)}Q_x^{(2)}h_1h_2h_3\Big(378h_1h_2^3h_3 \\
&\quad +216h_1h_2^2h_3^2 + 1008\,h_1^2h_2^2h_3 + 576h_1^2h_2h_3^2 + 672\,h_2h_1^3h_3 + 384h_1^3h_3^2 + 162h_1h_2^4 + 432h_1^2h_2^3 \\
&\quad +288h_1^3h_2^2\Big) + \left(Q_x^{(1)}\right)^2\Big(432\,h_3^3h_2^4h_1 + 144h_2^2h_3^4h_1^2 + 288\,h_2^3h_3^4h_1 + 216\,h_3^3h_2^3h_1^2 + 162\,h_3^2h_2^5h_1 \\
&\quad +81\,h_3^2h_2^6 + 216\,h_3^3h_2^5 + 81h_3^2h_2^4h_1^2 + 144h_3^4h_2^4\Big) + \left(Q_x^{(3)}\right)^2\Big(81h_1^2h_2^4h_3^2 + 216\,h_1^3h_2^3h_3^2 + 162\,h_1^2h_2^5h_3 \\
&\quad +432\,h_1^3h_2^4h_3 + 144\,h_2^2h_1^4h_3^2 + 288\,h_2^3h_1^4h_3 + 81\,h_1^2h_2^6 + 216h_1^3h_2^5 + 144h_1^4h_2^4\Big) + \left(Q_x^{(2)}\right)^2\Big[81\,h_1^2h_3^2h_2^4 \\
&\quad +576\,h_1^3h_3^3h_2^2 + 216h_2^3h_1^2h_3^2(h_1+h_3) + 144h_2^2h_1^2h_3^2\left(h_3^2+h_1^2\right) + 384h_2h_1^3h_3^3(h_1+h_3) + 256\,h_1^4h_3^4\Big]
\end{aligned} \tag{26}$$

4 GOVERNING EQUATIONS OF THE CIRCULAR FGM SANDWICH PLATE

The governing equations of the sandwich plate may be derived by using the principle of minimum total potential energy.

$$\delta\Pi = \delta U + \delta K - \delta W = 0 \tag{27}$$

where δU, δK, and δW are variations of the strain energy, kinetic energy, and energy of the externally applied loads, respectively:

$$\delta U = \int_V \delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}\,dV = \frac{1}{2}\int_V (\sigma_r\delta\varepsilon_r + \sigma_\theta\delta\varepsilon_\theta + \tau_{rz}\delta\gamma_{rz})dV, \tag{28}$$

$$\delta K = \int_V \rho\left(\ddot{u}\delta u + \ddot{w}\delta w\right)dV, \tag{29}$$

Employing this principle leads to the following three governing equations of the plate in the cylindrical coordinate system (*r, θ, z*):

$\delta u_0 \neq 0$:

$$\begin{aligned}
&\frac{N_r^{(1)}-N_\theta^{(1)}}{r} + N_{r,r}^{(1)} + \frac{N_r^{(2)}-N_\theta^{(2)}}{r} + N_{r,r}^{(2)} + \frac{N_r^{(3)}-N_\theta^{(3)}}{r} + N_{r,r}^{(3)} = \left(I_0^{(1)}+I_0^{(2)}+I_0^{(3)}\right)\ddot{u}_0 + \left(I_1^{(1)}+I_1^{(2)}+I_1^{(3)}\right)\ddot{\psi}_r \\
&+\left(I_1^{(1)} - \frac{h_2}{2}I_0^{(1)}\right)\ddot{\psi}_r^{(1)} + \left(\frac{h_2}{2}I_0^{(1)} + I_1^{(2)} - \frac{h_2}{2}I_0^{(3)}\right)\ddot{\psi}_r^{(2)} + \left(I_1^{(3)} + \frac{h_2}{2}I_0^{(3)}\right)\ddot{\psi}_r^{(3)}
\end{aligned} \tag{30}$$

$\delta\psi_r \neq 0$:

$$\frac{M_r^{(1)}-M_\theta^{(1)}}{r}+M_{r,r}^{(1)}+\frac{M_r^{(2)}-M_\theta^{(2)}}{r}+M_{r,r}^{(2)}+\frac{M_r^{(3)}-M_\theta^{(3)}}{r}+M_{r,r}^{(3)}-Q_r^{(1)}-Q_r^{(2)}-Q_r^{(3)}=\left(I_1^{(1)}+I_1^{(2)}+I_1^{(3)}\right)\ddot{u}_0$$
$$+\left(I_2^{(1)}+I_2^{(2)}+I_2^{(3)}\right)\ddot{\psi}_r+\left(I_2^{(1)}-\frac{h_2}{2}I_1^{(1)}\right)\ddot{\psi}_r^{(1)}-\left(\frac{h_2}{2}I_1^{(3)}-I_2^{(2)}-\frac{h_2}{2}I_1^{(1)}\right)\ddot{\psi}_r^{(2)}+\left(I_2^{(3)}+\frac{h_2}{2}I_1^{(3)}\right)\ddot{\psi}_r^{(3)} \tag{31}$$

$\delta\psi_r^{(1)} \neq 0$:

$$\frac{M_r^{(1)}-M_\theta^{(1)}}{r}+M_{r,r}^{(1)}-\frac{h_2}{2}\frac{N_r^{(1)}-N_\theta^{(1)}}{r}-\frac{h_2}{2}N_{r,r}^{(1)}-Q_r^{(1)}=\left(I_1^{(1)}-\frac{h_2}{2}I_0^{(1)}\right)\ddot{u}_0+\left(I_2^{(1)}-\frac{h_2}{2}I_1^{(1)}\right)\ddot{\psi}_r$$
$$+\left(\frac{h_2^2}{4}I_0^{(1)}-h_2I_1^{(1)}+I_2^{(1)}\right)\ddot{\psi}_r^{(1)}+\frac{h_2}{2}\left(I_1^{(1)}-\frac{h_2}{2}I_0^{(1)}\right)\ddot{\psi}_r^{(2)} \tag{32}$$

$\delta\psi_r^{(2)} \neq 0$:

$$\frac{M_r^{(2)}-M_\theta^{(2)}}{r}+M_{r,r}^{(2)}+\frac{h_2}{2}\frac{N_r^{(1)}-N_\theta^{(1)}}{r}+\frac{h_2}{2}N_{r,r}^{(1)}-\frac{h_2}{2}\frac{N_r^{(3)}-N_\theta^{(3)}}{r}-\frac{h_2}{2}N_{r,r}^{(3)}-Q_r^{(2)}=\left(\frac{h_2}{2}I_0^{(1)}+I_1^{(2)}+\frac{h_2}{2}I_0^{(3)}\right)\ddot{u}_0$$
$$+\left(\frac{h_2}{2}I_1^{(1)}+I_2^{(2)}+\frac{h_2}{2}I_1^{(3)}\right)\ddot{\psi}_r+\frac{h_2}{2}\left(I_1^{(1)}-\frac{h_2}{2}I_0^{(1)}\right)\ddot{\psi}_r^{(1)}+\left(\frac{h_2^2}{4}I_0^{(1)}+I_2^{(2)}-\frac{h_2^2}{4}I_0^{(3)}\right)\ddot{\psi}_r^{(2)}+\frac{h_2}{2}\left(I_1^{(3)}+\frac{h_2}{2}I_0^{(3)}\right)\ddot{\psi}_r^{(3)} \tag{33}$$

$\delta\psi_r^{(3)} \neq 0$:

$$\frac{M_r^{(3)}-M_\theta^{(3)}}{r}+M_{r,r}^{(3)}+\frac{h_2}{2}\frac{N_r^{(3)}-N_\theta^{(3)}}{r}+\frac{h_2}{2}N_{r,r}^{(3)}-Q_r^{(3)}=\left(\frac{h_2}{2}I_0^{(3)}+I_1^{(3)}\right)\ddot{u}_0+\left(\frac{h_2}{2}I_1^{(3)}+I_2^{(3)}\right)\ddot{\psi}_r$$
$$+\left(\frac{h_2^2}{4}I_0^{(3)}+I_2^{(3)}+h_2I_1^{(3)}\right)\ddot{\psi}_r^{(3)}-\frac{h_2}{2}\left(\frac{h_2}{2}I_0^{(3)}+I_1^{(3)}\right)\ddot{\psi}_r^{(2)} \tag{34}$$

$\delta w \neq 0$:

$$Q_{r,r}^{(1)}+\frac{Q_r^{(1)}}{r}+Q_{r,r}^{(2)}+\frac{Q_r^{(2)}}{r}+Q_{r,r}^{(3)}+\frac{Q_r^{(3)}}{r}+\frac{1}{r}\left(r\bar{N}_r w_{,r}\right)_{,r}=q+\left(I_0^{(1)}+I_0^{(2)}+I_0^{(3)}\right)\ddot{w} \tag{35}$$

The stress resultants M, N, and Q and the higher-order inertias are defined in the chosen cylindrical coordinate as:

$$\begin{Bmatrix}M_r^{(1)}\\M_\theta^{(1)}\end{Bmatrix}=\int_{\frac{h_2}{2}}^{\frac{h_2}{2}+h_1}\begin{Bmatrix}\sigma_r^{(1)}\\\sigma_\theta^{(1)}\end{Bmatrix}zdz,\quad \begin{Bmatrix}M_r^{(2)}\\M_\theta^{(2)}\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix}\sigma_r^{(2)}\\\sigma_\theta^{(2)}\end{Bmatrix}zdz,\quad \begin{Bmatrix}M_r^{(3)}\\M_\theta^{(3)}\end{Bmatrix}=\int_{-\frac{h_2}{2}-h_3}^{-\frac{h_2}{2}}\begin{Bmatrix}\sigma_r^{(3)}\\\sigma_\theta^{(3)}\end{Bmatrix}zdz,$$
$$\begin{Bmatrix}N_r^{(1)}\\N_\theta^{(1)}\end{Bmatrix}=\int_{\frac{h_2}{2}}^{\frac{h_2}{2}+h_1}\begin{Bmatrix}\sigma_r^{(1)}\\\sigma_\theta^{(1)}\end{Bmatrix}dz,\quad \begin{Bmatrix}N_r^{(2)}\\N_\theta^{(2)}\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix}\sigma_r^{(2)}\\\sigma_\theta^{(2)}\end{Bmatrix}dz,\quad \begin{Bmatrix}N_r^{(3)}\\N_\theta^{(3)}\end{Bmatrix}=\int_{-\frac{h_2}{2}-h_3}^{-\frac{h_2}{2}}\begin{Bmatrix}\sigma_r^{(3)}\\\sigma_\theta^{(3)}\end{Bmatrix}dz,$$
$$\begin{Bmatrix}N_r^{(1)}\\N_\theta^{(1)}\end{Bmatrix}=\int_{\frac{h_2}{2}}^{\frac{h_2}{2}+h_1}\begin{Bmatrix}\sigma_r^{(1)}\\\sigma_\theta^{(1)}\end{Bmatrix}dz,\quad \begin{Bmatrix}N_r^{(2)}\\N_\theta^{(2)}\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{Bmatrix}\sigma_r^{(2)}\\\sigma_\theta^{(2)}\end{Bmatrix}dz,\quad \begin{Bmatrix}N_r^{(3)}\\N_\theta^{(3)}\end{Bmatrix}=\int_{-\frac{h_2}{2}-h_3}^{-\frac{h_2}{2}}\begin{Bmatrix}\sigma_r^{(3)}\\\sigma_\theta^{(3)}\end{Bmatrix}dz, \tag{36}$$

$$Q_r^{(1)} = \int_{\frac{h_2}{2}}^{\frac{h_2}{2}+h_1} \sigma_{rz}^{(1)} dz, \quad Q_r^{(2)} = \int_{-h/2}^{h/2} \sigma_{rz}^{(2)} dz, \quad Q_r^{(3)} = \int_{-\frac{h_2}{2}-h_3}^{-\frac{h_2}{2}} \sigma_{rz}^{(3)} dz, \tag{37}$$

$$\begin{Bmatrix} I_0^{(1)} \\ I_1^{(1)} \\ I_2^{(1)} \end{Bmatrix} = \int_{-h_2/2}^{h_2/2+h_1} \rho^{(1)} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz, \; \begin{Bmatrix} I_0^{(2)} \\ I_1^{(2)} \\ I_2^{(2)} \end{Bmatrix} = \int_{-h_2/2}^{h_2/2} \rho^{(2)} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz, \; \begin{Bmatrix} I_0^{(3)} \\ I_1^{(3)} \\ I_2^{(3)} \end{Bmatrix} = \int_{-h_2/2-h_3}^{-h_2/2} \rho^{(3)} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz \tag{38}$$

Based on Eqs. (1-3), Eqs. (36-38) may be rewritten in the following form:

$$\begin{aligned}
&\begin{Bmatrix} M_r^{(1)} \\ N_r^{(1)} \end{Bmatrix} = \begin{Bmatrix} B^{(1)} \\ A^{(1)} \end{Bmatrix} \left[u_{0,r} + \frac{h_2}{2}\left(\psi_{r,r}^{(2)} - \psi_{r,r}^{(1)}\right) + \frac{\nu^{(1)}}{r}\left(u_0 + \frac{h_2}{2}\left(\psi_r^{(2)} - \psi_r^{(1)}\right)\right)\right] + \begin{Bmatrix} D^{(1)} \\ B^{(1)} \end{Bmatrix}\left(\psi_{r,r}^{(1)} + \psi_{r,r} + \nu^{(1)} \frac{\psi_r^{(1)} + \psi_r}{r} \right) \\
&\begin{Bmatrix} M_r^{(2)} \\ N_r^{(2)} \end{Bmatrix} = \begin{Bmatrix} B^{(2)} \\ A^{(2)} \end{Bmatrix} \left(u_{0,r} + \nu^{(2)} \frac{u_0}{r} \right) + \begin{Bmatrix} D^{(2)} \\ B^{(2)} \end{Bmatrix}\left(\psi_{r,r}^{(2)} + \psi_{r,r} + \nu^{(2)} \frac{\psi_r^{(2)} + \psi_r}{r} \right) \\
&\begin{Bmatrix} M_r^{(3)} \\ N_r^{(3)} \end{Bmatrix} = \begin{Bmatrix} B^{(3)} \\ A^{(3)} \end{Bmatrix} \left[u_{0,r} + \frac{h_2}{2}\left(\psi_{r,r}^{(3)} - \psi_{r,r}^{(2)}\right) + \frac{\nu^{(3)}}{r}\left(u_0 + \frac{h_2}{2}\left(\psi_r^{(3)} - \psi_r^{(2)}\right)\right)\right] + \begin{Bmatrix} D^{(3)} \\ B^{(3)} \end{Bmatrix}\left(\psi_{r,r}^{(3)} + \psi_{r,r} + \nu^{(3)} \frac{\psi_r^{(3)} + \psi_r}{r} \right) \\
&\begin{Bmatrix} M_\theta^{(1)} \\ N_\theta^{(1)} \end{Bmatrix} = \begin{Bmatrix} B^{(1)} \\ A^{(1)} \end{Bmatrix} \left[\frac{1}{r}\left(u_0 + \frac{h_2}{2}\left(\psi_r^{(2)} - \psi_r^{(1)}\right)\right) + \nu^{(1)} u_{0,r} + \frac{h_2}{2}\nu^{(1)}\left(\psi_{r,r}^{(2)} - \psi_{r,r}^{(1)}\right)\right] \\
&\qquad + \begin{Bmatrix} D^{(1)} \\ B^{(1)} \end{Bmatrix} \left[\frac{\psi_r^{(1)} + \psi_r}{r} + \nu^{(1)}(\psi_{r,r}^{(1)} + \psi_{r,r}) \right] \\
&\begin{Bmatrix} M_\theta^{(2)} \\ N_\theta^{(2)} \end{Bmatrix} = \begin{Bmatrix} B^{(2)} \\ A^{(2)} \end{Bmatrix} \left(\frac{u_0}{r} + \nu^{(2)} u_{0,r} \right) + \begin{Bmatrix} D^{(2)} \\ B^{(2)} \end{Bmatrix} \left[\frac{\psi_r^{(2)} + \psi_r}{r} + \nu^{(2)}\left(\psi_{r,r}^{(2)} + \psi_{r,r}\right) \right] \\
&\begin{Bmatrix} M_\theta^{(3)} \\ N_\theta^{(3)} \end{Bmatrix} = \begin{Bmatrix} B^{(3)} \\ A^{(3)} \end{Bmatrix} \left[\frac{1}{r}\left(u_0 + \frac{h_2}{2}\left(\psi_r^{(3)} - \psi_r^{(2)}\right)\right) + \nu^{(3)} u_{0,r} + \frac{h_2}{2}\nu^{(3)}\left(\psi_{r,r}^{(3)} - \psi_{r,r}^{(2)}\right)\right] \\
&\qquad + \begin{Bmatrix} D^{(3)} \\ B^{(3)} \end{Bmatrix} \left[\frac{\psi_r^{(3)} + \psi_r}{r} + \nu^{(3)}\left(\psi_{r,r}^{(3)} + \psi_{r,r}\right) \right]
\end{aligned} \tag{39}$$

$$\begin{aligned}
&Q_r^{(1)} = \left[\kappa^2 \frac{1-\nu^{(1)}}{2}\left(\psi_r^{(1)} + \psi_r + w,_r\right)\right] A^{(1)}, \qquad Q_r^{(2)} = \left[\kappa^2 \frac{1-\nu^{(2)}}{2}\left(\psi_r^{(2)} + \psi_r + w,_r\right)\right] A^{(2)}, \\
&Q_r^{(3)} = \left[\kappa^2 \frac{1-\nu^{(3)}}{2}\left(\psi_r^{(3)} + \psi_r + w,_r\right)\right] A^{(3)}
\end{aligned} \tag{40}$$

where

$$\begin{Bmatrix} A^{(1)} \\ B^{(1)} \\ D^{(1)} \end{Bmatrix} = \int_{\frac{h_2}{2}}^{\frac{h_2}{2}+h_1} \frac{E^{(1)}}{1-\nu^{(1)}} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz, \quad \begin{Bmatrix} A^{(2)} \\ B^{(2)} \\ D^{(2)} \end{Bmatrix} = \int_{-h_2/2}^{h_2/2} \frac{E^{(2)}}{1-\nu^{(2)}} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz, \quad \begin{Bmatrix} A^{(3)} \\ B^{(3)} \\ D^{(3)} \end{Bmatrix} = \int_{-\frac{h_2}{2}-h_3}^{-\frac{h_2}{2}} \frac{E^{(3)}}{1-\nu^{(3)}} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz \tag{41}$$

The governing Eqs .(30-35) may be expanded and rewritten based on Eqs. (39,40) as:

$$
\begin{aligned}
&\delta u_0 \neq 0:\\
&\left(A^{(1)}+A^{(3)}+A^{(2)}\right)\left(u_{0,rr}+\frac{u_{0,r}}{r}-\frac{u_0}{r^2}\right)+\left(B^{(1)}-\frac{h_2}{2}A^{(1)}\right)\left(\psi^{(1)}_{r,rr}+\frac{\psi^{(1)}_{r,r}}{r}-\frac{\psi^{(1)}_r}{r^2}\right)+\left(B^{(2)}+\frac{h_2}{2}A^{(1)}-\frac{h_2}{2}A^{(3)}\right)\\
&\left(\psi^{(2)}_{r,rr}+\frac{\psi^{(2)}_{r,r}}{r}-\frac{\psi^{(2)}_r}{r^2}\right)+\left(B^{(1)}+B^{(2)}+B^{(3)}\right)\left(\psi_{r,rr}+\frac{\psi_{r,r}}{r}-\frac{\psi_r}{r^2}\right)+\left(B^{(3)}+\frac{h_2}{2}A^{(3)}\right)\left(\psi^{(3)}_{r,rr}+\frac{\psi^{(3)}_{r,r}}{r}-\frac{\psi^{(3)}_r}{r^2}\right)=\\
&\left(I_0^{(1)}+I_0^{(2)}+I_0^{(3)}\right)\ddot{u}_0+\left(I_1^{(1)}+I_1^{(2)}+I_1^{(3)}\right)\ddot{\psi}_r+\left(I_1^{(1)}-\frac{h_2}{2}I_0^{(1)}\right)\ddot{\psi}^{(1)}_r+\left(\frac{h_2}{2}I_0^{(1)}+I_1^{(2)}-\frac{h_2}{2}I_0^{(3)}\right)\ddot{\psi}^{(2)}_r\\
&+\left(I_1^{(3)}+\frac{h_2}{2}I_0^{(3)}\right)\ddot{\psi}^{(3)}_r
\end{aligned}
\tag{42a}
$$

$$
\begin{aligned}
&\delta \psi_r \neq 0:\\
&\left(B^{(1)}+B^{(2)}+B^{(3)}\right)\left(u_{0,rr}+\frac{u_{0,r}}{r}-\frac{u_0}{r^2}\right)+\left(D^{(1)}-\frac{h_2}{2}B^{(1)}\right)\left(\psi^{(1)}_{r,rr}+\frac{\psi^{(1)}_{r,r}}{r}-\frac{\psi^{(1)}_r}{r^2}\right)\\
&+\left(\frac{h_2}{2}B^{(1)}+D^{(2)}-\frac{h_2}{2}B^{(3)}\right)\left(\psi^{(2)}_{r,rr}+\frac{\psi^{(2)}_{r,r}}{r}-\frac{\psi^{(2)}_r}{r^2}\right)+\left(D^{(1)}+D^{(2)}+D^{(3)}\right)\left(\psi_{r,rr}+\frac{\psi_{r,r}}{r}-\frac{\psi_r}{r^2}\right)\\
&+\left(\frac{h_2}{2}B^{(3)}+D^{(3)}\right)\left(\psi^{(3)}_{r,rr}+\frac{\psi^{(3)}_{r,r}}{r}-\frac{\psi^{(3)}_r}{r^2}\right)-\kappa^{(1)2}\frac{1-\nu^{(1)}}{2}A^{(1)}\left(\psi^{(1)}_r+\psi_r+w_{,r}\right)\\
&-\kappa^{(2)2}\frac{1-\nu^{(2)}}{2}A^{(2)}\left(\psi^{(2)}_r+\psi_r+w_{,r}\right)-\kappa^{(3)2}\frac{1-\nu^{(3)}}{2}A^{(3)}\left(\psi^{(2)}_r+\psi_r+w_{,r}\right)=\left(I_1^{(1)}+I_1^{(2)}+I_1^{(3)}\right)\ddot{u}_0\\
&+\left(I_2^{(1)}+I_2^{(2)}+I_2^{(3)}\right)\ddot{\psi}_r+\left(I_2^{(1)}-\frac{h_2}{2}I_1^{(1)}\right)\ddot{\psi}^{(1)}_r-\left(\frac{h_2}{2}I_1^{(3)}-I_2^{(2)}-\frac{h_2}{2}I_1^{(1)}\right)\ddot{\psi}^{(2)}_r+\left(I_2^{(3)}+\frac{h_2}{2}I_1^{(3)}\right)\ddot{\psi}^{(3)}_r
\end{aligned}
\tag{42b}
$$

$$
\begin{aligned}
&\delta \psi^{(1)}_r \neq 0:\\
&\left(B^{(1)}-\frac{h_2}{2}A^{(1)}\right)\left(u_{0,rr}+\frac{u_{0,r}}{r}-\frac{u_0}{r^2}\right)+\left(D^{(1)}+\frac{h_2^2}{4}A^{(1)}-h_2B^{(1)}\right)\left(\psi^{(1)}_{r,rr}+\frac{\psi^{(1)}_{r,r}}{r}-\frac{\psi^{(1)}_r}{r^2}\right)+\frac{h_2}{2}\left(B^{(1)}-\frac{h_2}{2}A^{(1)}\right)\\
&\left(\psi^{(2)}_{r,rr}+\frac{\psi^{(2)}_{r,r}}{r}-\frac{\psi^{(2)}_r}{r^2}\right)+\left(D^{(1)}-\frac{h_2}{2}B^{(1)}\right)\left(\psi_{r,rr}+\frac{\psi_{r,r}}{r}-\frac{\psi_r}{r^2}\right)-\left[\kappa^{(1)2}\frac{1-\nu^{(1)}}{2}\left(\psi^{(1)}_r+\psi_r+w_{,r}\right)\right]A^{(1)}=\\
&\left(I_1^{(1)}-\frac{h_2}{2}I_0^{(1)}\right)\ddot{u}_0+\left(I_2^{(1)}-\frac{h_2}{2}I_1^{(1)}\right)\ddot{\psi}_r+\left(\frac{h_2^2}{4}I_0^{(1)}-h_2I_1^{(1)}+I_2^{(1)}\right)\ddot{\psi}^{(1)}_r+\frac{h_2}{2}\left(I_1^{(1)}-\frac{h_2}{2}I_0^{(1)}\right)\ddot{\psi}^{(2)}_r
\end{aligned}
\tag{42c}
$$

$$
\begin{aligned}
&\delta \psi^{(2)}_r \neq 0:\\
&\left(B^{(2)}+\frac{h_2}{2}A^{(1)}-\frac{h_2}{2}A^{(3)}\right)\left(u_{0,rr}+\frac{u_{0,r}}{r}-\frac{u_0}{r^2}\right)+\left(D^{(2)}-\frac{h_2}{2}B^{(3)}+\frac{h_2}{2}B^{(1)}\right)\left(\psi_{r,rr}+\frac{\psi_{r,r}}{r}-\frac{\psi_r}{r^2}\right)\\
&+\left(D^{(2)}+\frac{h_2}{2}\frac{h_2}{2}A^{(1)}+\frac{h_2}{2}\frac{h_2}{2}A^{(3)}\right)\left(\psi^{(2)}_{r,rr}+\frac{\psi^{(2)}_{r,r}}{r}-\frac{\psi^{(2)}_r}{r^2}\right)+\left(B^{(1)}-\frac{h_2}{2}A^{(1)}\right)\left(\psi^{(1)}_{r,rr}+\frac{\psi^{(1)}_{r,r}}{r}-\frac{\psi^{(1)}_r}{r^2}\right)\\
&-\frac{h_2}{2}\left(B^{(3)}-\frac{h_2}{2}A^{(3)}\right)\left(\psi^{(3)}_{r,rr}+\frac{\psi^{(3)}_{r,r}}{r}-\frac{\psi^{(3)}_r}{r^2}\right)-\kappa^{(2)2}\frac{1-\nu^{(2)}}{2}A^{(2)}\left(\psi^{(2)}_r+\psi_r+w_{,r}\right)=\left(\frac{h_2}{2}I_0^{(1)}+I_1^{(2)}+\frac{h_2}{2}I_0^{(3)}\right)\ddot{u}_0\\
&+\left(\frac{h_2}{2}I_1^{(1)}+I_2^{(2)}+\frac{h_2}{2}I_1^{(3)}\right)\ddot{\psi}_r+\frac{h_2}{2}\left(I_1^{(1)}-\frac{h_2}{2}I_0^{(1)}\right)\ddot{\psi}^{(1)}_r+\left(\frac{h_2^2}{4}I_0^{(1)}+I_2^{(2)}-\frac{h_2^2}{4}I_0^{(3)}\right)\ddot{\psi}^{(2)}_r+\frac{h_2}{2}\left(I_1^{(3)}+\frac{h_2}{2}I_0^{(3)}\right)\ddot{\psi}^{(3)}_r
\end{aligned}
\tag{42d}
$$

$\delta\psi_r^{(3)} \neq 0$:

$$\left(B^{(3)}+\frac{h_2}{2}A^{(3)}\right)\left(u_{0,rr}+\frac{u_{0,r}}{r}-\frac{u_0}{r^2}\right)-\frac{h_2}{2}\left(B^{(3)}+\frac{h_2}{2}A^{(3)}\right)\left(\psi_{r,rr}^{(2)}+\frac{\psi_{r,r}^{(2)}}{r}-\frac{\psi_r^{(2)}}{r^2}\right)+\left(D^{(3)}+\frac{h_2}{2}B^{(3)}\right)$$
$$\left(\psi_{r,rr}+\frac{\psi_{r,r}}{r}-\frac{\psi_r}{r^2}\right)+\left(\frac{h_2^2}{4}A^{(3)}+\frac{h_2}{2}B^{(3)}+\frac{h_2}{2}B^{(3)}+D^{(3)}\right)\left(\psi_{r,rr}^{(3)}+\frac{\psi_{r,r}^{(3)}}{r}-\frac{\psi_r^{(3)}}{r^2}\right)-\kappa^{(3)2}A^{(3)}\frac{1-\nu^{(3)}}{2} \tag{42e}$$
$$\left(\psi_r^{(3)}+\psi_r+w,_r\right)=\left(\frac{h_2}{2}I_0^{(3)}+I_1^{(3)}\right)\ddot{u}_0+\left(\frac{h_2}{2}I_1^{(3)}+I_2^{(3)}\right)\ddot{\psi}_r+\left(\frac{h_2^2}{4}I_0^{(3)}+I_2^{(3)}+h_2I_1^{(3)}\right)\ddot{\psi}_r^{(3)}-\frac{h_2}{2}\left(\frac{h_2}{2}I_0^{(3)}+I_1^{(3)}\right)\ddot{\psi}_r^{(2)}$$

$\delta w \neq 0$:

$$\kappa^{(1)2}\frac{1-\nu^{(1)}}{2}A^{(1)}\left(\frac{\psi_r^{(1)}+\psi_r}{r}+\psi_{r,r}^{(1)}+\psi_{r,r}\right)+\kappa^{(2)2}\frac{1-\nu^{(2)}}{2}A^{(2)}\left(\frac{\psi_r^{(2)}+\psi_r}{r}+\psi_{r,r}^{(2)}+\psi_{r,r}\right)$$
$$+\kappa^{(3)2}\frac{1-\nu^{(3)}}{2}A^{(3)}\left(\frac{\psi_r^{(3)}+\psi_r}{r}+\psi_{r,r}^{(3)}+\psi_{r,r}\right)+\frac{1}{r}(r\bar{N}_r w,_r),_r \tag{42f}$$
$$+\left(\kappa^{(1)2}\frac{1-\nu^{(1)}}{2}A^{(1)}+\kappa^{(2)2}\frac{1-\nu^{(2)}}{2}A^{(2)}+\kappa^{(3)2}\frac{1-\nu^{(3)}}{2}A^{(3)}\right)\left(\frac{w,_r}{r}+w,_{rr}\right)=q+\left(I_0^{(1)}+I_0^{(2)}+I_0^{(3)}\right)\ddot{w}$$

5 BOUNDARY CONDITIONS

Three most common edge conditions of the solid circular plates are considered here to develop the semi-analytical solution:

Clamped immovable edge:

$$u_0=0,\ \psi_r=0,\ \psi_r^{(1)}=0,\ \psi_r^{(2)}=0,\ \psi_r^{(3)}=0,\ w=0 \tag{43}$$

Simply-supported immovable edge:

$$\begin{cases} u_0=0 \\ M_r^{(1)}+M_r^{(2)}+M_r^{(3}=0 \\ -\frac{h_2}{2}N_r^{(1)}+M_r^{(1)}=0 \\ \frac{h_2}{2}N_r^{(1)}+M_r^{(2)}-\frac{h_2}{2}N_r^{(3)}=0 \\ \frac{h_2}{2}N_r^{(3}+M_r^{(3)}=0 \\ w=0 \end{cases} \tag{44}$$

Roller-supported movable edge:

$$\begin{cases} N_r^{(1)} + N_r^{(2)} + N_r^{(3)} = 0 \\ M_r^{(1)} + M_r^{(2)} + M_r^{(3} = 0 \\ -\dfrac{h_2}{2} N_r^{(1)} + M_r^{(1)} = 0 \\ \dfrac{h_2}{2} N_r^{(1)} + M_r^{(2)} - \dfrac{h_2}{2} N_r^{(3)} = 0 \\ \dfrac{h_2}{2} N_r^{(3} + M_r^{(3)} = 0 \\ \delta w = 0 \end{cases} \tag{45}$$

6 THE POWER SERIES SOLUTION

By using series solutions for the unknown displacement parameters (i.e.,), $u_0(r,t)$, $w(r,t)$, $\psi(r,t)$, $\psi_r^{(1)}(r,t)$, $\psi_r^{(2)}(r,t)$, and $\psi_r^{(3)}(r,t)$ the governing differential equations and the relevant boundary conditions of the functionally graded viscoelastic sandwich plate are transformed into a set of algebraic equations in terms of the original functions. Assuming that the displacement parameters $u_0(r,t)$, $w(r,t)$, $\psi(r,t)$, $\psi_r^{(1)}(r,t)$, $\psi_r^{(2)}(r,t)$, and $\psi_r^{(3)}(r,t)$ are analytical functions in a domain *R*, these functions may be expressed by the following power series, for each mode of vibration:

$$\begin{aligned} &u_0 = \left(\sum_{k=0}^{\infty} U_k r^k\right) e^{i\omega t}, \;\; \psi_r = \left(\sum_{k=0}^{\infty} F_k r^k\right) e^{i\omega t}, \;\; \psi_r^{(1)} = \left(\sum_{k=0}^{\infty} F_k^{(1)} r^k\right) e^{i\omega t}, \;\; \psi_r^{(2)} = \left(\sum_{k=0}^{\infty} F_k^{(2)} r^k\right) e^{i\omega t}, \\ &\psi_r^{(3)} = \left(\sum_{k=0}^{\infty} F_k^{(3)} r^k\right) e^{i\omega t}, \;\; w = \left(\sum_{k=0}^{\infty} W_k r^k\right) e^{i\omega t} \end{aligned} \tag{46}$$

where ω is the natural frequency.

In practical applications, the functions must be expressed by means of finite series. By substituting Eq. (46) into the governing Eq. (42) and performing some manipulations, the transformed form of Eq. (42) may be obtained as:

$$\begin{aligned} \sum_{k=0}^{\infty} \Bigg\{ &\left(A^{(1)} + A^{(3)} + A^{(2)}\right) U_{k+2} + \left(B^{(1)} - \frac{h_2}{2} A^{(1)}\right) F_{k+2}^{(1)} + \left(B^{(2)} + \frac{h_2}{2} A^{(1)} - \frac{h_2}{2} A^{(3)}\right) F_{k+2}^{(2)} \\ &+ \left(B^{(1)} + B^{(2)} + B^{(3)}\right) F_{k+2} + \left(B^{(3)} + \frac{h_2}{2} A^{(3)}\right) F_{k+2}^{(3)} + \left(I_0^{(1)} + I_0^{(2)} + I_0^{(3)}\right) \omega^2 U_k + \left(I_1^{(1)} - \frac{h_2}{2} I_0^{(1)}\right) \omega^2 F_k^{(1)} \\ &+ \left(I_1^{(1)} + I_1^{(2)} + I_1^{(3)}\right) \omega^2 F_k + \left(\frac{h_2}{2} I_0^{(1)} + I_1^{(2)} - \frac{h_2}{2} I_0^{(3)}\right) \omega^2 F_k^{(2)} + \left(I_1^{(3)} + \frac{h_2}{2} I_0^{(3)}\right) \omega^2 F_k^{(3)} \Bigg\} r^k = 0 \end{aligned} \tag{47a}$$

$$\begin{aligned} &\sum_{k=0}^{\infty} \Bigg\{ \left(B^{(1)} + B^{(2)} + B^{(3)}\right)(k+3)(k+1) U_{k+2} + \left(D^{(1)} - \frac{h_2}{2} B^{(1)}\right)(k+3)(k+1) F_{k+2}^{(1)} + \left(\frac{h_2}{2} B^{(1)} + D^{(2)} - \frac{h_2}{2} B^{(3)}\right) \\ &(k+3)(k+1) F_{k+2}^{(2)} + \left(D^{(1)} + D^{(2)} + D^{(3)}\right)(k+3)(k+1) F_{k+2} + \left(\frac{h_2}{2} B^{(3)} + D^{(3)}\right)(k+3)(k+1) F_{k+2}^{(3)} \\ &-\kappa^2 \frac{1-\nu^{(1)}}{2} A^{(1)} \left[F_k^{(1)} + F_k + (k+1) W_{k+1}\right] - \kappa^2 \frac{1-\nu^{(2)}}{2} A^{(2)} \left[F_k^{(2)} + F_k + (k+1) W_{k+1}\right] - \kappa^2 \frac{1-\nu^{(3)}}{2} A^{(3)} \end{aligned} \tag{47b}$$

$$\left[F_k^{(3)}+F_k+(k+1)W_{k+1}\right]\left(I_1^{(1)}+I_1^{(2)}+I_1^{(3)}\right)\omega^2U_k+\left(I_2^{(1)}+I_2^{(2)}+I_2^{(3)}\right)\omega^2F_k+\left(I_2^{(1)}-\frac{h_2}{2}I_1^{(1)}\right)\omega^2F_k^{(1)}$$
$$+\left(I_2^{(3)}+\frac{h_2}{2}I_1^{(3)}\right)\omega^2F_k^{(3)}-\left(\frac{h_2}{2}I_1^{(3)}-I_2^{(2)}-\frac{h_2}{2}I_1^{(1)}\right)F_k^{(2)}\Bigg\}r^k=0$$

$$\sum_{k=0}^{\infty}\Bigg\{\left(B^{(1)}-\frac{h_2}{2}A^{(1)}\right)U_{k+2}+\left(-\frac{h_2}{2}B^{(1)}+D^{(1)}+\frac{h_2^2}{4}A^{(1)}-\frac{h_2}{2}B^{(1)}\right)F_{k+2}^{(1)}+\left(D^{(1)}-\frac{h_2}{2}B^{(1)}\right)F_{k+2}$$
$$+\left(\frac{h_2}{2}B^{(1)}-\frac{h_2^2}{4}A^{(1)}\right)F_{k+2}^{(2)}-\left[\kappa^2\frac{1-\nu^{(1)}}{2(k+3)(k+1)}\left(F_k^{(1)}+F_k+(k+1)W_{k+1}\right)\right]A^{(1)}\left(I_1^{(1)}-\frac{h_2}{2}I_0^{(1)}\right)\omega^2U_k \tag{47c}$$
$$+\left(I_2^{(1)}-\frac{h_2}{2}I_1^{(1)}\right)\omega^2F_k+\left(\frac{h_2^2}{4}I_0^{(1)}-h_2I_1^{(1)}+I_2^{(1)}\right)\omega^2F_k^{(1)}+\frac{h_2}{2}\left(I_1^{(1)}-\frac{h_2}{2}I_0^{(1)}\right)\omega^2F_k^{(2)}\Bigg\}r^k=0$$

$$\sum_{k=0}^{\infty}\Bigg\{\left(B^{(2)}+\frac{h_2}{2}A^{(1)}-\frac{h_2}{2}A^{(3)}\right)U_{k+2}+\left(D^{(2)}-\frac{h_2}{2}B^{(3)}+\frac{h_2}{2}B^{(1)}\right)F_{k+2}+\left(D^{(2)}+\frac{h_2^2}{4}A^{(1)}+\frac{h_2^2}{4}A^{(3)}\right)F_{k+2}^{(2)}$$
$$-\frac{h_2}{2}\left(\frac{h_2}{2}A^{(1)}-B^{(1)}\right)F_{k+2}^{(1)}-\left(\frac{h_2^2}{4}A^{(3)}+\frac{h_2}{2}B^{(3)}\right)F_{k+2}^{(3)}-\kappa^2\frac{1-\nu^{(2)}}{2(k+3)(k+1)}A^{(2)}\left[F_k^{(2)}+F_k+(k+1)W_{k+1}\right]$$
$$\left(\frac{h_2}{2}I_0^{(1)}+I_1^{(2)}+\frac{h_2}{2}I_0^{(3)}\right)\omega^2U_k+\left(\frac{h_2}{2}I_1^{(1)}+I_2^{(2)}+\frac{h_2}{2}I_1^{(3)}\right)\omega^2F_k+\frac{h_2}{2}\left(I_1^{(1)}-\frac{h_2}{2}I_0^{(1)}\right)\omega^2F_k^{(1)} \tag{47d}$$
$$+\left(\frac{h_2^2}{4}I_0^{(1)}+I_2^{(2)}-\frac{h_2^2}{4}I_0^{(3)}\right)\omega^2F_k^{(2)}+\frac{h_2}{2}\left(I_1^{(3)}+\frac{h_2}{2}I_0^{(3)}\right)\omega^2F_k^{(3)}\Bigg\}r^k=0$$

$$\sum_{k=0}^{\infty}\Bigg\{-\left(B^{(3)}+\frac{h_2}{2}A^{(3)}\right)U_{k+2}+\left(\frac{h_2}{2}B^{(3)}+\frac{h_2}{2}\frac{h_2}{2}A^{(3)}\right)F_{k+2}^{(2)}-\left(D^{(3)}+\frac{h_2}{2}B^{(3)}\right)F_{k+2}+\left(\frac{h_2}{2}I_0^{(3)}+I_1^{(3)}\right)\omega^2U_k$$
$$-\left(\frac{h_2^2}{4}A^{(3)}+\frac{h_2}{2}B^{(3)}+\frac{h_2}{2}B^{(3)}+D^{(3)}\right)F_{k+2}^{(3)}+\kappa^2A^{(3)}\frac{1-\nu^{(3)}}{2(k+3)(k+1)}\left[F_k^{(3)}+F_k+(k+1)W_{k+1}\right] \tag{47e}$$
$$+\left(\frac{h_2}{2}I_1^{(3)}+I_2^{(3)}\right)\omega^2F_k+\left(\frac{h_2^2}{4}I_0^{(3)}+I_2^{(3)}+h_2I_1^{(3)}\right)\omega^2F_k^{(3)}-\frac{h_2}{2}\left(\frac{h_2}{2}I_0^{(3)}+I_1^{(3)}\right)\omega^2F_k^{(2)}\Bigg\}r^k=0$$

$$\sum_{k=0}^{\infty}\Big\{\left[\left(1-\nu^{(1)}\right)\kappa^{(1)}A^{(1)}+\left(1-\nu^{(2)}\right)\kappa^{(2)}A^{(2)}+\left(1-\nu^{(3)}\right)\kappa^{(3)}A^{(3)}\right](k+2)W_{k+2}$$
$$+\left(1-\nu^{(1)}\right)\kappa^{(1)}A^{(1)}\left(F_{k+1}^{(1)}+F_{k+1}\right)+\left(1-\nu^{(2)}\right)\kappa^{(2)}A^{(2)}\left(F_{k+1}^{(2)}+F_{k+1}\right)+\left(1-\nu^{(3)}\right)\kappa^{(3)}A^{(3)}\left(F_{k+1}^{(3)}+F_{k+1}\right) \tag{47f}$$
$$+\left(I_0^{(1)}+I_0^{(2)}+I_0^{(3)}\right)\omega^2W_k\Big\}r^k=0$$

By solving Eqs. (46a) to (46f), an algebraic homogeneous system of equations including the unknown displacement parameters $U_{k+2}, F_{k+2}, F_{k+2}^{(1)}, F_{k+2}^{(2)}, F_{k+2}^{(3)}$ and W_{k+2} (*k=0,1,2, …*) is obtained. The boundary conditions (43-45) are not sufficient to make this system of equations a deterministic one. Additional conditions are required that may be extracted based on the regularity conditions at the center of the axisymmetric plate:

$$
\begin{aligned}
&u\big|_{r=0}=0 && \rightarrow U_0=0\\
&\psi\big|_{r=0}=0 && \rightarrow F_0=0\\
&\psi^{(1)}\big|_{r=0}=0 && \rightarrow F_0^{(1)}=0\\
&\psi^{(2)}\big|_{r=0}=0 && \rightarrow F_0^{(2)}=0\\
&\psi^{(3)}\big|_{r=0}=0 && \rightarrow F_0^{(3)}=0\\
&Q_r\big|_{r=0}=0 && \rightarrow W_1=0
\end{aligned}
\tag{48}
$$

By substituting Eq. (46) into the boundary condition (43) the transformed form of Eq. (43) may be written as:

$$
\begin{aligned}
&u\big|_{r=b}=\sum_{k=0}^{\infty}U_k b^k=0,\ w\big|_{r=b}=\sum_{k=0}^{\infty}W_k b^k=0,\ \psi_r\big|_{r=b}=\sum_{k=0}^{\infty}F_k b^k=0,\\
&\psi_r^{(1)}\Big|_{r=b}=\sum_{k=0}^{\infty}F_k^{(1)}b^k=0,\ \psi_r^{(2)}\Big|_{r=b}=\sum_{k=0}^{\infty}F_k^{(2)}b^k=0,\ \psi_r^{(3)}\Big|_{r=b}=\sum_{k=0}^{\infty}F_k^{(3)}b^k=0,
\end{aligned}
\tag{49}
$$

By substituting U_i, W_i, F_i, $F_i^{(1)}$, $F_i^{(2)}$, and $F_i^{(3)}$ ($i=2..n+2$) from Eq. (47) into Eq. (48) and applying Eq. (49), the following system of equations is obtained:

$$
\begin{bmatrix}
Y_{11}(\omega) & Y_{12}(\omega) & Y_{13}(\omega) & Y_{14}(\omega) & Y_{15}(\omega) & Y_{16}(\omega)\\
Y_{21}(\omega) & Y_{22}(\omega) & Y_{23}(\omega) & Y_{24}(\omega) & Y_{25}(\omega) & Y_{26}(\omega)\\
Y_{31}(\omega) & \mathrm{X}_{32}(\omega) & Y_{33}(\omega) & Y_{34}(\omega) & Y_{35}(\omega) & Y_{36}(\omega)\\
Y_{41}(\omega) & \mathrm{X}_{42}(\omega) & Y_{43}(\omega) & Y_{44}(\omega) & Y_{45}(\omega) & Y_{46}(\omega)\\
Y_{51}(\omega) & \mathrm{X}_{52}(\omega) & Y_{53}(\omega) & Y_{54}(\omega) & Y_{55}(\omega) & Y_{56}(\omega)\\
Y_{61}(\omega) & \mathrm{X}_{62}(\omega) & Y_{63}(\omega) & Y_{64}(\omega) & Y_{65}(\omega) & Y_{66}(\omega)
\end{bmatrix}
\begin{Bmatrix}
U_1\\ F_1\\ F_1^{(1)}\\ F_1^{(2)}\\ F_1^{(3)}\\ W_0
\end{Bmatrix}=0
\tag{50}
$$

Existence of non-trivial solutions of the resulted system of equations requires that:

$$
\begin{vmatrix}
Y_{11}(\omega) & Y_{12}(\omega) & Y_{13}(\omega) & Y_{14}(\omega) & Y_{15}(\omega) & Y_{16}(\omega)\\
Y_{21}(\omega) & Y_{22}(\omega) & Y_{23}(\omega) & Y_{24}(\omega) & Y_{25}(\omega) & Y_{26}(\omega)\\
Y_{31}(\omega) & \mathrm{X}_{32}(\omega) & Y_{33}(\omega) & Y_{34}(\omega) & Y_{35}(\omega) & Y_{36}(\omega)\\
Y_{41}(\omega) & \mathrm{X}_{42}(\omega) & Y_{43}(\omega) & Y_{44}(\omega) & Y_{45}(\omega) & Y_{46}(\omega)\\
Y_{51}(\omega) & \mathrm{X}_{52}(\omega) & Y_{53}(\omega) & Y_{54}(\omega) & Y_{55}(\omega) & Y_{56}(\omega)\\
Y_{61}(\omega) & \mathrm{X}_{62}(\omega) & Y_{63}(\omega) & Y_{64}(\omega) & Y_{65}(\omega) & Y_{66}(\omega)
\end{vmatrix}=0
\tag{51}
$$

7 RESULTS AND DISCUSSIONS

In the present section, various results of damped free vibration analyses of clamped circular sandwich plates are introduced. Verifications are performed by comparing some of the present results with results of the three-dimensional theory of elasticity derived from ABAQUS software. In this regard, the plate is discretized using 20-noded brick elements. Material properties listed in Table 1. are considered to extract the results.

Table 1
Properties of the chosen materials.

Material notation	Mat 1	Mat 2	Mat 3	Mat 4	Mat 5
Material name	Aluminum	Steel	Rubber	Ceramic	Hard rubber
E (GPa)	70	210	0.15	380	2.3
ρ (kg/m^3)	2700	7850	950	3800	2117
ν	0.3	0.3	0.3	0.3	0.4

7.1 Verifications and comparing present results with the conventional results

In the present example, while verifying results of the present research, a comparison is also made with results of the first-order shear deformation theory (FSDT). Results of the first-order shear deformation theory are extracted by substituting $\psi_r = \psi_r^{(1)} = \psi_r^{(2)} = \psi_r^{(3)}$ in the present formulation and using Mindlin's shear correction factor (*5/6*). Thick sandwich plates with $h_1/b = h_2/b = 0.1$ are considered to evaluated range of the validity of the results.

To extract more general conclusions, both symmetric and asymmetric layups are adopted to include effects of the extension-bending coupling. The considered three-layer sandwich plates are:

i) Sandwich plates with aluminum cores. Therefore, the core may not be considered as a soft core.
ii) Sandwich plates with soft cores made of a hard rubber (Mat 5).
iii) Sandwich plates with very soft cores made of a soft rubber (Mat 3)

Results of the first two natural frequencies of the mentioned sandwich plates are given in Tables 2. to 4 Some of the first-order shear deformation results have encountered convergence problems for thicker plates, higher vibration modes, or plates with softer cores. Indeed, even results of the present FSDT are more accurate than those of the conventional FSDT. Although the plates are thick, present results are generally in a good agreement with results of the three-dimensional theory of elasticity. Results of Tables 2. to 4 reveal that the discrepancies between present and FSDT results and results of the three-dimensional theory of elasticity are more considerable for the higher vibration modes. So that results of the FSDT may be unreliable in some cases.

However, it is known that the commercial finite element softwares such as ABAQUS use C^0-continuous formulations (Lagrangian shape functions) and therefore, *in-plane* continuity of the derivatives of the displacement components is not guaranteed in these softwares. Therefore, some of the differences appeared in Tables 2-4, are due the error sources of the ABAQUS software itself.

Table 2
A comparison among the natural frequencies predicted by the three-dimensional theory of elasticity, FSDT, and present results for the thick sandwich plates ($h_1/b = h_3/b = 0.1$).

h_2/b	Layup	Frequency (Hz)	3D (ABAQUS)	FSDT	Difference (%)	Present	Difference (%)
0.1	Mat 4-Mat 1-Mat 4	ω_1	614.86	598.87	2.60	607.14	1.26
		ω_2	1716.7	----	----	1640	4.46
	Mat 4-Mat 1-Mat 2	ω_1	768.37	----	----	755.41	1.68
		ω_2	2119.2	----	----	1988.4	6.17
	Mat 2-Mat 1-Mat2	ω_1	1004.1	869.75	13.38	927.05	7.67
		ω_2	2737.1	----	-----	----	----
0.2	Mat 4-Mat 1-Mat 4	ω_1	719.07	718.04	0.14	712	0.98
		ω_2	1870.04	----	-----	1852	0.96
	Mat 4-Mat 1-Mat 2	ω_1	860.04	853.17	0.79	852.88	0.83
		ω_2	2207.09	1911.5	13.39	2212.09	0.22
	Mat 2-Mat 1-Mat2	ω_1	1058.6	1194	12.79	1067	0.79
		ω_2	2728.3	2247.1	17.63	2864	4.97
0.3	Mat 4-Mat 1-Mat 4	ω_1	791.43	791.57	0.02	784.44	0.88
		ω_2	1979.9	1999	0.96	1968.3	0.58

Mat 4-Mat 1-Mat 2	ω_1	919.08	946.02	2.93	914.66	0.48
	ω_2	2264.8	1722.7	23.93	2275.9	0.49
Mat 2-Mat 1-Mat2	ω_1	1090.5	1139.8	4.52	1093.4	0.26
	ω_2	2731.4	1802.4	34.01	2531	7.33

Table 3
A comparison among the natural frequencies predicted by the three-dimensional theory of elasticity, FSDT, and present results for thick sandwich plates with soft cores ($h_1/b = h_3/b = 0.1$).

h_2/b	Layup	Frequency (Hz)	3D (ABAQUS)	FSDT	Difference (%)	Present	Difference (%)
0.1	Mat 2-Mat 5-Mat 2	ω_1	288.49	242.82	15.83	278.3	3.53
		ω_2	933.44	----	----	870.1	6.78
	Mat 4-Mat 5-Mat 2	ω_1	368.12	----	----	362.2	1.60
		ω_2	1147.2	----	----	1106.1	3.58
	Mat 4-Mat 5-Mat 4	ω_1	484.97	326.31	32.72	472.82	2.50
		ω_2	1642.9	----	----	1491.5	9.21
0.2	Mat 2-Mat 5-Mat 2	ω_1	280.43	227.78	18.77	263.32	6.10
		ω_2	896.78	----	------	775.91	13.48
	Mat 4-Mat 5-Mat 2	ω_1	348.89	303.68	12.96	335.69	3.78
		ω_2	1019.9	----	----	916.29	10.16
	Mat 4-Mat 5-Mat 4	ω_1	447.46	283.84	----	438.25	2.05
		ω_2	1492.2	----	----	1268.7	14.98
0.3	Mat 2-Mat 5-Mat 2	ω_1	276.4	219.38	20.63	253.5	8.28
		ω_2	869.22	----	----	785.42	9.64
	Mat 4-Mat 5-Mat 2	ω_1	336.24	271.97	19.11	321.43	4.40
		ω_2	908.54	----	----	1005.5	10.67
	Mat 4-Mat 5-Mat 4	ω_1	422.3	303.91	28.03	400.72	5.11
		ω_2	1359.7	1086.6	20.09	1186.4	12.75

Table 4
A comparison among the natural frequencies predicted by the three-dimensional theory of elasticity, FSDT, and present results for thick sandwich plates with very soft cores ($h_1/b = h_3/b = 0.1$).

h_2/b	Layup	Frequency (Hz)	3D (ABAQUS)	FSDT	Difference (%)	Present	Difference (%)
0.1	Mat 1-Mat 3-Mat 1	ω_1	238.16	197.34	17.14	236.01	0.90
		ω_2	841.06	770.33	8.41	815.08	3.09
	Mat 1-Mat 3-Mat 2	ω_1	243.84	221.64	9.10	242.33	0.62
		ω_2	872.64	745.91	14.52	823.26	5.66
	Mat 2-Mat 3-Mat 2	ω_1	246.01	222.99	9.36	244.37	0.67
		ω_2	891.44	----	----	870.95	2.3
0.2	Mat 1-Mat 3-Mat 1	ω_1	223.62	170.95	23.55	221.1	1.13
		ω_2	754.86	----	----	759.61	0.63
	Mat 1-Mat 3-Mat 2	ω_1	234.86	205.35	12.56	232.38	1.05
		ω_2	807.88	---	---	811.28	0.42

	Mat 2-Mat 3-Mat 2	ω_1	239.77	207.92	13.28	236.5	1.36
		ω_2	841.63	----	----	832.38	1.10
0.3	Mat 1-Mat 3-Mat 1	ω_1	212.13	249.26	17.50	206.22	2.79
		ω_2	633.94	----	---	604.43	4.65
	Mat 1-Mat 3-Mat 2	ω_1	226.59	191.18	15.63	223.18	1.50
		ω_2	665.06	----	----	620.03	6.78
	Mat 2-Mat 3-Mat 2	ω_1	234.02	----	----	231.04	1.27
		ω_2	689.61	----	----	647.22	6.15

7.2 A parametric study

It is known that the soft cores exhibit viscoelastic behavior more significantly than the stiff face sheets. To evaluate influence of the shear correction factors on the accuracy of the present results for sandwich plates with functionally graded face sheets and viscoelastic cores, it is intended to investigate these effects both individually and in combination.

First consider three-layer circular sandwich plates whose cores are made of aluminum or a soft rubber and the top and bottom layers of the entire plates are respectively, ceramic-rich and steel-rich so that the material properties of the FGM face sheets become corresponding to the aluminum at the interfaces between the face sheets and the core, through a linear material properties distribution in the thickness direction ($h_1/b = h_3/b = 0.1, g = 1$). The resulting fundamental natural frequencies are listed in Table 5. and compared with results of the three-dimensional theory of elasticity. Present results are in a good agreement with results of the three-dimensional theory of elasticity. The discrepancies between the results have increased in the thick plates. Some researchers have proved that in these circumstances, even results of the high-order local theories that are much more accurate in comparison with the global theories (for the sandwich plates), may show considerable deviations with respect to results of the 3D elasticity [30-36].

Table 5
A comparison between present results and results of the three-dimensional theory of elasticity for circular ceramic-aluminium /core/aluminium -steel sandwich plates with FGM face sheets ($h_1/b = h_3/b = 0.1, g = 1$).

h_2/b	Layup	Fundamental frequency (Hz)		
		3D (ABAQUS)	Present	Difference (%)
0.1	FGM 1-Mat 1-FGM 3	779.42	784.98	0.71
	FGM 1-Mat 3-FGM 3	279.45	279.81	0.13
0.15	FGM 1-Mat 1-FGM 3	833.11	833.84	-0.09
	FGM 1-Mat 3-FGM 3	269.66	259.8	3.66
0.2	FGM 1-Mat 1-FGM 3	875.57	879.59	0.46
	FGM 1-Mat 3-FGM 3	261.38	266.92	2.12
0.25	FGM 1-Mat 1-FGM 3	909.8	936.75	2.96
	FGM 1-Mat 3-FGM 3	253.86	233.42	8.05
0.3	FGM 1-Mat 1-FGM 3	937.86	1021.8	8.95
	FGM 1-Mat 3-FGM 3	247.19	227.94	7.79

Results of the first two natural frequencies and modal loss factors of thick sandwich plates ($h_2/b = 0.2, h_1/b = h_3/b = 0.1$) with FGM face sheets and viscoelastic cores are given in Table 6. for different loss factors of the materials. As it is mentioned before, the modal loss factors are the imaginary components of the derived general natural frequencies whereas the common natural frequencies are the real components. Although the natural frequencies have increased by increasing the material loss factor, these frequencies are more affected by the stiffness of the face sheets.

Table 6
The first two natural frequencies and modal loss factors of thick sandwich plates with FGM face sheets and viscoelastic cores, for different loss factors of the materials ($h_2/b=0.2, h_1/b=h_3/b=0.1$).

η		Mat 1-Mat 3-Mat 1		Mat 1-Mat 3-Mat 2		Mat 2-Mat 3-Mat 2	
		1st mode	2nd mode	1st mode	2nd mode	1st mode	2nd mode
0	ω_n	221.1	759.61	232.38	811.28	236.5	832.38
	η_n	0	0	0	0	0	0
0.1	ω_n	222.12	759.11	232.38	811.56	236.5	832.64
	η_n	1.47	1.93	0.83	1.05	0.58	0.73
0.2	ω_n	222.12	759.49	232.38	811.75	236.8	832.81
	η_n	2.94	3.85	1.66	2.11	1.16	1.45
0.3	ω_n	222.16	759.42	232.4	811.95	236.65	832.85
	η_n	4.4	5.72	2.48	3.24	1.74	2.18
0.4	ω_n	222.2	759.87	232.41	812.08	236.81	832.99
	η_n	5.86	7.75	3.31	4.22	2.33	2.95
0.5	ω_n	222.24	760.82	232.41	812.1	236.89	833.07
	η_n	7.33	9.7	4.15	5.22	2.91	3.66
0.6	ω_n	222.34	760.98	232.45	812.1	237.01	833.12
	η_n	8.79	11.57	4.975	6.33	3.49	4.38
0.7	ω_n	222.4	760.14	232.48	812.17	237.08	833.17
	η_n	10.25	13.48	5.81	7.42	4.09	5.13
0.8	ω_n	22.51	760.87	232.49	812.24	238.13	833.23
	η_n	11.71	15.44	6.63	8.84	4.65	5.85

Effects of the material properties indices of the FGM face sheets on the natural frequencies are reported in Table 7. ($h_2/b=0.2, h_1/b=h_3/b=0.1$). $g_1=0$ and $g_2=0$ cases are corresponding to a ceramic top face sheet and an aluminum bottom face sheet, respectively. By increasing the mentioned indices, material properties of the top and bottom face sheets approach to those of the aluminum and steel, respectively.

Some of the plates of Table 7. are selected and effects of the material loss factor of the core on the natural frequencies and modal loss factors are evaluated. Results are given in Table 8. Since the core is made of a hard rubber, effects of the material loss factor are more remarkable on the natural frequencies of the sandwich plates.

Table 7
Effects of indices of the material properties of the FGM face sheets on the natural frequencies of the circular sandwich plates ($h_2/b=0.2, h_1/b=h_3/b=0.1$).

Material indices	FGM 1-Mat 1-FGM 3		FGM 1-Mat 5-FGM 3	
	ω_1	ω_2	ω_1	ω_2
$g_1=g_3=0$	891.52	2367.9.2	391.74	1212.3
$g_1=0, g_3=0.5$	903.43	2341.5	379.17	1176.6
$g_1=0.5, g_3=0$	885.03	2331.6	365.04	1088.7
$g_1=g_3=0.5$	895.44	2303.5	355.3	1073.2
$g_1=0, g_3=1$	899.61	2320.2	371.15	1149.8
$g_1=1, g_3=0$	874.72	2306.5	352.72	1050.7
$g_1=g_3=1$	879.68	2252.7	337.73	1008.2
$g_1=g_3=2$	850.08	2168.5	323.53	929.62
$g_1=0, g_3=5$	879.94	2252.2	356.03	1121.9
$g_1=5, g_3=0$	815.38	2172.9	336.9	943.53
$g_1=g_3=5$	800.1	2071.4	312.68	909.41
$g_1=g_3=10$	767	2007.1	306.72	899.74
$g_1=g_3=100$	720.14	1917	294.28	868.29

Table 8
Influences of the material loss factor of the core on the natural frequencies and modal loss factors for of thick ($h_2/b=0.2, h_1/b=h_3/b=0.1$) circular FGM 1-Mat 5-FGM 3 sandwich plates.

Material indices		$\eta=0$		$\eta=0.4$		$\eta=0.8$	
		ω_n	η_n	ω_n	η_n	ω_n	η_n
$g_1=g_3=0$	1st mode	391.74	0	393.49	28.23	398.68	55.67
	2nd mode	1213.2	0	1215.1	44.02	1220.2	94.34
$g_1=g_3=0.5$	1st mode	355.3	0	357.53	31.29	363.54	61.37
	2nd mode	1073.2	0	1059.1	46.77	1069	92.552
$g_1=g_3=1$	1st mode	337.73	0	340.18	32.03	346.58	62.40
	2nd mode	1008.2	0	998.22	51.29	1004.5	117.92
$g_1=g_3=2$	1st mode	323.53	0	325.79	32.03	332.57	62.459
	2nd mode	929.62	0	943.73	42.73	956.65	98.39
$g_1=g_3=5$	1st mode	312.68	0	314.92	30.43	321.46	59.44
	2nd mode	909.41	0	916.86	52.47	919.41	101.81
$g_1=g_3=10$	1st mode	306.72	0	308.95	29.03	315.19	56.7
	2nd mode	899.74	0	899.54	48.98	905.27	93.7
$g_1=g_3=100$	1st mode	294.28	0	296.35	27	302.24	52.71
	2nd mode	868.29	0	869.37	44.68	875.84	90.8

8 CONCLUSIONS

In the present paper, five main tasks are accomplished:
- Analysis of the sandwich circular functionally graded viscoelastic plates by a global-local (zigzag) theory, for the first time.
- Proposing a zigzag theory with local shear correction factors. It is the first time that the concept of local shear correction factor is introduced.
- Including the viscoelastic behavior of the material properties in the correction factors, for the first time.
- Presenting a semi-analytical solution for sandwich plates with different boundary conditions.

Results are validated by results of the three-dimensional theory of elasticity, even when the sandwich plate is thick. Influences of various geometric and material properties parameters on the free vibration of the circular sandwich plates are investigated in detail. To extract more general conclusions, sandwich plates with symmetric and asymmetric layups are considered.

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