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Vibrations of Circular Plates with Guided Edge and Resting on Elastic Foundation

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ABSTRACT
In this paper, transverse vibrations of thin circular plates with guided edge and resting on Winkler foundation have been studied on the basis of Classical Plate Theory. Parametric investigations on the vibration of circular plates resting on elastic foundation have been carried out with respect to various foundation stiffness parameters. Twelve vibration modes are presented. The location of the stepped region with respect to foundation stiffness parameter is presented.

Keywords: Plate; Frequency; Guided edge; Elastic foundation

1 INTRODUCTION

CIRCULAR plate systems [1-4] have wide applications in various fields of engineering; they are used as structural elements resting on foundations. The problems of plates on elastic foundation are important in engineering design and have been the focus of attention of many researchers. Some of the recent studies have reestablished the efficiency of the classical approach in analyzing the vibrations of variety of structures. Although the circular symmetry of the problem allows for its significant simplification, many difficulties very often arise due to complexity and uncertainty of boundary conditions. This uncertainty could be due to practical engineering applications where the edge of the plate does not fall into the classical boundary conditions such as clamped, simply supported or free. When the boundary conditions of the plate deviate from classical cases, elastic translational restraints should be considered. A recent survey of literature shows that very few studies exist on the study of circular plates resting on elastic foundation. Wang and Wang [5], who observed the switching between axisymmetric and asymmetric vibration modes, recently investigated the effect of internal elastic translational supports.

The vibration characteristics of plates resting on an elastic medium are different from those of the plates supported only on the boundary. Leissa [6] discussed the vibration of a plate supported laterally by an elastic foundation. Leissa deduced that the effect of Winkler foundation merely increases the square of the natural frequency of the plate by a constant. Salari et al. [7] speculated the same conclusion. Ascione and Grimaldi [8] studied unilateral frictionless contact between a circular plate and a Winkler foundation using a variational formulation. One of the earliest formulations of this problem was presented by Leissa [6], who tabulated values of frequency parameter for four vibration modes of simply supported circular plate with varying rotational stiffness.

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Kang and Kim [9] presented an extensive review of the modal properties of the elastically restrained beams and plates. Zheng and Zhou [10] studied the large deflection of a circular plate resting on Winkler foundation. Ghosh [11] studied the free and forced vibration of circular plates on Winkler foundation by exact analytical method. The most general soil model used in practical applications is the Winkler [12] model in which the elastic medium below a structure is represented by a system of identical but mutually independent elastic linear springs. Recent investigations have reiterated the efficiency of the classical approach [13] in analyzing the behavior of structures under vibrations. There are other papers [Weisman [14], Dempsey et al. [15], Celep et al. [16]] dealing with the study of plates on a Winkler foundation. In general, papers dealing with vibrating plates, shells and beams are concerned primarily with the determination of eigenvalues and mode shapes [1-4]. The present study deals with obtaining exact solutions to the most important practical case of transverse vibrations of circular plates resting on Winkler foundation with guided edge conditions at the periphery of the plate. The results are presented for the non-dimensional frequency parameter of the plate for a wide range of values of Winkler foundation modulus parameter for use in design of such systems in micro or macro electro-mechanical devices.

2 MATHEMATICAL FORMULATION OF THE SYSTEM

The considered elastic thin circular plate is supported on a Winkler foundation as shown in Fig. 1. In the classical plate theory [1-4], the following fourth order differential equation describes free flexural vibrations of a thin circular uniform plate.

$$D\nabla^4 w(r,\theta,t) + \rho h \partial^2 w(r,\theta,t)/\partial t^2 = 0 \tag{1}$$

where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of a plate and a, h, ρ, E, ν are the plate's radiuses, thickness, density, Young's modulus and Poisson ratio's respectively.

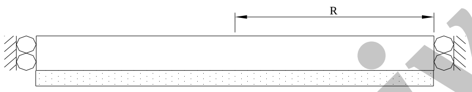

Fig. 1
A thin circular plate with guided edge and supported on full elastic foundation.

The homogeneous equation for Kirchoff's plate on one parameter elastic foundation is given by the following equation [11]

$$D\nabla^4 w(r,\theta,t) + K_w w(r,\theta,t) + \rho h \partial^2 w(r,\theta,t)/\partial t^2 = 0 \tag{2}$$

Displacement in Eq. (2) can be presented as a combination of spatial and time dependent components as follows; Let

$$w(r,\theta,t) = W(r,\theta)e^{i\omega t} \tag{3}$$

Now substitute the Eq. (3) in Eq. (2)

$$D\nabla^4 W(r,\theta) + (K_w - \rho h\omega^2)W(r,\theta) = 0 \tag{4}$$

The solution to the above equation takes the following form

$$W_{mn}(r,\theta) = A_{mn}\left[J_n\left(\frac{\lambda_{mn} r}{a}\right) + C_{mn} I_n\left(\frac{\lambda_{mn} r}{a}\right)\right]\cos n\theta \quad \text{where} \quad n = 0,\ 1,\ 2,\ 3\ldots\ ,\quad m = 0,\ 1,\ 2,\ 3,\ldots \tag{5}$$

where J_n is Bessel function of the first kind of first order and I_n is modified Bessel function of the first kind of first order. The boundary conditions can be formulated at $r = a$, as follows:

$$\frac{\partial w}{\partial r}(a,\theta) = 0 \tag{6}$$

$$V_r(a,\theta) = 0 \tag{7}$$

From Eqs. (6) and (7) yield the following:

$$\frac{\partial w}{\partial r}(a,\theta) = 0 \tag{8}$$

$$-D\left[\frac{\partial}{\partial r}\nabla^2 W + (1-\nu)\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial^2 W}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial W}{\partial\theta}\right)\right] = 0 \tag{9}$$

From Eqs. (5), (8) and (9), we derived the following:

$$C_{mn} = \frac{J_{m1}}{I_{P1}} \tag{10}$$

$$C_{mn} = \frac{J_{m3} - \dfrac{2J_{P2}}{\lambda_{mn}} - \left[3 + \dfrac{4+4(2-\nu)n^2}{\lambda_{mn}{}^2}\right]J_{m1} - \left[\dfrac{8(3-\nu)n^2}{\lambda^3} - \dfrac{4}{\lambda_{mn}}\right]J_n(\lambda_{mn})}{I_{P3} + \dfrac{2I_{P2}}{\lambda_{mn}} + \left[3 - \dfrac{4-4(2-\nu)n^2}{\lambda_{mn}{}^2}\right]I_{P1} + \left[\dfrac{8(3-\nu)n^2}{\lambda_{mn}{}^3} + \dfrac{4}{\lambda_{mn}}\right]I_n(\lambda_{mn})} \tag{11}$$

$$J_{m1} = J_{n+1}(\lambda_{mn}) - J_{n-1}(\lambda_{mn}); J_{P2} = J_{n+2}(\lambda_{mn}) + J_{n-2}(\lambda_{mn}); J_{m3} = J_{n+3}(\lambda_{mn}) - J_{n-3}(\lambda_{mn});$$
$$I_{P1} = I_{n+1}(\lambda_{mn}) + I_{n-1}(\lambda_{mn}); I_{P2} = I_{n+2}(\lambda_{mn}) + I_{n-2}(\lambda_{mn}); I_{P3} = I_{n+3}(\lambda_{mn}) + I_{n-3}(\lambda_{mn})$$

The frequency equation can be calculated From Eqs. (10) and (11), which allows determining eigenvalues λ_{mn}. The mode shape parameters C_{mn} can be determined corresponding to these eigenvalues by using either Eq. (10) or Eq. (11). The amplitude of each vibration mode in Eq. (5) is set by the normalization constant A_{mn} determined from the following condition [14].

$$\int_0^{2\pi}\int_0^{a} W_{mn}(r,\theta).W_{pq}(r,\theta) r dr d\theta = M_{mn}\delta_{mp}\delta_{nq} \tag{12}$$

where, M_{mn} is a mass of the plate, $\delta_{mp} = \delta_{nq} = 1$ if $m = p, n = q$ and $\delta_{mp}\delta_{nq} = 0$ if $m \neq p$ or $n \neq q$.

Dimensionless normalization constant A_{mn} can be determined from Eqs. (5) and (12) and it is given by following:

$$A_{mn} = \left[\frac{1}{\pi a^2}.\int_0^{2\pi}\int_0^{a}\left(\left(J_n\left(\frac{\lambda_{mn} r}{a}\right) + C_{mn}.I_n\left(\frac{\lambda_{mn} r}{a}\right)\right).\cos n\theta\right)^2 r dr d\theta\right] \tag{13}$$

In Eq (4), ω_{mn} is the natural frequency of vibrations:

$$\left(\frac{\lambda_{mn}{}^2}{a^2}\right)\sqrt{\frac{D}{\rho h}} \tag{14}$$

It is clear from the Eq. (14) the natural frequency of vibrations is dependent on the plate radius and eigenvalues from Eq. (14)

$$\lambda_{mn}{}^{4} = \frac{\rho h a^{4} \omega_{mn}{}^{2}}{D} \tag{15}$$

$$\lambda_{mn}{}^{4} = \frac{\rho h a^{4} \omega_{mn}{}^{2}}{D} - \frac{K_w a^4}{D} = \frac{\rho h a^{4} \omega_{mn}{}^{2}}{D} - \xi^2 \tag{16}$$

where

$$\xi^2 = \frac{K_w a^4}{D} \tag{17}$$

$$\lambda_{mn}{}^{*4} = \lambda_{mn}{}^{4} + \xi^2 \tag{18}$$

$$\lambda_{mn}{}^{*} = \left[\lambda_{mn}{}^{4} + \xi^2\right]^{\frac{1}{4}} \tag{19}$$

where λ_{mn} is eigenvalue without foundation and $\lambda_{mn}{}^{*}$ is eigenvalue with Winkler foundation

3 SOLUTION

Writing appropriate program and using Matlab computer software with symbolic capabilities, the above set of equations are solved for obtaining values of non-dimensional frequency parameter ($\lambda_{mn}{}^{*}$), for a given range of values Winkler foundation parameter. The following are the input parameters to the program; (i) Foundation ratio (ξ) (ii) Poisson ratio (ν) (iii) Upper bound for eigenvalues (N) (iv) Suggested for eigenvalues (d) (v) Number of mode shape parameters (n). The program finds eigenvalues $\lambda_{mn}{}^{*}$ by using Matlab root finding function.

4 RESULTS AND DISCUSSIONS

The Matlab programming code is also implanted for various plate materials by adjusting the Poisson ratio.The Poisson ratio used in this case is 0.3. Results are presented for a wide range of foundation parameter as they are not presently available in the literature. The eigenvalues for the plate with guided edge and fully resting on the elastic foundation are computed. The effects of the foundation stiffness ratios on eigenvalues are plotted in Fig. 2. It has been observed from Fig. 2, that eigenvalues increases with an increment in the foundation stiffness ratio, and the plate become unstable in the region when the foundation stiffness ratio exceeds a certain value. Twelve vibration modes are presented in Fig. 2. The smoothened stepped variation is observed in Fig. 2. The stepped region increases with increase in foundation stiffness ratio and vibration modes. The location of the stepped region with respect to ξ changed gradually from the range of 0.059745 † [0.988372] ‡– 1.053192 [14.67192] to 2 [9.91694] – 1.9966 [15.3779]. It † represents foundation stiffness ratio and ‡ represents eigenvalues through out the text. The eigenvalues for different plate materials, for various values of foundation parameters are computed and the results are given in Table 1. It is observed that for any value of foundation parameter (ξ), eigenvalues are independent on Poisson ratio, as presented in Fig. 3.

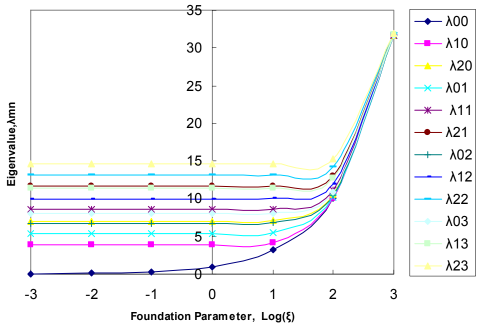

Fig. 2
Effect of foundation constraint, ξ on eigenvalues, λ_{mn} .

Table 1
Eigenvalues for different Poisson's ratio and foundation parameter

ν	$\xi=1$	$\xi=50$	$\xi=100$	$\xi=1000$	$\xi=1e+04$	$\xi=1e+05$
0	3.83614	7.2188	10.053	31.62448	100.0001	316.22777
0.1	3.83614	7.2188	10.053	31.62448	100.0001	316.22777
0.2	3.83614	7.2188	10.053	31.62448	100.0001	316.22777
0.3	3.83614	7.2188	10.053	31.62448	100.0001	316.22777
0.4	3.83614	7.2188	10.053	31.62448	100.0001	316.22777
0.5	3.83614	7.2188	10.053	31.62448	100.0001	316.22777

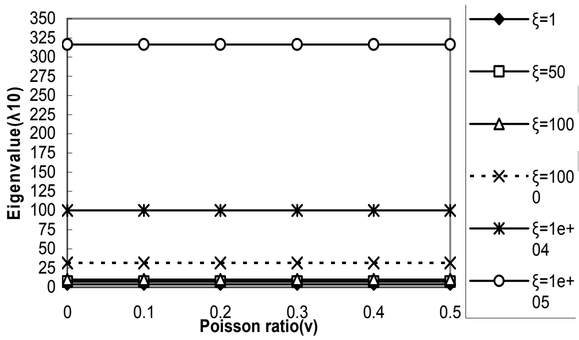

Fig. 3
Effect of poisson ratio, ν on eigenvalues, λ_{mn} .

5 CONCLUSIONS

The paper introduced a Matlab code for eigenvalues, of a circular plate with guided edge supports and resting on Winkler foundation. Two-dimensional plots of eigenvalues were drawn for different values of foundation stiffness ratios. It has been observed that the eigenvalues changes desperately only in a limited range of constraints specific to each vibration mode and are stable elsewhere. By knowing the position of the region where eigenvalues change rigorously is rudiment for structural design in the field of civil, marine, mechanical and aeronautical engineering applications and vibration control. It is also observed that the influence of foundation stiffness ratio on eigenvalues is more predominant.

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