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# Plane Wave Propagation Through a Planer Slab

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**ABSTRACT**

An approximation technique is considered for computing transmission and reflection coefficients for propagation of an elastic pulse through a planar slab of finite width. The propagation of elastic pulse through a planar slab is derived from first principles using straightforward time-dependent method. The paper ends with calculations of enhancement factor for the elastic plane wave and it is shown that it depends on the velocity ratio of the wave in two different media but not the incident wave form. The result, valid for quite arbitrary incident pulses and quite arbitrary slab inhomogeneities, agrees with that obtained by time-independent methods, but uses more elementary methods. 

**Keywords:** Inhomogeneous media; Plane waves; Time-independent methods; Navier's equations



## 1 INTRODUCTION

THE wave travelling through an elastic solid with finite velocity is known as elastic wave. If a body contains a sufficiently large number of molecules, so that the distance distances between two neighbouring molecules are negligible in comparison with the dimensions of the body, then the body is said to be the continuous body and it behaves in accordance with the law of mechanics. To simplify mathematical analysis, it is convenient to disregard the actual discrete molecular structure of the body, and treat the matter as uniformly and continuously distributed in the region of space without voids occupied by the body. There are mainly two themes to perform the study of mechanics of continuous matter: (a) Derivation of fundamental equations and (b) Derivation of constitutive relations. Fundamental equations for continuous body are based upon universal laws of physics such as conservation of mass, momentum and principle of energy etc., while the constitutive relations characterize the material properties of the matter, e.g., mechanical and thermal properties. These equations are in fact, the key points around which the various studies in the field of continuum mechanics proceed. Mathematically, the fundamental equations of the continuum mechanics are developed in two separate but essentially equivalent formulations. One, the integral or global form, derives from a consideration of the basic principles being applied to a finite volume of the material. The other, a differential or field approach, leads to equations resulting from the basic principles being applied to a very small (infinitesimal) element of volume. Under the continuum assumption, the field quantities such as density and volume which reflect the mechanical or kinematical properties of continuum bodies are expressed mathematically as continuous functions or at worst as piecewise continuous functions of space and time variable. Mathematical theory of continuous media is built upon the basic concepts of stress, motion and deformation, the law of conservation of mass, linear momentum, moment of momentum, energy and on the constitutive relations.

The problems of wave propagation through continuous bodies have been a subject of keen interest since long. The theory of wave propagation in elastic solids was developed during 19th and 20th centuries by Poisson [1], Kelvin [2], Rayleigh [3-5], Stoneley [6], Spencer [7], Love [8], Biot [9] and many others. Wave is a mode of energy transfer from one place to another, in a medium, often with little or no permanent displacement of the particles of

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the medium, i.e., little or no associated mass transport; instead there are oscillations around almost fixed positions. The mechanical waves require a medium to travel, while the electromagnetic waves can travel through the vacuum. Thus, in wave propagation, the particles in the medium do not change their original positions but they oscillate about their mean positions and usually periodic in nature with a finite velocity, e.g., water waves, sound waves, elastic waves. It is interesting to note that all wave motions have two important characteristics in common. First, energy is propagated to distant point. Second, the disturbance travels through the medium without giving the medium, as a whole, any permanent displacement. Each successive particle of the medium performs a motion similar to its predecessors but later in time returns to its original.

Wave propagation in inhomogeneous medium is a challenge for both theoretical research and engineering practice. With the rapid development of science and technology, wave motion study of the heterogeneous medium (atmosphere, ocean, earth-crust, functionally graded materials and cycle grid structure, etc.) seems much more important. Epstien [10] investigated reflection wave in an inhomogeneous absorbing medium by solving wave equation with variable coefficient based on hypergeometric function. The procedure represented that the reflection is always very insignificant, except the case when conductivity is small and where we have conditions very near to total reflection, which is the same as mechanism of transmission of acoustic or electromagnetic wave in earth atmosphere. Researchers had discussed the theory of plane waves Sinha [11] studied the transmission of elastic waves through a homogenous layer sandwiched in homogenous media. Tooly et al., [12] discussed reflection and transmission of plane compressional waves. Gupta [13] solved the problem of reflection of elastic waves from a linear transition layer. Agemi [14] studied the problem on the global existence of nonlinear elastic waves.

Kakar and Kakar [15] discussed propagation of Love waves in a non-homogeneous elastic media. In this work, we shall derive the solution of Navier's equations by using time-dependent methods. We shall consider first the case of homogeneous slab then inhomogeneous slab. The velocity is constant for homogeneous case but it is continuously varying for non-homogeneous case. The time-dependent methods are applied to solve the transmitted and reflected pulses. These methods are much easier than the earlier methods used.

## 2 GOVERNING EQUATIONS

The forces per unit area set up inside the body to resist deformation are called stresses. The deformation of the body accompanying stress is called strain. Thus stress and strain occur together. The strain set up in a body in such a way that there is a change in volume but no change in shape, is called dilatation. There are two kinds of dilatation: compression, in which volume is reduced; and rarefaction, in which the volume is increased. The second type of elastic deformation is a change of shape without a change in volume and is called shear. Consider a surface element, $\Delta S$ situated either in the interior or on the boundary of a medium, and let the force acting on this surface element be $T\Delta S$ .

$$\lim_{\Delta\to 0}\frac{T\Delta S}{\Delta S}=(x_i:\upsilon) \tag{1}$$

where the vector $T$ is called stress vector and represents the surface force per unit area of the surface element acting at the point $(x_i)$ whose orientation is specified by a unit normal vector, $\upsilon$ .The stress force depends not only on the position of the surface element but also on the orientation of the surface element. The state of stress at any point of a medium is completely characterized by the nine quantities, called stress tensors, $\tau_{ij}$ (sokolnikoff [16]). In more precise form, if $T^{\upsilon}$ be the stress vector acting at a point of a surface to which $\upsilon$ is normal, then the stress tensor can be written as:

$$T^{\upsilon}=\tau_{ij}\upsilon_j \qquad (i,j=1,2,3) \tag{2}$$

where, $\tau_{ij}$ is the j$^{th}$ component of the stress vector acting on a surface element to which $x_i$ axis is normal.

The relation of the strain tensor $e_{ij}$ with the components of displacement vector $(u,v,w)$ for a continuous deformable medium, is given by (sokolnikoff [16])

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \tag{3}$$

In the classical theory of elasticity, the relation between the stress components, $\tau_{ij}$ and the strain components, $e_{ij}$ for an elastic solid continuum is given by Hooke's law. It states that, within the elastic limit, the stress is a linear function of strain. That is:

$$\tau_{ij} = c_{ijkl} e_{kl} \qquad (i, j, k, l = 1, 2, 3) \tag{4}$$

where $c_{ijkl}$ are the elastic constants or elastic moduli, which characterize the elastic properties of the body. These constants are 81 in number. If the elastic constants vary from point to point of the medium, i.e., the elastic constants are the functions of the position, the body is said to be elastically non homogeneous or inhomogeneous. On the other hand if the elastic constants are same for all points of the medium, then the body is called elastically homogeneous The equation of motion for a homogeneous elastic body, in the presence of body forces

$$\tau_{ji,j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad (i, j = x, y, z) \tag{5}$$

For a homogeneous isotropic elastic medium, the coefficient $c_{ijkl}$ can be expressed by only two elastic constants $\lambda$ and $\mu$ called Lame's parameters and the stress-strain relation (Hooke's law) is given by:

$$\tau_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij}, \tag{6}$$

where $\delta_{ij}$ is Kronecker delta.

Substituting Eq. (3) and Eq. (6) in Eq. (5), we get

$$(\lambda + \mu)\nabla\nabla \cdot u + \mu \nabla^2 u = \rho \frac{\partial u^2}{\partial t^2} \tag{7}$$

where, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a Laplacian operator.

Using the vector identity $\nabla^2 u = \nabla\nabla . u - \nabla \times (\nabla \times u)$ in Eq. (7), we get

$$(\lambda + 2\mu)\nabla\nabla \cdot u + \mu \nabla \times (\nabla \times u) = \rho \frac{\partial u^2}{\partial t^2} \tag{8}$$

## 3 FORMULATION OF THE PROBLEM

Consider an infinite absolutely rigid plane plate (screen/surface), which is well wielded contact with the surrounding elastic medium. Let x-y-plane coincide with the plate (where central part of the plane is shown). The z-axis is taken normal to the plate in the upward direction. As horizontal section of the interface is shown and the media are taken in the x-y-plane $(-\infty < x < \infty,\ -\infty < y < \infty)$. If we disturb the plate sufficiently rapid in such a manner that it remains parallel to itself (plane parallel moment; horizontal plane), then at any instant of time the displacement of any point of the interface will be same. The displacement vector $u_i$ is taken to be independent of *x* and *y*. Rapid the medium in front of the interface will of course be compressed, while behind it, on the negative z-axis will be stretched. The state will be transmitted in the medium in directions parallel to z-axis. The problem is formulated by assuming the following assumptions.

Media are taken to be continuous at the interface due to perfect welded contact, with surrounding elastic medium, during the transmission of motion through the interface. The media do not slip relative to each other, so that at the interface resultant horizontal motions above and below are equal in pairs.

The condition of the interaterrestrial contacts for the vertical motions are analogous, there can be neither exploitation nor formation if intermediately cavities at the interface during motion, then $\overline{w}_1 - \overline{w}_2 = 0$, where $\overline{w}_1$ and $\overline{w}_2$ are the resultant vertical motions in the lower and upper media respectively.

The solutions of Eq. (8) are given by:

$$w = \alpha_z \left[ f_1\left(t - \frac{z}{a}\right) + f_2\left(t + \frac{z}{a}\right)\right] \qquad or \qquad w = \alpha_z \left[ f_1(z - at) + f_2(z + at)\right] \tag{9}$$

$$(u,v) = (\alpha_x, \alpha_y)\left[ f_1\left(t - \frac{z}{b}\right) + f_2\left(t + \frac{z}{b}\right)\right] \qquad or \qquad (u,v) = (\alpha_x, \alpha_y)\left[ f_1(z - bt) + f_2(z + bt)\right] \tag{10}$$

The first term $f_1(z-at), f_1(z-bt), f_1\left(t-\frac{z}{a}\right), f_1\left(t-\frac{z}{b}\right)$ in the above expressions represents the transmission of waves in the positive z-direction i.e. outgoing wave or advance wave and the second term $f_2(z+at), f_2(z+bt), f_2\left(t+\frac{z}{a}\right), f_2\left(t+\frac{z}{b}\right)$ represents the transmission in the negative z-direction i.e. incoming wave or retarding wave. Here $u,v,w$ are the components of $u_i$ and they vary with time but they differ only in the cosine of angles made by $u_i$ with the axis of co-ordinates $\alpha_x, \alpha_y, \alpha_z$. For sake of convenience, the coefficient of $\alpha$ 's$(\alpha_x, \alpha_y, \alpha_z)$ are taken to be unity as they do not affect the general behavior of the field variables. Since the terms of the above solution functions are arbitrary therefore they have bounded derivatives up to second order.

In case of the present problem, the displacements are assumed as:

Incident wave; $(z = \overline{a})$ in the medium $M_1$ $(-\infty < z \le a, -\infty < x, y < \infty)$: $W_I = W_I(z - c_0 t)$
where $c_0$ is the velocity of propagation in medium $M_1$

Reflected wave; $(z = \overline{a})$ in the medium $M_1$ , $W_R = W_R(z + c_0 t)$

Transmitted wave into the slab $S$ $(a \le z \le b, -\infty < x, y < \infty)$: $W_+ = W_+(z - c_1 t)$
where, $c_0$ is the velocity of propagation in $M_2$

Wave reflected from the upper boundary $(z = \overline{b})$ of slab into the slab: $W_- = W_-(z + c_1 t)$

Wave transmitted into the medium $M_2$ from slab: $W_T = W_T(z - c_0 t)$ i.e. medium $M_1$ is similar to $M_2$

## 4 CASE OF SINGLE-LAYER SLAB

We shall assume that the slab lies perpendicular throughout to the z-axis in $\Re^3$ , with faces at $z = a > 0$ and $z = b > a$ , and is isotropic in the horizontal *x* and *y* directions for slab $'S'$ $(a \le z \le b, -\infty < x, y < \infty)$. The incident wave has finite energy and propagates in the positive *z*-direction, normal to the slab and incident from below. Under the above assumptions, the problem essentially becomes one directional. The propagation velocity is $c_0$ outside the slab and $c_1$ inside the slab, where $c_0$ and $c_1$ are constants with $0 < c_1 < c_0$ (see Fig.1) . The general form of the solution is taken as:

$$W(z,t) = \begin{vmatrix} W_I(z-c_0t)+W_R(z+c_0t) & if(-\infty < z \le a) \quad for(M_1) \\ W_+(z-c_1t)+W_-(z+c_1t) & if(a \le z \le b) \quad for(S) \\ W_T = W_T(z-c_0t) & if(b \le z < \infty) \quad for(M_2) \end{vmatrix} \tag{11}$$

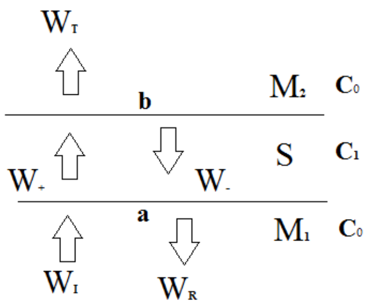


**Fig. 1**
Single layer slab.

*4.1 Solution of the problem*

The field variables $W_R$, $W_+$, $W_-$ and $W_T$ for the given value of $W_I$ can be found from the displacement and stress-boundary conditions at the interfaces. But in this case, we have taken the coefficient of $\alpha$ 's$(\alpha_x,\alpha_y,\alpha_z)$ equal to unity. Therefore, we apply the displacement boundary conditions coupled with travel-time of wave and using the lag in time for the waves travelling in the same direction with different velocities of propagation.

At the interfaces, $z=a$ and $z=b$ we assume that $W(z,t)$ is continuous at all times *t*. Therefore, at $z=a$ this leads to:

$$\frac{1}{c_0}W_I(z-c_0t)+\frac{1}{c_0}W_R(z+c_0t)=\frac{1}{c_0}W_+(z-c_1t)+\frac{1}{c_0}W_-(z+c_1t) \tag{12}$$

$$\frac{1}{c_0}W_I(z-c_0t)-\frac{1}{c_0}W_R(z+c_0t)=\frac{1}{c_1}W_+(z-c_1t)-\frac{1}{c_1}W_-(z+c_1t) \tag{13}$$

Adding Eq. (12) and Eq. (13), we get

$$\frac{2}{c_0}W_I(z-c_0t)=\frac{c_0+c_1}{c_0c_1}W_+(z-c_1t)-\frac{c_0-c_1}{c_0c_1}W_-(z+c_1t) \tag{14}$$

or

$$W_+(z-c_1t)=\frac{2c_1}{c_0+c_1}W_I(z-c_0t)+\frac{c_0-c_1}{c_0+c_1}W_-(z+c_1t) \tag{15}$$

Subtracting Eq. (12) and Eq. (13), we get

$$\frac{2}{c_0}W_R(z+c_0t)=\frac{c_1-c_0}{c_0c_1}W_+(z-c_1t)+\frac{c_1+c_0}{c_0c_1}W_-(z+c_1t) \tag{16}$$

or

$$W_R\left(z+c_0t\right)=\frac{c_1-c_0}{2c_1}W_+\left(z-c_1t\right)+\frac{c_1+c_0}{2c_1}W_-\left(z+c_1t\right) \tag{17}$$

Combining Eq. (15) and Eq. (17), we get

$$W_R\left(z+c_0t\right)=\frac{c_0-c_1}{c_0+c_1}W_I\left(z-c_0t\right)+\frac{2c_0}{c_0+c_1}W_-\left(z+c_1t\right) \tag{18}$$

Now Eq. (15) and Eq. (18) must hold at all times *t*. therefore put $u=a-c_1t,$ then $t=\left(a-u\right)/c_1$ and Eq. (15) becomes:

$$W_+(u)=\frac{2c_1}{c_0+c_1}W_I\left(a-\frac{c_0}{c_1}(a-ut)\right)+\frac{c_0-c_1}{c_0+c_1}W_-(2a-u) \tag{19}$$

Since this holds for all *u,* we can put $u=z-c_1t$ and get

$$W_+(z-c_1t)=\frac{2c_1}{c_0+c_1}W_I\left(a+\frac{c_0}{c_1}(z-a-c_1t)\right)+\frac{c_0-c_1}{c_0+c_1}W_-(a-z+c_1t). \tag{20}$$

Similarly, if we put $v=a+c_0t,$ then $t=\left(v-a\right)/c_0$ and Eq. (18) becomes:

$$W_R(v)=\frac{c_0-c_1}{c_0+c_1}W_I(2a-v)+\frac{2c_0}{c_0+c_1}W_-\left(a-\frac{c_1}{c_0}(a-v)\right) \tag{21}$$

when $v=z+c_0t$ , we get

$$W_R(z+c_0t)=\frac{c_0-c_1}{c_0+c_1}W_I(2a-z-c_0t)+\frac{2c_0}{c_0+c_1}W_-\left(a+\frac{c_1}{c_0}(z-a-c_0t)\right) \tag{22}$$

Eq. (20) and Eq. (22) give $W_R$ and $W_+$ in terms of $W_I$ and $W_-$ It must be noted that all the above relations are hold for values of *z* and *t*

Similarly, at the other interface $z=b,$ we get

$$W_-(z+c_1t)=\frac{c_0-c_1}{c_0+c_1}W_+(2b-z-c_1t), \tag{23}$$

$$W_T(z-c_0t)=\frac{2c_0}{c_0+c_1}W_+\left(b+\frac{c_1}{c_0}(z-b-c_0t)\right) \tag{24}$$

Giving $W_-$ and $W_T$ in terms of $W_+$ for all *z* and *t*

Now if we combine Eq. (20) and Eq. (23) we get

$$W_+(z-c_1t)=W_0(z-c_1t)+\left(\frac{c_0-c_1}{c_0+c_1}\right)^2W_+(2b-2a+z-c_1t), \tag{25}$$

where

$$W_0(z-c_1t)=\frac{2c_1}{c_0+c_1}W_I\left(a+\frac{c_0}{c_1}(z-a-c_1t)\right). \tag{26}$$

Eq. (25) can be solved for $W_+$ by iteration, we get

$$W_+(z-c_1t)=\sum_{n=0}^{\infty}\left(\frac{c_0-c_1}{c_0+c_1}\right)^{2n}W_0(2n(b-a)+z-c_1t) \tag{27}$$

$$W_+(z-c_1t)=\frac{2c_1}{c_0+c_1}\sum_{n=0}^{\infty}\left(\frac{c_0-c_1}{c_0+c_1}\right)^{2n}W_I\left(a+2n\frac{c_0}{c_1}(b-a)+\frac{c_0}{c_1}(z-a-c_1t)\right). \tag{28}$$

Also, Eq. (23) can be solved for $W_-$ by iteration, we get

$$W_-(z+c_1t)=\left(\frac{c_0-c_1}{c_0+c_1}\right)W_+(2b-z-c_1t)=\sum_{n=0}^{\infty}\left(\frac{c_0-c_1}{c_0+c_1}\right)^{2n+1}W_0(2n(b-a)+2b-z-c_1t). \tag{29}$$

Using, Eq. (27) and Eq. (28) to find $W_T$ from Eq. (19) and $W_R$ from Eq. (17)

*4.2 Discussion*

If the incident wave is $W_I$ bounded, then $W_0$ in the Eq. (26) is also bounded, and hence series in Eq. (27) and in Eq. (28) are convergent.

If the incident wave $W_I$ is a periodic having time period $(2b-2a)/c_0$ then $W_0$ will also be periodic with time period $(2b-2a)/c_1$. Hence, Eq. (27) reduces to:

$$W_+(z-c_1t)=\sum_{n=0}^{\infty}\left(\frac{c_0-c_1}{c_0+c_1}\right)^{2n}W_0(z-c_1t)$$

$$\Rightarrow W_+(z-c_1t)=\frac{(c_0+c_1)^2}{4c_0c_1}W_0(z-c_1t) \quad or \quad W_+(z-c_1t)=\frac{c_0+c_1}{2c_0}W_I\left(a+\frac{c_0}{c_1}(z-a-c_1t)\right) \tag{30}$$

The factor $\frac{(c_0+c_1)^2}{4c_0c_1}$ in Eq. (29) is called an amplitude enhancement factor. The enhancement factor depends on the ratio of $c_1/c_0$ but not the incident wave form. Hence, enhancement factor can be written as: $\xi=\frac{(c_0+c_1)^2}{4c_0c_1}=\frac{(1+\eta)^2}{4\eta}$ where, $\eta=\frac{c_1}{c_0}$

As, $\eta=\frac{c_1}{c_0}$ increases, the amplitude enhancement factor decreases and vice-versa.

Using Eq. (29) and Eq. (24), the transmitted wave is:

$$W_T(z-c_0t)=\frac{2c_0}{c_0+c_1}\frac{(c_0+c_1)^2}{4c_0c_1}W_0(z-c_1t)=W_I\left(a+\frac{c_0}{c_1}(b-a)+z-c_0t\right) \tag{31}$$

Using Eq. (29) and Eq. (22), the reflected wave is:

$$W_R(z+c_0t)=\frac{c_0-c_1}{c_0+c_1}\left(W_I(2a-z-c_0t)-\frac{4c_0c_1}{(c_0+c_1)^2}\frac{(c_0+c_1)^2}{4c_0c_1}W_I(2a-z-c_0t)\right)=0 \tag{32}$$

We observe that the transmitted wave has the same amplitude as that of the incident wave but it lags in time due to the width of the slab. The amplitude of the reflected wave is zero. This means that the slab is transparent to any pulse train with resonant time period.

## 5 CASE OF MULTIPLE-LAYER SLAB

We now consider a multiple-layer slab having $n$ layers with interfaces $a_i$, $0<a_0<a_1<\cdots\cdots\cdots\cdots<a_n$ and propagation velocity $c_j$ in the $j^{th}$ layer.

The general form of the solution is taken as:

$$W(z,t)=\begin{vmatrix} W_I(z-c_0t)+W_R(z+c_0t)\cdots\cdots\cdots if(-\infty<z\le a_0) \\ W_+^j(z-c_jt)+W_-^j(z+c_jt)\cdots\cdots\cdots if(a_{j-1}\le z\le a_j) \\ W_T=W_T(z-c_{n+1}t)\cdots\cdots\cdots if(a_n\le z<\infty)\cdots for(M_2) \end{vmatrix} \tag{33}$$

*5.1 Solution of the problem*

The solution has been found in the same way as it is done in the previous case for single layer; here we just piece together the solutions of previous section. Now from Eq. (25) we have

$$W_+^{(k)}(z-c_kt)=\frac{2c_k}{c_k+c_{k-1}}W_+^{(k-1)}\left(a_{k-1}+\frac{c_{k-1}}{c_k}(z-a_{k-1}-c_kt)\right)+\frac{c_k-c_{k-1}}{c_k+c_{k-1}}W_-^{(k)}(a_{k-1}-(z-a_{k-1}+c_kt)). \tag{34}$$

A similar expression gives $W_+^{(k-1)}$ in terms of $W_+^{(k-2)}$ and $W_-^{(k-1)}$ if we combine the expressions obtained for $W_+^{(j)}$ $(1\le j\le m)$ and set $W_+^{(0)}=W_I$, we get

$$\begin{aligned} W_+^{(m)}(z-c_mt)&=\prod_{i=0}^{m-1}\left(\frac{2c_{i+1}}{c_{i+1}+c_i}\right)W_I\left(a_0+\sum_{i=0}^{m-2}\frac{c_0}{c_{i+1}}\Delta a_i+\frac{c_0}{c_m}(z-a_{m-1}+c_mt)\right)\\ &-\sum_{j=1}^{m}\left(\prod_{i=j}^{m-1}\frac{2c_{i+1}}{c_{i+1}+c_i}\right)\left(\frac{\Delta c_{j-i}}{c_j-c_{j-i}}\right)\times W_-^{(j)}\left(a_{j-1}-\sum_{i=j}^{m-1}\frac{c_j}{c_i}\Delta a_i-\frac{c_j}{c_m}(z-a_{m-1}+c_mt)\right). \end{aligned} \tag{35}$$

Here, we have $\Delta a_j=a_{j+1}-a_j$ and $\Delta c_j=c_{j+1}-c_j$ and for simplicity we put $\Sigma_{l=m}^{m-i}=0$. In the same way, we have

$$W_-^{(k)}(z+c_kt)=\frac{2c_k}{c_k+c_{k-1}}W_-^{(k-1)}\left(a_k+\frac{c_{k+1}}{c_k}(z-a_k+c_kt)\right)-\frac{c_{k+1}-c_k}{c_{k+1}+c_k}W_+^{(k)}(a_k-(z-a_k+c_kt)). \tag{36}$$

Similar expression gives $W_{-}^{(j)}$ in terms of $W_{-}^{(j+1)}$ and $W_{-}^{(k-1)}$ for $W_{+}^{(j)}$ $(k \le j \le n)$ and combining these expressions and setting $W_{-}^{(n+1)} = 0$, we get

$$W_{-}^{(k)}(z-c_k t) = \sum_{j=k}^{n}\left(\prod_{j=k}^{j-1}\frac{2c_i}{c_{i+1}+c_i}\right)\left(\frac{\Delta c_j}{c_{j+1}+c_j}\right)\times W_{-}^{(j)}\left(a_j+\sum_{i=k}^{k-1}\frac{c_j}{c_i}\Delta a_i-\frac{c_j}{c_k}(z-a_k+c_k t)\right). \tag{37}$$

Put Eq. (36) and Eq. (34), we get

$$\begin{aligned} W_{+}^{(m)}(z-c_m t) &= C_{m-1}W_I\left(a_j+\sum_{i=0}^{m-2}\frac{c_0}{c_{i+1}}\Delta a_i+\frac{c_0}{c_m}(z-a_{m-1}+c_m t)\right)\\ &-\sum_{k=1}^{m}\sum_{j=k}^{n}(C_{m-1}/C_{k-1})(D_{j-1}/D_{k-1})\left(\frac{\Delta c_{k-i}}{c_k+c_{k-i}}\right)\left(\frac{\Delta c_j}{c_{j+1}+c_j}\right)\\ &\times W_{+}^{(j)}\left(a_j+\sum_{i=k+1}^{j-1}\frac{c_j}{c_i}\Delta a_j+\sum_{i=k}^{m-1}\frac{c_j}{c_i}\Delta a_j+\frac{c_j}{c_m}(z-a_{m-1}+c_m t)\right). \end{aligned} \tag{38}$$

where

$$C_{j-1} = \prod_{i=0}^{j-1}\frac{2c_{i+1}}{c_{i+1}+c_i} \qquad and \qquad D_{j-1} = \prod_{i=0}^{j-1}\frac{2c_{i+1}}{c_{i+1}+c_i} \tag{39}$$

Eq. (38) can be solved by iteration, as we did for Eq. (25), the solution is very complicated therefore for sake of convenience, we developed, as for Eq. (27), the series solution is:

$$W_{+}^{(m)}(z-c_m t) = \sum_{p=0}^{\infty}W_{2p}^{(m)}(z-c_m t) \tag{40}$$

where

$$W_{0}^{(m)}(z-c_m t) = C_{m-1}W_I\left(a_0+\sum_{i=0}^{m-2}\frac{c_0}{c_{i+1}}\Delta a_i+\frac{c_0}{c_m}(z-a_{m-1}+c_m t)\right) \tag{41}$$

There is no reflection at the interface, and

$$\begin{aligned} W_{2p}^{(m)}(z-c_m t) &= -\sum_{k=1}^{m}\sum_{j=k}^{n}(C_{m-1}/C_{k-1})(D_{j-1}/D_{k-1})\left(\frac{\Delta c_{k-i}}{c_k+c_{k-i}}\right)\left(\frac{\Delta c_j}{c_{j+1}+c_j}\right)\\ &\times W_{2p}^{(m)}\left(a_j+\sum_{i=k+1}^{j-1}\frac{c_j}{c_i}\Delta a_j+\sum_{i=k}^{m-1}\frac{c_j}{c_i}\Delta a_j+\frac{c_j}{c_m}(z-a_{m-1}+c_m t)\right).D_{j-1} = \prod_{i=0}^{j-1}\frac{2c_{i+1}}{c_{i+1}+c_i} \end{aligned} \tag{42}$$

Involves 2p reflections at interfaces within the slab (see Fig. 2). $W_R$ and $W_T$ are calculated as done in Eq. (22) and Eq. (24).

$$W_R(z+c_0t)=\frac{\Delta c_0}{c_0+c_1}W_I(2a-z-c_0t)+\frac{2c_0}{c_0+c_1}W_-\left(a+\frac{c_1}{c_0}(z-a-c_0t)\right). \tag{43}$$

$$W_T(z-c_{n+1}t)=\frac{2c_{n+1}}{c_{n+1}+c_n}W_+\left(b+\frac{c_{n+1}}{c_{n+1}+c_n}(z-b-c_{n+1}t)\right). \tag{44}$$

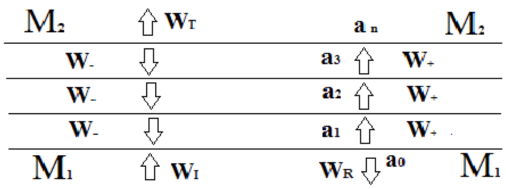


**Fig. 2**
Multiple layers slab.

*5.2 Discussion*

If the incident wave $W_I$ is a periodic having time period $(2\Delta a_j)/c_0$ then $j^{th}$ layer is resonant, and will appear transparent to the waveforms $W_+^{j-1}$ and $W_-^{j+1}$. The delay in each pulse time is:

$$a_0+\sum(2\Delta a_j)/c_j \tag{45}$$

## 6 CASE OF CONTINUOUS SLAB

Finally, we take the case of continuous slab in which the wave velocity varies continuously and differentially across the slab.

$$c=c(z)\begin{vmatrix}c(a)\cdots\cdots\cdots if\ (-\infty<z\le a)\\ c(z)\cdots\cdots\cdots if\ (a\le z\le b)\\ c(b)\cdots\cdots\cdots if\ (b\le z<\infty)\end{vmatrix} \tag{46}$$

*6.1 Solution of the problem*

This can be treated as the limiting case of multiple slab of preceding section and it can be solved by replacing $a_i$ by z, and let $n\to\infty$ then $\frac{\Delta c_i}{\Delta a_i}\to\frac{dc(z)}{dz}$, but $a=a_0$ and $b=a_n$ remain constant. Therefore for limiting case,

$$\sum_{i=0}^{m-1}\frac{c_0}{c_i}\Delta a_i\to\int_a^z\frac{c(a)}{c(z)}dz \tag{47}$$

$$\frac{\Delta c_i/\Delta a_i}{c_{i+1}+c_i}\to\frac{c'(z)}{2c(z)} \tag{48}$$

Also,

$$\frac{2c_{i+1}}{c_{i+1}+c_i}=\left(1-\frac{\Delta c_j}{2c_{i+1}}\right)^{-1} \tag{49}$$

Hence,

$$\begin{aligned}(C_{m-1})^{-1}&=\left(\prod_{i=0}^{m-1}\frac{2c_{i+1}}{c_{i+1}+c_i}\right)^{-1}=\prod_{i=0}^{m-1}\left(1-\frac{1}{2c_{i+1}}\frac{\Delta c_i}{\Delta a_i}\Delta a_i\right)\\&=1-\sum_{i=0}^{m-1}\frac{1}{2c_{i+1}}\Delta a_i+\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\left(\frac{1}{2c_{i+1}}\frac{\Delta c_i}{\Delta a_i}\Delta a_i\right)\left(\frac{1}{2c_{i+1}}\frac{\Delta c_j}{\Delta a_j}\Delta a_j\right)\end{aligned} \tag{50}$$

For $n\to\infty$ Eq. (50) reduces to:

$$\begin{aligned}(C_{m-1})^{-1}&\to 1-\int_a^z\frac{c^{/}(u)}{2c(u)}du+\int_a^z\int_a^u\frac{c^{/}(u)}{2c(u)}\frac{c^{/}(v)}{2c(v)}dvdu-\cdots\\&=\exp\left(-\frac{1}{2}\int_a^z\frac{c^{/}(u)}{2c(u)}du\right)=\exp\left(-\frac{1}{2}(\log c(z)-\log c(a))\right)=\left(\frac{c(a)}{c(z)}\right)^{1/2}\end{aligned} \tag{51}$$

Similarly, we can find that

$$\frac{2c_{i+1}}{c_{i+1}+c_i}=\left(1+\frac{\Delta c_j}{2c_{i+1}}\right)^{-1} \tag{52}$$

From Eq. (51) , it follows that

$$(D_{m-1})^{-1}\to\left(\frac{c(a)}{c(z)}\right)^{1/2} \tag{53}$$

Therefore, Eq. (34) becomes by using Eq. (52) and Eq. (53)

$$W_+(z-c(z)t)=\left(\frac{c(a)}{c(z)}\right)^{1/2}W_I\left(a+\int_a^z\frac{c(u)}{c(u)}du-c(a)t\right)-\int_a^z\left(\frac{c(z)}{c(y)}\right)^{1/2}\frac{c^{/}(y)}{2c(y)}duW_-\left(y-\int_y^z\frac{c(y)}{c(u)}du+c(y)t\right)dy \tag{54}$$

and Eq. (36) becomes by using Eq. (52) and Eq. (53)

$$W_-(y-c(y)t)=+\int_y^b\left(\frac{c(y)}{c(x)}\right)^{1/2}\frac{c^{/}(x)}{2c(x)}duW_+\left(x+\int_y^x\frac{c(x)}{c(u)}du-c(x)t\right)dx \tag{55}$$

Comparing Eq. (54) and Eq. (55)

$$\begin{aligned}W_+(z-c(z)t)&=\left(\frac{c(a)}{c(z)}\right)^{1/2}W_I\left(a+\int_a^z\frac{c(u)}{c(u)}du-c(a)t\right)\\&-\int_a^z\int_y^b\left(\frac{c(z)}{c(x)}\right)^{1/2}\frac{c^{/}(y)}{2c(y)}\frac{c^{/}(x)}{2c(x)}\times W_+\left(x+\int_y^x\frac{c(x)}{c(u)}du+\int_y^z\frac{c(x)}{c(v)}du-c(x)t\right)dxdy\end{aligned} \tag{56}$$

Eq. (56) can be solved for $W_+$ by iteration. Hence, we have

$$W_+(z-c(z)t)=\sum_{p=0}^{\infty}W_{2p}(z-c(z)t)\left(\frac{c(a)}{c(z)}\right)^{1/2} \tag{57}$$

where

$$W_0(z-c(z)t)=\left(\frac{c(a)}{c(z)}\right)^{1/2}W_I\left(a+\int_a^z\frac{c(u)}{c(u)}du-c(a)t\right) \tag{58}$$

Involves no reflections, and

$$\sum_{p=0}^{\infty}W_{2p}(z-c(z)t)=-\int_a^z\int_y^b\left(\frac{c(z)}{c(x)}\right)^{1/2}\frac{c^{/}(y)}{2c(y)}\frac{c^{/}(x)}{2c(x)}\times W_{2p-2}\left(x+\int_y^x\frac{c(x)}{c(u)}du+\int_y^z\frac{c(x)}{c(v)}du-c(x)t\right)dxdy \tag{59}$$

Involves 2p reflections.
Eq. (43) and Eq. (44) gives

$$W_R(z+c(a)t)=C_-(z+c(a)t) \qquad \text{if } z\le a \tag{60}$$

$$W_T(z-c(a)t)=W_+(z-c(a)t) \qquad \text{if } z\ge b \tag{61}$$

## 7 CONCLUSIONS

The incident wave $W_I$ is a periodic having time period $(2\Delta a_j)/c_0$ for multiple slab. The j$^{th}$ layer is resonant in the multiple slab, and will appear transparent to the waveforms $W_+^{j-1}$ and $W_-^{j+1}$. The delay in each pulse time $a_0+\sum(2\Delta a_j)/c_j$. We observe that the transmitted wave has the same amplitude as that of incident wave but it lags due to finite width of the slab. The amplitude of the reflected wave is zero. This means that slab is transparent to any pulse train with resonant time period.

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