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Thermal Vibration of Composites and Sandwich Laminates Using Refined Higher Order Zigzag Theory

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ABSTRACT

Vibration of laminated composite and sandwich plate under thermal loading is studied in this paper. A refined higher order theory has been used for the purpose. In order to avoid stress oscillations observed in the implementation of a displacement based finite element, the stress field derived from temperature (initial strains) have been made consistent with total strain field. So far no study has been reported in literature on the thermal vibration problem based on the refined higher order theory using a FE model. Numerical results are presented for thermal vibration problems to study the influence of boundary conditions, ply orientation and plate geometry on the natural frequencies of these structures.

Keywords: Finite element; Higher order; Laminated composites; Thermal load

1 INTRODUCTION

DUE to increasing use of fibrous composite materials in various structural members subjected to high temperatures, it is desirable to accurately predict the vibration response of thermally stressed laminated composite and sandwich plates. This study is motivated by the lack of finite element results for natural frequencies of laminated composite and sandwich plates under thermal loading. Only few publications exist for the vibration of laminated composite and sandwich plates under thermal loading. Noor and Burton [1] presented a three dimensional analytical solution for the vibration of thermally stressed laminated angle-ply composite plates. Matsunaga [2] studied free vibration and stability of angle-ply composite and sandwich plates using two dimensional global higher order shear deformation theory subjected to thermal loading. Several two dimensional refined higher order shear deformation theories have been proposed to analyze the response of laminated composite plates. Global higher order plate theories (HSDT) were proposed by Reddy and Phan [3], Putcha and Reddy [4], Tessler et al. [5], Ganapathi and Makhecha [6]), Shi et al. [7] and Khare et al. [8]. Various layer wise and individual theories have been presented to obtain accurate solution at ply level by Carrera [9], Nosier et al. [10] and Cho et al. [11]. Although such types of theories have been proved to be one of the best alternatives to three dimensional elasticity solutions, these theories are computationally expensive to obtain the accurate solutions. Since the total number of unknowns increases dramatically as the number of layers increases. A number of third order theories [12-16] in which the in-plane displacement are assumed to be a cubic expression of the thickness coordinate and out of plane displacement to be a quadratic expression. Kapuria and Achary [16] presented a new zigzag theory based on zigzag third-order variation of in-plane displacements for laminated plates subjected to thermal loading. Matsunaga [17] evaluated interlaminar stresses and displacements in cross-ply multilayered composite and sandwich plates subjected to

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thermal loadings using two-dimensional global higher-order deformation theory. Wang et al. [18] studied the response of dynamic interlaminar stresses in laminated composite plates with piezoelectric layers using an analytical approach based on the layer wise theories.

Zhen and Wanji [19] presented global-local higher order shear deformation theory which satisfies the free surface conditions and the geometric and stress continuity conditions at interfaces for angle ply laminated plates. Thermal buckling behavior of cross and angle-ply laminated composite and sandwich plates subjected to thermal loading have been analyzed by Matsuanga [20, 21]. Vibration and stability problem of angle-ply laminated composite and sandwich plates subjected to thermal loading using global higher order shear deformation theory is presented by Matsuanga [22]. However, finite element based solutions are also needed to validate the analytical solutions. On the other hand many publications were reported in the literatures on thermal buckling of conventional isotropic material, composite and sandwich laminates. However, there is lack of study on vibration under thermal loading based on the refined plate theory especially using a finite element approach.

The present study is focused to fill the above void in the literature. In the present paper natural frequencies of laminated composite and sandwich plates subjected to thermal loading have been studied for the first time by using FE method based on a refined higher order zigzag theory.

2 FORMULATION

The in-plane displacement fields Fig. 1 are taken as below:

$$u_\alpha = u_\alpha^0 + \sum_{k=0}^{n_u-1} S_\alpha^k (z - z_k^u) H(z - z_k^u) + \sum_{k=0}^{n_l-1} T_\alpha^k (z - z_k^l) H(z - z_k^l) + \xi_\alpha z^2 + \varphi_\alpha z^3 \tag{1}$$

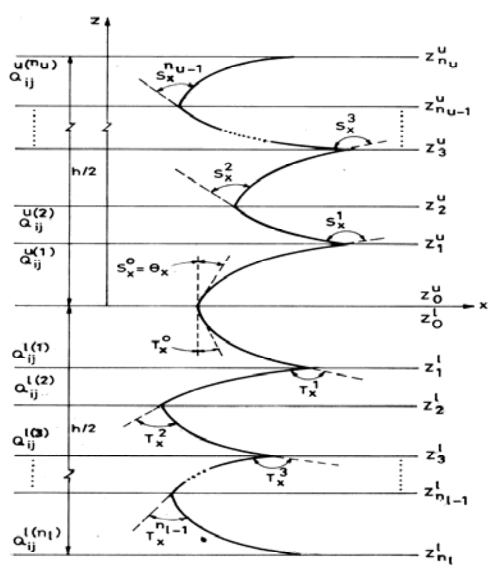

Fig. 1
Lamination layup and in-plane displacement across the cross-section of a plate.

where u_α^0 denotes the in-plane displacement of any point on mid surface, n_u and n_l represent number of upper and lower layers respectively. S_α^k, T_α^k are the slopes of k-th layer corresponding to upper and lower layers respectively, $\xi_\alpha, \varphi_\alpha$ are the higher order unknown terms, $H(z - z_k^u)$, $H(z - z_k^l)$ are unit step functions and the subscript α represents the co-ordinate directions [α = 1, 2 i.e., x, y in this case] respectively and

$$u_3 = w(x, y) \tag{2}$$

The stress-strain relationship of a lamina, say k^{th} lamina having any fiber orientation with respect to structural axes system (*x-y-z*) may be expressed as:

$$\bar{\sigma} = [\bar{Q}]_k \{\bar{\varepsilon}\} \tag{3}$$

The rigidity matrix $[\bar{Q}]_k$ can be evaluated by material properties and fibre orientation following usual techniques for laminated composites.

Now by utilizing the transverse shear stress free boundary condition at the top and bottom of the plate, $\sigma_{3\alpha|z=\pm h/2} = 0$, the components ξ_α and φ_α could be expressed as:

$$\varphi_\alpha = -\frac{4}{3h^2}\{w,_\alpha + \frac{1}{2}(\sum_{k=0}^{n_u-1} S_\alpha^k + \sum_{k=0}^{n_l-1} T_\alpha^k)\} \tag{4}$$

and

$$\xi_\alpha = -\frac{1}{2}(\sum_{k=0}^{n_u-1} S_\alpha^k - \sum_{k=0}^{n_l-1} T_\alpha^k) \tag{5}$$

Similarly by imposing the transverse shear stress continuity conditions at the layer interfaces the following expressions for S_α^k and T_α^k are obtained as below:

$$S_\alpha^k = a_{\alpha\gamma}^k (w,_\gamma + \psi_\gamma) + b_{\alpha\gamma}^k w,_\gamma \tag{6}$$

and

$$T_\alpha^k = c_{\alpha\gamma}^k (w,_\gamma + \psi_\gamma) + d_{\alpha\gamma}^k w,_\gamma \tag{7}$$

where $a_{\alpha\gamma}^k, b_{\alpha\gamma}^k, c_{\alpha\gamma}^k, d_{\alpha\gamma}^k$ are constants depending on material and geometric properties of individual layers, $w,_\gamma$ is the derivatives of transverse displacement while $\gamma = 1, 2$ and $S_\alpha^0 = \psi_\alpha$ is the rotation of normal at the mid surface about co-ordinate directions.

By using Eqs. (2)-(6) the strain vector can be evaluated as:

$$\{\bar{\varepsilon}\} = [H]\{\varepsilon\} \tag{8}$$

where $\{\bar{\varepsilon}\}$ is the strain field vector (5x1) and $\{\varepsilon\}$ is the strain vector (17×1) at the reference plane (i.e., at the mid plane), [*H*] is the matrix (5×17) consists of terms containing *z* and in order to avoid the difficulties associated with C^1 continuity, the derivatives of *w* with respect to *x* and *y* are expressed in terms of two new variables as follows:

$$w,_1 = \frac{\partial w}{\partial x} = w_1 \qquad and \qquad w,_2 = \frac{\partial w}{\partial y} = w_2 \tag{9}$$

which helps to define all the variables as C^0 continuous.

$$\{\varepsilon\}^T = \{\frac{\partial u_1^0}{\partial x}, \frac{\partial u_2^0}{\partial y}, \frac{\partial u_2^0}{\partial x} + \frac{\partial u_1^0}{\partial y}, \frac{\partial w_1}{\partial x}, \frac{\partial w_2}{\partial y}, \frac{\partial w_2}{\partial x}, \frac{\partial w_1}{\partial y}, \frac{\partial \psi_1}{\partial x}, \frac{\partial \psi_2}{\partial y}, \frac{\partial \psi_2}{\partial x}, \frac{\partial \psi_1}{\partial y}, \psi_1, \psi_2, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, w_1, w_2 \tag{10}$$

The strain displacement relation may be written as below,

$$\{\varepsilon\} = [B]\{\delta\} \tag{11}$$

where $[B]$ is the strain-displacement matrix and $\{\delta\}$ is the element nodal unknown vector.

Thermal strain due to temperature change is given by:

$$\{\varepsilon_{th}\} = \left\{\alpha_x \Delta T,\ \alpha_y \Delta T,\ \alpha_{xy} \Delta T,\ 0,\ 0\right\}^T \tag{12}$$

In which ΔT is the change of temperature/moisture concentration with respect to reference temperature/moisture concentration, $\alpha_x, \alpha_y, \alpha_{xy}$ are the thermal expansion coefficients in the structural axis (*x*-*y*) system. Therefore, the net strain may be written as:

$$\{\bar{\varepsilon}_n\} = \{\bar{\varepsilon}\} - \{\varepsilon_{th}\} \tag{13}$$

For the present study, a nine noded C[0] continuous isoparametric element with seven nodal unknowns (i.e., $u_1, u_2, w, \Psi_1, \Psi_2, w_1, w_2$) are used to develop the proposed finite element model. The generalized displacements included in the present theory can be expressed as follows,

$$u_1 = \sum_{i=1}^{9} N_i u_i, u_2 = \sum_{i=1}^{9} N_i v_i, w = \sum_{i=1}^{9} N_i w_i, \psi_1 = \sum_{i=1}^{9} N_i \psi_{1i}, \psi_2 = \sum_{i=1}^{9} N_i \psi_{2i}, w_1 = \sum_{i=1}^{9} N_i w_{1i}, w_2 = \sum_{i=1}^{9} N_i w_{2i} \tag{14}$$

where N_i are the shape functions for the nine noded isoparametric finite element.

By applying virtual work method one can get,

$$[k]\{\delta\} = \{P\} \tag{15}$$

where $[k]$ is the element stiffness matrix and $\{P\}$ is the element nodal load vector as written below,

$$[k] = \iint [B]^T [D][B] dxdy \tag{16a}$$

where $[B]$ is the strain displacement matrix,

$$[D] = \sum_{k=1}^{n} \int [H]^T [\bar{Q}_k][H] dz \tag{16b}$$

Is the rigidity matrix and n is the number of layers, respectively.

$$\{P\} = \iint [N]^T q\, dxdy \tag{17}$$

where [N] is the shape functions matrix and q is the intensity of transverse load, respectively.

Thermal loading may also be obtained as below,

$$[F] = \iint [B]^T [H]^T [F^N] dxdy \tag{18}$$

where

$$\{F^N\}^T = [N_x^N\ \ N_y^N\ \ N_{xy}^N\ \ 0\ 0\ 0\ ....] \tag{19a}$$

and

$$\{N_x^N \; N_y^N \; N_{xy}^N \}^T = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \{\overline{Q}_{ij}\}_k \{\varepsilon_{th}\} dz \tag{19b}$$

where i, j =1, 2, 6 and $\{\varepsilon_{th}\}$ is the thermal strain components.

It can be observed that the total strain field is always interpolated to a lower order when compared to the thermal strain fields. Hence, thermal strain fields have been consistently reconstituted to the order of in-plane normal strain field to get accurate strains and stresses over the element domain [23]. The following thermal cases are considered:

Case 1
Temperature uniform across the depth

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} .\Delta T \tag{20}$$

where

$$\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix} [Q]_k \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_{12} \end{Bmatrix}_k \tag{21}$$

where α_1, α_2, α_{12} are thermal expansion coefficient in the material axis system and $c = \cos\theta$, $s = \sin\theta$ and θ is the angle between principal material axis and structural axis system.

In this case, $\Delta T = T$, therefore, thermal force

$$\{F\} = \iiint [B]^T [H]^T \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} .T dv \tag{22}$$

Case 2
Temperature varying across the depth, thermal force,

$$\{F\} = \iiint [B]^T [H]^T \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} .[\frac{1}{2}(T_U + T_L) + \frac{z}{h}(T_U - T_L)] dv \tag{23}$$

where, T_U = Temperature at the top surface and T_L = Temperature at the bottom surface.

The element mass matrix can be written as:

$$[m^e] = \sum_{k=1}^{n_u + n_l} \int \rho_k [C]^T [F]^T [F][C] dxdydz \tag{24}$$

where ρ_k is the mass density of the kth layer and [C] is the shape function matrix.

Also the element geometric strain vector may be expressed as:

$$\{\varepsilon_G\} = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \bar{u}}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \bar{v}}{\partial x}\right)^2 \\ \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial y}\right)^2 \\ \left(\frac{\partial \bar{w}}{\partial x}\right)\left(\frac{\partial \bar{w}}{\partial y}\right) + \left(\frac{\partial \bar{u}}{\partial x}\right)\left(\frac{\partial \bar{u}}{\partial y}\right) + \left(\frac{\partial \bar{v}}{\partial x}\right)\left(\frac{\partial \bar{v}}{\partial y}\right) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{\partial \bar{w}}{\partial x} & 0 & \frac{\partial \bar{u}}{\partial x} & 0 & \frac{\partial \bar{v}}{\partial x} & 0 \\ 0 & \frac{\partial \bar{w}}{\partial y} & 0 & \frac{\partial \bar{u}}{\partial y} & 0 & \frac{\partial \bar{v}}{\partial y} \\ \frac{\partial \bar{w}}{\partial y} & \frac{\partial \bar{w}}{\partial x} & \frac{\partial \bar{u}}{\partial y} & \frac{\partial \bar{u}}{\partial x} & \frac{\partial \bar{v}}{\partial y} & \frac{\partial \bar{v}}{\partial x} \end{bmatrix} \begin{Bmatrix} \frac{\partial \bar{w}}{\partial x} \\ \frac{\partial \bar{w}}{\partial y} \\ \frac{\partial \bar{u}}{\partial x} \\ \frac{\partial \bar{u}}{\partial y} \\ \frac{\partial \bar{v}}{\partial x} \\ \frac{\partial \bar{v}}{\partial y} \end{Bmatrix} \tag{25}$$

$$= \frac{1}{2}[A_G][\theta] = \frac{1}{2}[A_G][G][\Delta] \tag{26}$$

With the matrix [G] in the above equation, the geometric stiffness matrix [k_g] of an element can be derived and may be written as:

$$[k_g] = \sum_{i=1}^{n_u+n_l} \iiint [G]^T [S^k][G]dxdydz \tag{27}$$

where [S^k] is the in-plane stress components of the k-th layer .

For the linear thermal buckling problems, the stability equation can be expressed as:

$$([K] - \lambda[K_G])\{\delta\} = 0 \tag{28}$$

In which [K] and [K_G] are the global elastic and geometric stiffness matrix and λ is the critical temperature parameter, respectively.

In the first step, a static problem is solved to calculate thermal stresses at the gauss points of different elements for assumed temperature rise. These thermal stresses are then used to form the matrix [S^k] of the geometric stiffness matrix and the linear thermal buckling problem is solved to calculate the critical buckling temperature. Finally, thermal vibration problem is solved as a Eigen value problem by taking different temperatures just below the calculated critical buckling temperature.

The equation of thermal vibration may be written as:

$$([K'] - \omega^2[M])\{\delta\} = 0 \tag{29}$$

where $[K'] = [K] - \lambda[K_G]$ and, [K'] is the reduced stiffness matrix, [M] is the mass matrix, λ is a fraction of critical buckling temperature and ω is the frequency of thermal vibration, respectively.

A computer program has been written as per the above formulation. The boundary conditions used in different cases are as follows:

1. Simply supported boundary conditions on all sides (SSSS)

$u_1 = u_2 = w = \Psi_1 = w_1 = 0$ at $x = 0$, a

$u_1 = u_2 = w = \Psi_2 = w_2 = 0$ at y= 0, b

2. Clamped boundary conditions on all sides (CCCC)

$u_1 = u_2 = w = \Psi_1 = \Psi_2 = w_1 = w_2 = 0$ at $x = 0$, a and y= 0, b

3 NUMERICAL RESULTS AND DISCUSSIONS

In this section, various numerical examples have been solved on the vibration of laminated composite and sandwich plates under thermal loading. A number of numerical problems with uniform temperature in the plane of the plate have been solved considering different boundary conditions, ply orientations, thickness ratio and aspect ratio. The results obtained by using the proposed finite element method is first validated with the published results and many new results are generated for future reference as there is very few results available in the literature on the present problem based on the refined theories.

3.1 Ten layer simply supported angle-ply laminated composite square plates

Example 1: In this example, 10 layer angle-ply ($\theta = 45^0$) laminated composite plate subjected to uniform temperature rise throughout the thickness has been investigated. The corresponding material properties (dimensionless) for the angle-ply composite plate are as follows:

$$E_1 / E_2 = 15\ ,\ G_{12} = G_{13} = 0.5E_2\ ,\ G_{23} = 0.3356E_2\ ,\ \nu_{12} = 0.3\ ,\ \nu_{23} = 0.49\ ,\ \alpha_1 / \alpha_0 = 0.015\ ,\ \alpha_2 / \alpha_0 = 1.0\ ,\ \rho = 1$$

The normalized critical temperature is defined as follows:

$\lambda_{cr} = \alpha_0\ T$ where T is the critical temperature, α_0 is the normalization factor which is taken as = 10^{-6}., the normalized natural frequency has been defined as $\Omega = \omega h\sqrt{\rho / E_2}$. Normalized natural frequencies and critical temperatures (first mode) obtained for different thickness ratio (a/h) are shown in Table 1. , where a is the planar dimension of the plate in x-direction and h is the total thickness of the plate. The present results (Mesh: 12 x12) are very close to Matsuanaga [22].

Table 1
Normalized natural frequency and critical temperature of 10 layer angle-ply [θ/- θ/../- θ]$_{10}$ square plate with different thickness ratio ($\theta = 45^0$)

a/h	*Ω*		λ_{cr}	
	Present	Matsunaga [22]	Present	Matsunaga [22]
100	0.1605x10^{-2}	0.1595 x10^{-2}	0.1697 x10^{-2}	0.1675 x10^{-2}
50	0.6368 x10^{-2}	0.6342 x10^{-2}	0.6675 x10^{-2}	0.6620 x10^{-2}
20	0.3805 x10^{-1}	0.3811 x10^{-1}	0.3810 x10^{-1}	0.3826 x10^{-1}
10	0.1339	0.1355	0.1172	0.1209
5	0.3961	0.4017	0.2531	0.2656

Example 2: This problem has been solved for 10 layer angle-ply ($\theta = 15^0$) laminated composite plate subjected to uniform temperature rise throughout the thickness has been investigated. The material properties are same as in Example 1. Normalized natural frequencies and critical temperatures obtained for different thickness ratio (a/h) are shown in Table 2. The present results (Mesh: 12x12) are in good agreement with Matsuanaga [22].

Table 2
Normalized natural frequency and critical temperature of 10 layer angle-ply [θ/- θ/.../- θ]$_{10}$ square plate with different thickness ratio ($\theta = 15^0$)

a/h	*Ω*		λ_{cr}	
	Present	Matsunaga [22]	Present	Matsunaga [22]
100	0.1336 x10^{-2}	0.1328 x10^{-2}	0.1123 x10^{-2}	0.1161 x10^{-2}
50	0.5306 x10^{-2}	0.5286 x10^{-2}	0.4419 x10^{-2}	0.4600 x10^{-2}
20	0.3208 x10^{-1}	0.3302 x10^{-1}	0.2633 x10^{-1}	0.2700 x10^{-1}
10	0.1163	0.1163	0.8894 x10^{-1}	0.8899 x10^{-1}
5	0.3588	0.3592	0.2045	0.2124

3.2 Simply supported square sandwich [θ/- θ/.../- θ]₁₀ [Core][- θ/ θ/…/θ]₁₀) plate

In this example, a 21 layer angle-ply ($\theta = 45^0$) sandwich plate has been analyzed with the same temperature loadings mentioned above. The material properties (dimensionless) of individual layers are as given below: Face sheets (total thickness h_f)

$$E_1 / E_2 = 19\ ,\ G_{12} = G_{13} = 0.52E_2\ ,\ G_{23} = 0.338E_2\ ,\ \nu_{12} = 0.32\ ,\ \nu_{23} = 0.49\ ,\ \alpha_1 / \alpha_0 = 0.001\ ,\ \alpha_2 / \alpha_0 = 1.0\ ,\ \rho =$$

Core (thickness h_c)

$$E_1 = 3.2\times10^{-5}\ ,\ \mathrm{E2} = 2.9\times10^{-5}\ ,\ E_3 = 0.4\ ,\ \mathrm{G12} = 2.4\times10^{-3}\ ,\ G_{13} = 0.7.9\times10^{-2}\ ,\ G_{23} = 6.6\times10^{-2}\ ,$$
$$\nu_{12} = 0.99\ ,\ \nu_{23} = \nu_{23} = 3.0\times10^{-5}\ ,\ \alpha_1 = \alpha_2 = 1.36\alpha_0\ ,\ \rho_c = 0.07\rho$$

The Normalized natural frequencies and critical temperatures obtained for different thickness ratio by varying the face thickness (h_f) to total thickness (h) ratio are shown in Table 3. It may be observed in Table 3. that the present results are again closer to the results given by Matsunaga [22].

Table 3
Normalized natural frequency and critical temperature of angle-ply [θ/- θ/.../- θ]₁₀ [Core][- θ/ θ/…/θ]₁₀ sandwich square plate ($\theta = 45^0$) with simply supported boundary conditions with different thickness ratio (a/h)

h_f/h	a/h	Ω		λ_{cr}	
		Present	Matsunaga [22]	Present	Matsunaga [22]
0.3	100	0.2463 x10^{-2}	0.2426 x10^{-2}	0.5297 x10^{-2}	0.5130 x10^{-2}
	50	0.9631 x10^{-2}	0.9509 x10^{-2}	0.2024 x10^{-1}	0.1973 x10^{-1}
	20	0.5291 x10^{-1}	0.5261 x10^{-1}	0.9773 x10^{-1}	0.9664 x10^{-1}
	10	0.1580	0.1587	0.2178	0.2198
	5	0.3858	0.3895	0.3295	0.3306
0.2	100	0.2482 x10^{-2}	0.2462 x10^{-2}	0.5808 x10^{-2}	0.5805 x10^{-2}
	50	0.9758 x10^{-2}	0.9692 x10^{-2}	0.2282 x10^{-1}	0.2252 x10^{-1}
	20	0.5505 x10^{-1}	0.5487 x10^{-1}	0.1161	0.1154
	10	0.1714	0.1718	0.2813	0.2826
	5	0.4303	0.4325	0.4425	0.4471
0.1	100	0.2326 x10^{-2}	0.2322 x10^{-2}	0.6576 x10^{-2}	0.6358 x10^{-2}
	50	0.9204 x10^{-2}	0.9201 x10^{-2}	0.2571 x10^{-1}	0.2570 x10^{-1}
	20	0.5408 x10^{-1}	0.5410 x10^{-1}	0.1418	0.1419
	10	0.1825	0.1826	0.4026	0.4033
	5	0.5005	0.5003	0.7551	0.7542

3.3 Simply supported 1 layer orthotropic (0^0) and 3 layer orthotropic ($0^0/90^0/0^0$) plates

In this section, simply supported 1 layer (0^0) and 3 layer orthotropic plate ($0^0/90^0/0^0$) subjected to uniform temperature loading conditions has been analyzed. The Normalized natural frequencies and critical temperatures obtained for different thickness ratio are presented in Table 4. All these are new results.

3.4 Simply supported square sandwich ([$0^0/90^0/0^0/\ldots]_{10}$[Core][$90^0/0^0/90^0/\ldots/0^0]_{10}$) plate

In this example, a 21 layer sandwich plate has been analyzed with the same temperature loadings mentioned above. The material properties (dimensionless) of individual layers are as given below:

Face sheets (total thickness h_f)

$$E_1 / E_2 = 19\ ,\ G_{12} = G_{13} = 0.52E_2\ ,\ G_{23} = 0.338E_2\ ,\ \nu_{12} = 0.32\ ,\ \nu_{23} = 0.49\ ,\ \alpha_1 / \alpha_0 = 0.001\ ,\ \alpha_2 / \alpha_0 = 1.0\ ,\ \rho =$$

Core (thickness h_c)

$E_1 = 3.2\times10^{-5}$, $E2 = 2.9\times10^{-5}$, $E_3 = 0.4$, $G12 = 2.4\times10^{-3}$, $G_{13} = 0.7.9\times10^{-2}$, $G_{23} = 6.6\times10^{-2}$,
$\nu_{12} = 0.99$, $\nu_{23} = \nu_{23} = 3.0\times10^{-5}$, $\alpha_1 = \alpha_2 = 1.36\alpha_0$, $\rho_c = 0.07\rho$

Table 4
Normalized natural frequency and critical temperature of 1 layer cross-ply (0^0) and 3 layer cross-ply ($0^0/90^0/0^0$) laminated composite square plates

a/h	0^0		$0^0/90^0/0^0$	
	Ω	λ_{cr}	*Ω*	λ_{cr}
100	0.1231 x10^{-2}	0.7491 x10^{-3}	0.1231 x10^{-2}	0.9946 x10^{-3}
50	0.4898 x10^{-2}	0.2960 x10^{-2}	0.4893 x10^{-2}	0.3927 x10^{-2}
20	0.2968 x10^{-1}	0.1736 x10^{-1}	0.2951 x10^{-1}	0.2290 x10^{-1}
10	0.1079	0.5746 x10^{-1}	0.1061	0.7417 x10^{-1}
5	0.3354	0.1414	0.3254	0.1743

Table 5
Normalized natural frequency and critical temperature of 21 layer cross-ply $[0^0/90^0/ .../90^0]_{10}$[Core]$[90^0/90^0/ .../0^0]_{10}$ sandwich square plate with simply supported boundary conditions at different thickness ratio

h_f/h	*a/h*							
	100		50		20		10	
	Ω	λ_{cr}	*Ω*	λ_{cr}	*Ω*	λ_{cr}	*Ω*	λ_{cr}
0.05	0.5207 x10^{-3}	0.3995 x10^{-2}	0.2053 x10^{-2}	0.1551 x10^{-1}	0.1257 x10^{-1}	0.9260 x10^{-1}	0.4717 x10^{-1}	0.3211
0.1	0.7148 x10^{-3}	0.3804 x10^{-2}	0.2817 x10^{-2}	0.1447 x10^{-1}	0.1697 x10^{-1}	0.8540 x10^{-1}	0.6083 x10^{-1}	0.2715
0.15	0.8482 x10^{-3}	0.3583 x10^{-2}	0.8377 x10^{-2}	0.1398 x10^{-1}	0.1991 x10^{-1}	0.7866 x10^{-1}	0.6889 x10^{-1}	0.2338
0.2	0.9515 x10^{-3}	0.3388 x10^{-2}	0.3756 x10^{-2}	0.1319 x10^{-1}	0.2206 x10^{-1}	0.7266 x10^{-1}	0.7433 x10^{-1}	0.2050
0.3	0.1101 x10^{-2}	0.9870 x10^{-2}	0.4339 x10^{-2}	0.1176 x10^{-1}	0.2506 x10^{-1}	0.6264 x10^{-1}	0.8143 x10^{-1}	0.1648

The Normalized natural frequencies and critical temperatures obtained for different thickness ratio by varying the face thickness (h_f) to total thickness (h) ratio and using the present finite element are shown in Table 5. All these results are new.

3.5 Effect of aspect ratio, thickness ratio on cross-ply $[90^0/-90^0/ .../-90^0]$

In this example, a 10 layer plate has been analyzed with the same temperature loadings mentioned above. The material properties (dimensionless) of individual layers are as given below:

$E_1 / E_2 = 15$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.3356E_2$, $\nu_{12} = 0.3$, $\nu_{23} = 0.49$, $\alpha_1 / \alpha_0 = 0.015$, $\alpha_2 / \alpha_0 = 1.0$, $\rho = 1$

The Normalized natural frequencies and critical temperatures obtained with different aspect ratio (*a*/*b*) using the present finite element approach are shown in Table 6. All these results are presented for the first time.

3.6 Clamped cross-ply ($0^0/90^0/0^0/90^0$) anti-symmetric laminate

The present problem of composite square plate ($0^0/90^0/0^0/90^0$) has been solved with the uniform temperature rise throughout the thickness. The material properties (dimensionless) of individual layers are as given below:

$E_1 / E_2 = 15$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.3356E_2$, $\nu_{12} = 0.3$, $\nu_{23} = 0.49$, $\alpha_1 / \alpha_0 = 0.015$, $\alpha_2 / \alpha_0 = 1.0$, $\rho = 1$

Example 1: In this section, anti-symmetric cross-ply ($0^0/90^0/0^0/90^0$) with clamped boundary conditions subjected to uniform temperature rise throughout the thickness has been considered. The normalized natural frequencies with different thickness ratio (*a*/*h*) are plotted in Fig. 2.

Table 6
Normalized natural frequency and critical temperature of 10 layer cross-ply [90^0/-90^0/ .../-90^0] laminated composite square plate with simply supported boundary conditions with different aspect ratio (*a*/*b*)

a/*h*	*a*/*b*	Ω	λ_{cr}
100	1	0. 1232 x10^{-2}	0.7496 x10^{-3}
	1.5	0.2591 x10^{-2}	0.1690 x10^{-2}
	2	0.4503 x10^{-2}	0.3026 x10^{-2}
50	1	0.4899 x10^{-2}	0.2960 x10^{-2}
	1.5	0.1025 x10^{-1}	0.6580 x10^{-2}
	2	0.1769 x10^{-1}	0.1153 x10^{-1}
20	1	0.2268 x10^{-1}	0.1736 x10^{-1}
	1.5	0.6002 x10^{-1}	0.3592 x10^{-1}
	2	0.9915 x10^{-1}	0.5764 x10^{-1}
10	1	0.1079	0.5745 x10^{-1}
	1.5	0.2006	0.1018
	2	0.3059	0.1416
5	1	0.3354	0.1413
	1.5	0.5488	0.2019
	2	0.7720	0.2440

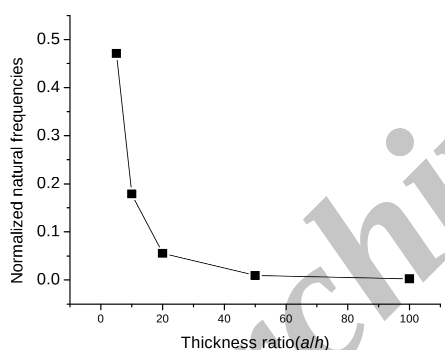

Fig. 2
Variation of natural frequencies of laminated composite clamped plates ($0^0/90^0/0^0/90^0$) with different thickness ratio (*a*/*h*).

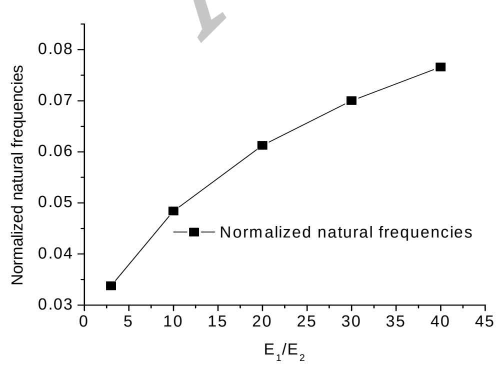

Fig. 3
Variation of natural frequencies of laminated composite clamped plates ($0^0/90^0/0^0/90^0$) with different modular ratio (E_1/E_2).

Example 2: In this example anti-symmetric cross-ply ($0^0/90^0/0^0/90^0$) with clamped boundary conditions subjected to uniform temperature rise throughout the thickness has been considered. The Normalized natural frequencies with different modular ratio (E_1/E_2) with thickness ratio (a/h =20) are plotted in Fig. 3.

3.7 Cross-ply sandwich square plate with simply supported at all the edges

In this section, five layer cross-ply sandwich plate ($0^0/90^0$/Core/$90^0/0^0$) with different boundary conditions for h_f/h =0.3 subjected to uniform temperature rise throughout the thickness has been investigated. Results of normalized frequencies and critical temperatures are presented in Table 7. All the results are new.

Table 7
Normalized Natural frequency (Ω) and critical temperature (λ_{cr}) of a simply supported laminated square sandwich plate ($0^0/90^0$/Core/$90^0/0^0$) for different thickness ratio (a/h)

Boundary Condition	Parameter	a/h				
		100	50	20	10	5
SSSS	Ω	$0.1099\text{x}10^{-2}$	$0.4340\text{x}10^{-2}$	$0.2502\text{x}10^{-1}$	$0.8102\ \text{x}10^{-1}$	0.2114
	λ_{cr}	$0.3026\text{x}10^{-2}$	$0.1177\text{x}10^{-1}$	$0.6243\text{x}10^{-1}$	0.1631	0.2777
SSSC	Ω	$0.1367\text{x}10^{-2}$	$0.5346\text{x}10^{-2}$	$0.2938\text{x}10^{-1}$	$0.8867\ \text{x}10^{-1}$	0.2205
	λ_{cr}	$0.4288\text{x}10^{-2}$	$0.1644\text{x}10^{-1}$	$0.8080\text{x}10^{-1}$	0.1874	0.2910
SSCC	Ω	$0.1748\text{x}10^{-2}$	$0.6732\text{x}10^{-2}$	$0.3462\text{x}10^{-1}$	$0.9733\text{x}10^{-1}$	0.2328
	λ_{cr}	$0.6740\text{x}10^{-2}$	$0.2515\text{x}10^{-1}$	0.1090	0.2194	0.3124
SSSF	Ω	$0.8064\text{x}10^{-3}$	$0.3182\text{x}10^{-2}$	$0.3108\text{x}10^{-1}$	$0.5978\ \text{x}10^{-1}$	0.1571
	λ_{cr}	$0.3285\text{x}10^{-2}$	$0.1277\ \text{x}10^{-1}$	$0.6795\text{x}10^{-1}$	0.1791	0.3086
SSFF	Ω	$0.7856\text{x}10^{-3}$	$0.3099\text{x}10^{-2}$	$0.1792\text{x}10^{-1}$	$0.5833\ \text{x}10^{-1}$	0.1527
	λ_{cr}	$0.3178\text{x}10^{-2}$	$0.1236\text{x}10^{-1}$	$0.6606\text{x}10^{-1}$	0.1748	0.2999

4 CONCLUSIONS

Natural frequencies of laminated composite and sandwich plates subjected to thermal loading have been analyzed by using the higher order zigzag theory for the first time in this paper. Since very few results are available in the literature based on a refined theory like the present one, the accuracy of natural frequencies and critical temperatures of laminated composite and sandwich plates subjected to thermal stresses has been studied by using present zigzag theory in detail. It has been established through various numerical examples that the present higher order zigzag theory can accurately predict the natural frequencies of general laminated composite and sandwich plates under thermal loading. As such the present model may be recommended for generating further results in the field.

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