Analytical Solution for Response of Piezoelectric Cylinder Under Electro–Thermo–Mechanical Fields

S. Golabi[*], J. Jafari Fesharaki

Department of Mechanical Engineering, University of Kashan, Kashan, Iran

ABSTRACT

This paper presents an analytical solution for response of a piezoelectric hollow cylinder under two-dimensional electro thermo mechanical fields. The solution is based on a direct method and the Navier equations were solved using the complex Fourier series. The advantage of this method is its generality and from mathematical point of view, any type of the thermo mechanical and electrical boundary conditions can be considered without any restrictions. The thermo mechanical and electrical displacements are assumed that vary in radial and circumferential directions. Finally, three examples were considered to confirm the results and investigate the effect of in-phase and opposite-phase electro thermo mechanical boundary loads on two-dimensional electro thermo mechanical behavior of piezoelectric hollow cylinder. The results are compared with the previous work and FEM analysis. The main result of this study is that, by applying a proper distribution of thermal, electrical and mechanical fields, the distributions of electric and mechanical displacement, thermal and mechanical stresses can be controlled.

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1 INTRODUCTION

PIEZOELECTRIC materials have attracted widespread attention due to their direct and inverse effects on many industrials including aerospace, automobile and marine as a sensor or actuator [1] in the last decade. But the combinations of mechanical, electrical and thermal loads have confronting effects on piezoelectric materials which has been the subjects of many researches in these years. Ying and Zhifei [2] presented an exact solution for a long thick walled double layers piezo-thermo-elastic hollow cylinder. They investigated the effects of some coupled loading on piezo-thermo-elastic cylinder. Haojiang et al. [3] presented a general solution for coupled equations for piezoelectric media. An exact solutions for a piezoelectric plane beam subjected to electromechanical loads was investigated by Zhang et al. [4]. The thermo piezoelectric behavior of a functionally graded hollow cylinder due to electric, thermal and mechanical loads is investigated by Khoshgoftar et al. [5]. They reported by applying specific thermo mechanical boundary conditions, the distribution of electro mechanical field and displacements in the cylinder can be controlled. By using the double Fourier sinusoidal series expansions Xu and Zhou studies the elasticity solution of a transversely isotropic rectangular plates with variable thickness [6]. Jafari Fesharaki et al. [7] investigated the two-dimensional solution for electro mechanical behavior of FGPM Hollow Cylinder. Tarn and Jiann [8] depicted an exact solutions for a piezoelectric circular tube or bar under torsion, extension, shearing, pressuring, uniform electric loading and temperature changes. Zhang et al. [9] presented an exact solution of coupled electro thermo elastic behavior of piezoelectric laminates. Dumir et al. [10] investigated the solution of piezo elastic

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[*] Corresponding author. Tel.: +98 361 5919.

E-mail address: golabi-s@kashanu.ac.ir (S. Golabi).

simply-supported orthotropic circular cylindrical panel under cylindrical bending. Galic and Horgan [11, 12] developed an analytical solution for a radially polarized piezoelectric hollow cylinder in rotating and static conditions. Dai and Wang [13] depicted the responses of piezoelectric hollow spheres and cylinders due to transient electro thermo mechanical loads. Guan and He [14] presented the two-dimensional analysis of piezoelectric/piezomagnetic and elastic media. The effective elastic properties of a piezoelectric composite cylinders with radially polarized cylinder is investigated by Gu et al. [15]. They showed that, by increasing the piezoelectric cylinder constants, the effective elastic properties can be enhanced or reduced. By using the infinitesimal theory, Dai et al. [16], depicted an analytical solution for electro magneto thermo mechanic behavior of a functionally graded hollow cylinder. Dai and Wang [17] presented an exact expression for the dynamic responses of a piezoelectric hollow cylinder in an axial magnetic field. Yee and Moon [18] presented a plane thermal stress analysis of an orthotropic cylinder subjected to an arbitrary transient asymmetric temperature distribution.

In this paper, the complex Fourier series is employed to present an exact electro-thermo-elastic solution of a piezoelectric hollow cylinder subjected to two-dimensional steady state thermal, electrical and mechanical loads. All electro thermo mechanical fields are assumed to vary in radial and circumferential directions which do not have any limitations to handle the general types of thermal, mechanical and electrical boundary condition. With a direct solution method, the Navier equations in terms of displacements are derived and solved analytically. After the solution is completed, three examples were considered to confirm the results. In the first example, the effect of two phases -mechanical and thermal field -are investigated and the results are compared with the given data in previous works. In the second example, the effect of two phases electro mechanical fields were depicted and the result was confirmed with the FEM analysis. Finally, in the third example, four general boundary conditions were considered to investigate and control the three phases electro thermo mechanical response of a piezoelectric hollow cylinder with the combination of any types of boundary conditions.

2   BASIC FORMULATION OF THE PROBLEM

Consider a thick piezoelectric hollow cylinder with inside and outside radius of "*a*" and "*b*" respectively subjected to two-dimensional electro thermo mechanical loads (Fig. 1). The cylinder is polarized in radial direction.
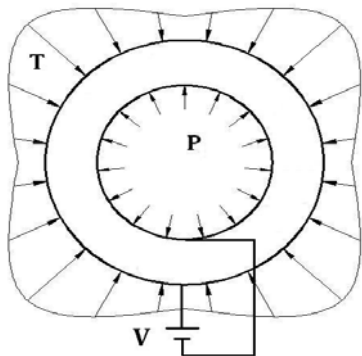
Fig. 1
Hollow cylinder subjected to two-dimensional electro thermo mechanical loads.

The relations between strain and displacement components in cylindrical coordinate system are:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}, \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \qquad \varepsilon_{r\theta} = \frac{1}{2}(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}).$$ (1)

"u" and *"v"* are the displacement components in the radial and circumferential directions, respectively. The constitutive relations for a piezoelectric cylinder with thermal effect are [19]:

$$\sigma_{rr} = c_{11}\varepsilon_{rr} + c_{12}\varepsilon_{\theta\theta} + g_{11}\frac{\partial\varphi}{\partial r} - d_{11}T(r,\theta)$$

$$\sigma_{\theta\theta} = c_{21}\varepsilon_{rr} + c_{22}\varepsilon_{\theta\theta} + g_{21}\frac{\partial\varphi}{\partial r} - d_{21}T(r,\theta)$$

$$\sigma_{r\theta} = 2c_{31}\varepsilon_{r\theta} + g_{31}\frac{1}{r}\frac{\partial\varphi}{\partial\theta} \tag{2}$$

$$D_r = g_{11}\varepsilon_{rr} + g_{21}\varepsilon_{\theta\theta} - h_{11}\frac{\partial\varphi}{\partial r} + b_{11}T(r,\theta)$$

$$D_\theta = 2g_{31}\varepsilon_{r\theta} - h_{21}\frac{1}{r}\frac{\partial\varphi}{\partial\theta} + b_{21}T(r,\theta)$$

where σ_{ij} and ε_{ij} $(i, j = r, \theta)$ are stress and strain tensors, respectively. D_i $(i=r, \theta)$ are electric and displacements in radial and circumferential directions and φ is electric potential. c_{ij}, g_{ij}, h_{ij}, and b_{ij} $(i, j=1, 2, 3)$ are elastic, piezoelectric, dielectric and pyroelectric coefficients respectively and finally d_{ij} $(i, j=1, 2)$ are thermal modules. The equilibrium equations in cylindrical coordinates, in radial and circumferential directions, irrespective of body forces, inertia terms and electrostatic charge equation, are expressed as [19]:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2\sigma_{r\theta}}{r} = 0 \tag{3}$$

$$\frac{\partial D_r}{\partial r} + \frac{1}{r}\frac{\partial D_\theta}{\partial\theta} + \frac{D_r}{r} = 0.$$

Substituting Eq. (1) into (2) and then into equilibrium Eqs. (3), three coupled governing differential equations for two dimensional electro thermo mechanical problem in cylindrical coordinates are obtained:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{w_1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{w_2}{r^2}u + w_3\frac{\partial^2 v}{\partial r\partial\theta} + \frac{w_4}{r^2}\frac{\partial v}{\partial\theta} + w_5\frac{\partial^2\varphi}{\partial r^2} + \frac{w_6}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{w_7}{r}\frac{\partial\varphi}{\partial r} = w_8\frac{\partial T}{\partial r} + w_9\frac{T}{r}$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{w_{10}}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{w_{11}}{r^2}v + \frac{w_{12}}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{w_{13}}{r^2}\frac{\partial u}{\partial\theta} + \frac{w_{14}}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} + \frac{w_{15}}{r^2}\frac{\partial\varphi}{\partial\theta} = \frac{w_{16}}{r}\frac{\partial T}{\partial\theta} \tag{4}$$

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{w_{17}}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + w_{18}\frac{\partial^2 u}{\partial r^2} + \frac{w_{19}}{r}\frac{\partial u}{\partial r} + \frac{w_{20}}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{w_{21}}{r}\frac{\partial^2 v}{\partial r\partial\theta} + \frac{w_{22}}{r^2}\frac{\partial v}{\partial\theta} = \frac{w_{23}}{r}\frac{\partial T}{\partial\theta} + w_{24}\frac{\partial T}{\partial r} + w_{24}\frac{T}{r}$$

where constants w_1 to w_{24} are given in the Appendix.

3 HEAT CONDUCTION PROBLEM

Before solving the governing Eqs. (4), the steady-state heat conduction equations for a two-dimensional problem with no heat generation in cylindrical coordinates and the thermal boundary conditions for a hollow cylinder are considered as [20]:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} = 0, \qquad a \le r \le b, \qquad -\pi \le \theta \le +\pi. \tag{5}$$

$$m_{11}T(a,\theta) + m_{12}\frac{\partial T(a,\theta)}{\partial r} = f_1(\theta)$$

$$m_{21}T(b,\theta) + m_{22}\frac{\partial T(b,\theta)}{\partial r} = f_2(\theta)$$

(6)

where m_{ij} (i, j=1,2) are the thermal constant parameters related to the conduction and convection coefficients. The functions $f_1(\theta)$ and $f_2(\theta)$ are known on the inner and outer surfaces of the cylinder, respectively. Because of $T(r, \theta)$ is a periodic function of θ, the complex Fourier series may be used for temperature function as [21]:

$$T(r,\theta) = \sum_{n=-\infty}^{\infty} T_n(r)e^{in\theta}$$

(7)

where $T_n(r)$ is the coefficient of the complex Fourier series of $T(r,\theta)$:

$$T_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} T(r,\theta)e^{-in\theta}d\theta$$

(8)

Substituting Eq. (7) into Eq.(5) yields:

$$T_n'' + \frac{1}{r}T_n' - \frac{n^2}{r^2}T_n = 0$$

(9)

The solutions of Euler Eq. (9) may be considered in the form of:

$$T_n(r) = Q_n.r^{\beta_n}$$

(10)

Substituting Eq. (10) into Eq. (9) yields:

$$\beta_{n1,2} = \pm n$$

(11)

Thus

$$T_n(r) = Q_{n1}r^{\beta_{n1}} + Q_{n2}r^{\beta_{n2}}$$

(12)

By substituting Eq. (12) into Eq. (7), the temperature function is obtained:

$$T(r,\theta) = \sum_{n=-\infty}^{\infty} (Q_{n1}r^{\beta_{n1}} + Q_{n2}r^{\beta_{n2}})\, e^{in\theta}$$

(13)

To determine the constants Q_{n1} and Q_{n2}, one should note that the right hand sides of Eq. (6) are the complex Fourier series coefficients of left hand one, so substituting boundary conditions (6) into Eq. (13) yields:

$$(m_{11}a^{\beta_{n1}} + m_{12}\beta_{n1}a^{\beta_{n1}-1}).Q_{n1} + (m_{11}a^{\beta_{n2}} + m_{12}\beta_{n2}a^{\beta_{n2}-1}).Q_{n2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\theta)e^{-in\theta}d\theta$$

(14)

$$(m_{21}b^{\beta_{n1}} + m_{22}\beta_{n1}b^{\beta_{n1}-1}).Q_{n1} + (m_{21}b^{\beta_{n2}} + m_{22}\beta_{n2}b^{\beta_{n2}-1}).Q_{n2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_2(\theta)e^{-in\theta}d\theta$$

By using the Cramer's method to solve the system of algebraic Eqs. (14), the constants Q_{n1} and Q_{n2} is expressed as:

$$Q_{n1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[(m_{21} b^{\beta_{n2}} + m_{22} \beta_{n2} b^{\beta_{n2}-1}).f_1(\theta) - (m_{11} a^{\beta_{n2}} + m_{12} \beta_{n2} a^{\beta_{n2}-1}).f_2(\theta) \right].e^{-in\theta} d\theta / (L_1 - L_2)$$

$$Q_{n2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[(m_{11} a^{\beta_{n1}} + m_{12} \beta_{n1} a^{\beta_{n1}-1}).f_2(\theta) - (m_{21} b^{\beta_{n1}} + m_{22} \beta_{n1} b^{\beta_{n1}-1}).f_1(\theta) \right].e^{-in\theta} d\theta / (L_1 - L_2)$$

(15)

where

$$L_1 = (m_{11} a^{\beta_{n1}} + m_{12} \beta_{n1} a^{\beta_{n1}-1}).(m_{21} b^{\beta_{n2}} + m_{22} \beta_{n2} b^{\beta_{n2}-1})$$

$$L_2 = (m_{11} a^{\beta_{n2}} + m_{12} \beta_{n2} a^{\beta_{n2}-1}).(m_{21} b^{\beta_{n1}} + m_{22} \beta_{n1} b^{\beta_{n1}-1})$$

(16)

4 SOLUTION OF THE PROBLEM

To solve the Eq.(4) consider the complex Fourier series for displacements $u(r, \theta)$ and $v(r, \theta)$ and electric potential $\varphi(r, \theta)$ as:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} u_n(r)e^{in\theta} \quad , \quad v(r,\theta) = \sum_{n=-\infty}^{\infty} v_n(r)e^{in\theta} \quad , \quad \varphi(r,\theta) = \sum_{n=-\infty}^{\infty} \varphi_n(r)e^{in\theta}$$

(17)

where u_n (r), v_n (r) and φ_n (r) are the coefficients of the complex Fourier series of $u_n(r, \theta)$, $v_n(r, \theta)$ and $\varphi_n(r, \theta)$ respectively, and are:

$$u_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r,\theta)e^{-in\theta} d\theta \quad , \quad v_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(r,\theta)e^{-in\theta} d\theta \quad , \quad \varphi_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r,\theta)e^{-in\theta} d\theta$$

(18)

Substituting Eq. (17) into Eq. (4) and using Eq. (7) one obtains:

$$u_n'' + \frac{u_n'}{r} - \frac{w_1}{r^2} n^2 u_n + \frac{w_2}{r^2} u_n + \frac{w_3}{r} inv_n' + \frac{w_4}{r^2} inv_n + w_5 \varphi_n'' - \frac{w_6}{r^2} n^2 \phi_n + \frac{w_7}{r} \phi_n' = w_8 T' + \frac{w_9}{r} T$$

(19)

$$v_n'' + \frac{1}{r} v_n' - \frac{w_{10}}{r^2} n^2 v_n + \frac{w_{11}}{r^2} v_n + \frac{w_{12}}{r} inu_n' + \frac{w_{13}}{r^2} inu_n + \frac{w_{14}}{r} in\phi_n' + \frac{w_{15}}{r^2} in\phi_n = \frac{w_{16}}{r} inT$$

(20)

$$\phi_n'' + \frac{1}{r} \phi_n' - \frac{w_{17}}{r^2} n^2 \phi_n + w_{18} u_n'' + \frac{w_{19}}{r} u_n' - \frac{w_{20}}{r^2} n^2 u_n + \frac{w_{21}}{r} inv_n' + \frac{w_{22}}{r^2} inv_n = \frac{w_{23}}{r} inT + w_{24} T' + \frac{w_{24}}{r} T$$

(21)

Eqs. (19) to (21) are a system of ordinary differential equations with non-constant coefficients that have the general and particular solutions. The general solution of the this system are assumed as:

$$u_n^g(r) = Ar^\xi, \quad v_n^g(r) = Br^\xi, \quad \varphi_n^g(r) = Cr^\xi$$

(22)

Substituting Eqs. (22) into Eqs. (19), (20) and (21) and neglecting the right hand sides of equations for general solution lead to:

$$[\xi(\xi-1)+\xi-w_1n^2+w_2]A+[w_3\xi+w_4]inB+[w_5\xi(\xi-1)-w_6n^2+w_7\xi]C=0$$
$$[w_{12}\xi+w_{13}]inA+[\xi(\xi-1)+\xi-w_{10}n^2+w_{11}]B+[w_{14}\xi+w_{15}]inC=0 \tag{23}$$
$$[w_{18}\xi(\xi-1)+w_{19}\xi-w_{20}n^2]A+[w_{21}\xi+w_{22}]inB+[\xi(\xi-1)+\xi-w_{17}n^2]C=0$$

For obtaining the nontrivial solution of the system of algebraic Eqs. (23), the determinant of coefficients should be equal to zero. So six roots, ξ_{n1} to ξ_{n6} for system of equations are achieved and the general solutions are expressed as:

$$u_n^g(r)=\sum_{j=1}^{6}A_{nj}r^{\xi_{nj}} \qquad , \qquad v_n^g(r)=\sum_{j=1}^{6}M_{nj}A_{nj}r^{\xi_{nj}} \qquad , \qquad \varphi_n^g(r)=\sum_{j=1}^{6}N_{nj}A_{nj}r^{\xi_{nj}} \tag{24}$$

where M_{nj} is the relation between constant A_{nj} and B_{nj}, and N_{nj} is the relation between constants A_{nj} and C_{nj} obtained from Eq. (23):

$$M_{nj}=\frac{[\xi_{nj}(\xi_{nj}-1)+\xi_{nj}-w_1n^2+w_2].[w_{14}\xi_{nj}+w_{15}]in-[w_5\xi_{nj}(\xi_{nj}-1)-w_6n^2+w_7\xi_{nj}].[w_{12}\xi_{nj}+w_{13}]in}{[w_5\xi_{nj}(\xi_{nj}-1)-w_6n^2+w_7\xi_{nj}].[\xi_{nj}(\xi_{nj}-1)+\xi_{nj}-w_{10}n^2+w_{11}]-[w_3\xi_{nj}+w_4]in.[w_{14}\xi_{nj}+w_{15}]in} \tag{25}$$

$$N_{nj}=\frac{[w_3\xi_{nj}+w_4]in.[w_{12}\xi_{nj}+w_{13}]in-[\xi_{nj}(\xi_{nj}-1)+\xi_{nj}-w_1n^2+w_2].[\xi_{nj}(\xi_{nj}-1)+\xi_{nj}-w_{10}n^2+w_{11}}{[w_5\xi_{nj}(\xi_{nj}-1)-w_6n^2+w_7\xi_{nj}].[\xi_{nj}(\xi_{nj}-1)+\xi_{nj}-w_{10}n^2+w_{11}]-[w_3\xi_{nj}+w_4]in.[w_{14}\xi_{nj}+w_{15}]i} \tag{26}$$
$$j=1,2,3,4,5,6$$

The particular solution of Eqs. (19), (20) and (21) are assumed as:

$$u_n^p(r)=P_{n1}r^{\beta_{n1}+1}+P_{n2}r^{\beta_{n2}+1} \qquad , \qquad v_n^p(r)=R_{n1}r^{\beta_{n1}+1}+R_{n2}r^{\beta_{n2}+1} \qquad , \qquad \varphi_n^p(r)=S_{n1}r^{\beta_{n1}+1}+S_{n2}r^{\beta_{n2}+1} \tag{27}$$

Using Eq. (12), substituting Eq.(27) into Eqs. (19) to (21), and equating the coefficients of identical powers, the following systems for power $r^{\beta_{n1}+1}$ and $r^{\beta_{n2}+1}$ are obtained:

$$\begin{cases} X_1.P_{n1}+X_2.R_{n1}+X_3.S_{n1}=Y_1.Q_{n1} \\ X_7.P_{n1}+X_8.R_{n1}+X_9.S_{n1}=Y_3.Q_{n1} \\ X_{13}.P_{n1}+X_{14}.R_{n1}+X_{15}.S_{n1}=Y_5.Q_{n1} \end{cases} \tag{28}$$

$$\begin{cases} X_4.P_{n2}+X_5.R_{n2}+X_6.S_{n2}=Y_2.Q_{n2} \\ X_{10}.P_{n2}+X_{11}.R_{n2}+X_{12}.S_{n2}=Y_4.Q_{n2} \\ X_{16}.P_{n2}+X_{17}.R_{n2}+X_{18}.S_{n2}=Y_6.Q_{n2} \end{cases} \tag{29}$$

where, the constants X_1 to X_{18} and Y_1 to Y_6 are given in the appendix. Eqs. (28) and (29) are a system of algebraic equations and using the Cramer's method, the solutions for unknown constants P_{n1}, P_{n2}, R_{n1}, R_{n2}, S_{n1} and S_{n2} can be obtained.

Summation of the general and particular solution for $u_n(r)$, $v_n(r)$ and $\varphi_n(r)$, leads to the complete solutions:

$$u_n(r) = u_n^g(r) + u_n^p(r) = \sum_{j=1}^{6} A_{nj} r^{\xi_{nj}} + \sum_{j=1}^{2} P_{nj} \cdot r^{\beta_{nj}+1}$$

$$v_n(r) = v_n^g(r) + v_n^p(r) = \sum_{j=1}^{6} M_{nj} A_{nj} r^{\xi_{nj}} + \sum_{j=1}^{2} R_{nj} \cdot r^{\beta_{nj}+1} \tag{30}$$

$$\varphi_n(r) = \varphi_n^g(r) + \varphi_n^p(r) = \sum_{j=1}^{6} N_{nj} A_{nj} r^{\xi_{nj}} + \sum_{j=1}^{2} S_{nj} \cdot r^{\beta_{nj}+1}$$

If $n = 0$, since Eq. (20) is independent of Eqs. (19) and (21), the coefficients M_{nj} and N_{nj} are not defined. Thus, Eqs. (19) to (21) are reduced to:

$$u_0'' + \frac{u_0'}{r} + \frac{w_2}{r^2} u_0 + w_5 \varphi_0'' + \frac{w_7}{r} \phi_0' = w_8 T' + \frac{w_9}{r} T \tag{31}$$

$$v_0'' + \frac{1}{r} v_0' + \frac{w_{11}}{r^2} v_0 = 0 \tag{32}$$

$$\phi_0'' + \frac{1}{r} \phi_0' + w_{18} u_0'' + \frac{w_{19}}{r} u_0' = w_{24} T' + \frac{w_{24}}{r} T \tag{33}$$

where zero subscript denoted for n=0. Eqs. (31) and (33) are a system of ordinary differential equations and have general and particular solutions. The general solution is assumed as:

$$u_0(r) = A_0 r^{\xi_0}, \qquad \varphi_0(r) = C_0 r^{\xi_0}. \tag{34}$$

Substituting Eq. (34) into Eqs. (31) and (33) yield:

$$[\xi_0(\xi_0 - 1) + \xi_0 + w_2] A_0 + [w_5 \xi_0(\xi_0 - 1) + w_7 \xi_0] C_0 = 0$$
$$[w_{18} \xi_0(\xi_0 - 1) + w_{19} \xi_0] A_0 + [\xi_0(\xi_0 - 1) + \xi_0] C_0 = 0 \tag{35}$$

To obtain the nontrivial solution of Eq. (35), the determinant of coefficients should be equal to zero. So the four roots, ξ_{01} to ξ_{04}, are achieved and the general solutions are:

$$u_0^g(r) = \sum_{j=1}^{4} A_{0j} r^{\xi_{0j}}, \qquad \varphi_0^g(r) = \sum_{j=1}^{4} N_{0j} A_{0j} r^{\xi_{0j}}. \tag{36}$$

where N_{0j} is the relation between constants A_0 and C_0 and is obtained from the first Eq. (35) as:

$$N_{0j} = \frac{-[\xi_{0j}(\xi_{0j} - 1) + \xi_{0j} + w_2]}{[w_5 \xi_{0j}(\xi_{0j} - 1) + w_7 \xi_{0j}]} \tag{37}$$

For $n = 0$, Eq. (32) is a decoupled ordinary differential equation and the solution of this equation is considered in the form of:

$$v_0^g = A_{05} r + \frac{A_{06}}{r} \tag{38}$$

The particular solution of Eqs. (31) and (33) are assumed as:

$$u_0^p(r) = P_{01} r^{\beta_{01}+1} + P_{02} r^{\beta_{02}+1} \qquad , \qquad \varphi_0^p(r) = S_{01} r^{\beta 0+1} + S_{02} r^{\beta_{02}+1} \tag{39}$$

For $n = 0$ in Eq. (12), substituting Eq. (39) into Eqs. (31) and (33), and equating the coefficients of the identical powers, the following systems for power $r^{\beta_{n1}+1}$ and $r^{\beta_{n2}+1}$ are obtained:

$$\begin{cases} X_{19}.P_{01} + X_{20}.S_{01} = Y_7.Q_{01} \\ X_{23}.P_{01} + X_{24}.S_{01} = Y_9.Q_{01} \end{cases} \tag{40}$$

$$\begin{cases} X_{21}.P_{02} + X_{22}.S_{02} = Y_8.Q_{02} \\ X_{25}.P_{02} + X_{26}.S_{02} = Y_{10}.Q_{02} \end{cases} \tag{41}$$

where the constants X_{19} to X_{26} and Y_7 *to* Y_{10} are given in the appendix. Eqs. (40) and (41) are a system of algebraic equations and using the Cramer's method, the solutions for unknown constants $P_{01},\ P_{02},\ S_{01}$ and S_{02} can be obtained.

Considering the addition of general and particular solutions for every value of *"n"* and using Eq.(17), the complete solution for $u(r, \theta)$, $v(r, \theta)$ and $\varphi(r, \theta)$ are expressed as:

$$u(r,\theta) = \sum_{j=1}^{4} A_{0j} r^{\xi_{0j}} + \sum_{j=1}^{2} P_{0j}.r^{\beta_{0j}+1} + \sum_{n=-\infty,\,n\neq0}^{\infty}\left[\sum_{j=1}^{6} A_{nj} r^{\xi_{nj}} + \sum_{j=1}^{2} P_{nj}.r^{\beta_{nj}+1}\right] e^{in\theta} \tag{42}$$

$$v(r,\theta) = A_{05}.r + \frac{A_{06}}{r} + \sum_{n=-\infty,\,n\neq0}^{\infty}\left[\sum_{j=1}^{6} M_{nj}.A_{nj} r^{\xi_{nj}} + \sum_{j=1}^{2} R_{nj}.r^{\beta_{nj}+1}\right] e^{in\theta} \tag{43}$$

$$\varphi(r,\theta) = \sum_{j=1}^{4} N_{0j}.A_{0j} r^{\xi_{0j}} + \sum_{j=1}^{2} S_{0j}.r^{\beta_{0j}+1} + \sum_{n=-\infty,\,n\neq0}^{\infty}\left[\sum_{j=1}^{6} N_{nj}.A_{nj} r^{\xi_{nj}} + \sum_{j=1}^{2} S_{nj}.r^{\beta_{nj}+1}\right] e^{in\theta} \tag{44}$$

Substituting Eqs. (42), (43) and (44) into Eq. (1), the strains are obtained as:

$$\varepsilon_{rr} = \sum_{j=1}^{4} A_{0j}\xi_{0j} r^{\xi_{0j}-1} + \sum_{j=1}^{2} P_{0j}.(\beta_{0j}+1).r^{\beta_{oj}} + \sum_{n=-\infty,\,n\neq0}^{\infty}\left[\sum_{j=1}^{6} A_{nj}\xi_{nj} r^{\xi_{nj}-1} + \sum_{j=1}^{2} P_{nj}.(\beta_{nj}+1).r^{\beta_{nj}}\right] e^{in\theta} \tag{45}$$

$$\varepsilon_{\theta\theta} = \sum_{j=1}^{4} A_{0j} r^{\xi_{0j}-1} + \sum_{j=1}^{2} P_{0j}.r^{\beta_{oj}} + \sum_{n=-\infty,\,n\neq0}^{\infty}\left[\sum_{j=1}^{6}(in.M_{nj}+1)A_{nj}.r^{\xi_{nj}-1} + \sum_{j=1}^{2}(in.R_{nj}+P_{nj}).r^{\beta_{nj}}\right] e^{in\theta} \tag{46}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left\{\sum_{n=-\infty,\,n\neq0}^{\infty}\left[\sum_{j=1}^{6}(in+M_{nj}.\xi_{nj}-M_{nj}).A_{nj} r^{\xi_{nj}-1} + \sum_{j=1}^{2}(in.P_{nj}+\beta_{nj}.R_{nj}).r^{\beta_{nj}}\right] e^{in\theta}\right\} \tag{47}$$

Substituting Eqs.(45), (46) and (47) into Eq.(2) and utilizing Eq.(13) and (44), the stress components are obtained as:

$$\sigma_{rr} = \sum_{j=1}^{4}(c_{11}.\xi_{oj} + c_{12} + g_{11}.N_{0j}.\xi_{0j}).A_{0j}.r^{\xi_{0j}-1} + \sum_{j=1}^{2}\begin{bmatrix} c_{11}.P_{0j}.(\beta_{oj}+1)+c_{12}.P_{0j}+ \\ g_{11}.S_{0j}.(\beta_{0j}+1) \end{bmatrix}.r^{\beta_{0j}}$$

$$+ \sum_{n=-\infty,\,n\neq0}^{\infty}\left\{\begin{array}{l}\sum_{j=1}^{6}(c_{11}.\xi_{nj} + c_{12}.M_{nj}.in + c_{12} + g_{11}.N_{nj}.\xi_{nj}).A_{nj}.r^{\xi_{nj}-1} \\ + \sum_{j=1}^{2}\left[c_{11}.P_{bj}.(\beta_{nj}+1) + c_{12}.R_{nj} + c_{12}.P_{nj} + g_{11}S_{nj}.(\beta_{nj}+1)\right].r^{\beta_{nj}}\end{array}\right\}.e^{in\theta} - d_{11}\sum_{n=-\infty}^{\infty}\left(\sum_{j=1}^{2} Q_{nj}.r^{\beta_{nj}}\right).e^{in\theta} \tag{48}$$

$$\sigma_{\theta\theta} = \sum_{j=1}^{4}(c_{21}.\xi_{oj}+c_{22}+g_{21}.N_{0j}.\xi_{0j}).A_{0j}.r^{\xi_{0j}-1} + \sum_{j=1}^{2}\begin{bmatrix}c_{21}.P_{0j}.(\beta_{oj}+1)+c_{22}.P_{0j}\\+g_{21}.S_{0j}.(\beta_{0j}+1)\end{bmatrix}.r^{\beta_{0j}}$$

$$+ \sum_{n=-\infty,\,n\neq0}^{\infty}\left\{\begin{matrix}\sum_{j=1}^{6}(c_{21}.\xi_{nj}+c_{22}.M_{nj}.in+c_{22}+g_{21}.N_{nj}.\xi_{nj}).A_{nj}.r^{\xi_{nj}-1}\\+\sum_{j=1}^{2}\left[c_{21}.P_{bj}.(\beta_{nj}+1)+c_{22}.R_{nj}+c_{22}.P_{nj}+g_{21}S_{nj}.(\beta_{nj}+1)\right].r^{\beta_{nj}}\end{matrix}\right\}.e^{in\theta}-d_{21}\sum_{n=-\infty}^{\infty}\left(\sum_{j=1}^{2}Q_{nj}.r^{\beta_{nj}}\right).e^{in\theta} \tag{49}$$

$$\sigma_{r\theta} = \sum_{n=-\infty,\,n\neq0}^{\infty}\begin{bmatrix}\sum_{j=1}^{6}(c_{31}.in.+c_{31}.M_{nj}.\xi_{nj}-c_{31}.M_{nj}+g_{31}.N_{nj}.in).A_{nj}.r^{\xi_{nj}-1}\\+\sum_{j=1}^{2}(c_{31}.P_{nj}.in+c_{31}.R_{nj}.\beta_{nj}+g_{31}.S_{nj}.in).r^{\beta_{nj}}\end{bmatrix}.e^{in\theta} \tag{50}$$

And electrical displacement components in radial and circumferential directions are:

$$D_r = \sum_{j=1}^{4}(g_{11}.\xi_{oj}+g_{21}-h_{11}.N_{0j}.\xi_{0j}).A_{0j}.r^{\xi_{0j}-1} + \sum_{j=1}^{2}\begin{bmatrix}g_{11}.P_{0j}.(\beta_{oj}+1)+g_{21}.P_{0j}\\-h_{11}.S_{0j}.(\beta_{0j}+1)\end{bmatrix}.r^{\beta_{0j}}$$

$$+ \sum_{n=-\infty,\,n\neq0}^{\infty}\left\{\begin{matrix}\sum_{j=1}^{6}(g_{11}.\xi_{nj}+g_{21}.M_{nj}.in+g_{21}-h_{11}.N_{nj}.\xi_{nj}).A_{nj}.r^{\xi_{nj}-1}\\+\sum_{j=1}^{2}\left[g_{11}.P_{bj}.(\beta_{nj}+1)+g_{21}.R_{nj}+g_{21}.P_{nj}-h_{11}S_{nj}.(\beta_{nj}+1)\right].r^{\beta_{nj}}\end{matrix}\right\}.e^{in\theta}+b_{11}\sum_{n=-\infty}^{\infty}\left(\sum_{j=1}^{2}Q_{nj}.r^{\beta_{nj}}\right).e^{in\theta} \tag{51}$$

$$D_\theta = \sum_{n=-\infty,\,n\neq0}^{\infty}\begin{bmatrix}\sum_{j=1}^{6}(g_{31}.in.+g_{31}.M_{nj}.\xi_{nj}-g_{31}.M_{nj}-h_{21}.N_{nj}.in).A_{nj}.r^{\xi_{nj}-1}\\+\sum_{j=1}^{2}(g_{31}.P_{nj}.in+g_{31}.R_{nj}.\beta_{nj}-h_{21}.S_{nj}.in).r^{\beta_{nj}}\end{bmatrix}.e^{in\theta}+b_{21}\sum_{n=-\infty}^{\infty}\left(\sum_{j=1}^{2}Q_{nj}.r^{\beta_{nj}}\right).e^{in\theta} \tag{52}$$

It is recalled that Eqs. (42) - (52) contain six unknown constants A_{nj} ($j=1,...,6$) and therefore to evaluate these constants, six boundary conditions from displacements, stresses, or combinations are required. Expanding the given boundary conditions in complex fourier series gives:

$$Z_j(\theta) = \sum_{n=-\infty}^{\infty}Z_j(n)e^{in\theta} \qquad j=1,...,6 \tag{53}$$

where

$$Z_j(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}Z_j(\theta)e^{-in\theta}d\theta \qquad j=1,...,6 \tag{54}$$

Using Eqs. (53) and (54) and substituting the six boundary conditions into Eqs. (42) to (52), all unknown constants A_{nj} are calculated.

5 FUNCTIONALLY GRADED MATERIAL FORMULATION

Consider that the functionally graded material proposed in here are radially graded and the material properties are the function of *r*:

$$X_{ii} = X_{ii}^0 r^\beta, \qquad (i, j = 1,\ 2,\ 3) \tag{55}$$

where the superscript zero denotes the corresponding value at the outer surface of the functionally graded piezoelectric material hollow cylinder, and β is the power-law indices of the material in-homogeneity. "X" is the material property such as elastic coefficient, electric constant etc. Considering the presented above solution, the stresses along the radial and circumferential direction are:

$$
\sigma_{rr} = c_{11}^0 r^\beta \left[\sum_{j=1}^{4} A_{0j}\xi_{0j} r^{\xi_{0j}-1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} A_{nj}\xi_{nj} r^{\xi_{nj}-1} \right)e^{in\theta} \right] + c_{12}^0 r^{\beta-1}\left[\sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} N_{nj}A_{nj} r^{\xi_{nj}} in \right)e^{in\theta} \right]
$$
$$
+ c_{12}^0 r^{\beta-1}\left[\sum_{j=1}^{4} A_{0j} r^{\xi_{0j}} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} A_{nj} r^{\xi_{nj}} \right)e^{in\theta} \right] + g_{11}^0 r^\beta \left[\sum_{j=1}^{4} R_{0j}A_{0j}\xi_{0j} r^{\xi_{0j}-1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} M_{nj}A_{nj}\xi_{nj} r^{\xi_{nj}-1} \right)e^{in\theta} \right] \tag{56}
$$

$$
\sigma_{\theta\theta} = c_{21}^0 r^\beta \left[\sum_{j=1}^{4} A_{0j}\xi_{0j} r^{\xi_{0j}-1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} A_{nj}\xi_{nj} r^{\xi_{nj}-1} \right)e^{in\theta} \right] + c_{22}^0 r^{\beta-1}\left[\sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} N_{nj}A_{nj} r^{\xi_{nj}} in \right)e^{in\theta} \right]
$$
$$
+ c_{22}^0 r^{\beta-1}\left[\sum_{j=1}^{4} A_{0j} r^{\xi_{0j}} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} A_{nj} r^{\xi_{nj}} \right)e^{in\theta} \right] + g_{21}^0 r^\beta \left[\sum_{j=1}^{4} R_{0j}A_{0j}\xi_{0j} r^{\xi_{0j}-1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} M_{nj}A_{nj}\xi_{nj} r^{\xi_{nj}-1} \right)e^{in\theta} \right] \tag{57}
$$

$$
\sigma_{r\theta} = c_{31}^0 r^{\beta-1} \left[\sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} A_{nj} in r^{\xi_{nj}} \right)e^{in\theta} \right] + c_{31}^0 r^\beta \left[\sum_{j=5}^{6} \beta_{0j}\xi_{0j} r^{\xi_{0j}-1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} N_{nj}A_{nj}\xi_{nj} r^{\xi_{nj}-1} \right)e^{in\theta} \right]
$$
$$
- c_{31}^0 r^{\beta-1} \left[\sum_{j=5}^{6} \beta_{0j} r^{\xi_{0j}} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} N_{nj}A_{nj} r^{\xi_{nj}} \right)e^{in\theta} \right] + g_{31}^0 r^{\beta-1} \left[\sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{6} M_{nj}A_{nj} in r^{\xi_{nj}} \right)e^{in\theta} \right] \tag{58}
$$

And the relations for electrical displacements along the radial and circumferential direction in cylinder are presented as:

$$
D_r = g_{11}^0 r^\beta \left[\begin{array}{l} \sum_{j=1}^{4} A_{0j}\xi_{0j}r^{\xi_{0j}-1} + \\ \sum_{n=-\infty,n\neq0}^{\infty}(\sum_{j=1}^{6}A_{nj}\xi_{nj}r^{\xi_{nj}-1})e^{in\theta} \end{array} \right] + g_{21}^0 r^{\beta-1}\left[\sum_{n=-\infty,n\neq0}^{\infty}(\sum_{j=1}^{6}N_{nj}A_{nj}r^{\xi_{nj}}in)e^{in\theta} \right]
$$

$$
+g_{21}^0 r^{\beta-1}\left[\begin{array}{l} \sum_{j=1}^{4} A_{0j}r^{\xi_{0j}} + \\ \sum_{n=-\infty,n\neq0}^{\infty}(\sum_{j=1}^{6}A_{nj}r^{\xi_{nj}})e^{in\theta} \end{array} \right] - h_{41}^0 r^\beta \left[\begin{array}{l} \sum_{j=1}^{4} R_{0j}A_{0j}\xi_{0j}r^{\xi_{0j}-1} + \\ \sum_{n=-\infty,n\neq0}^{\infty}(\sum_{j=1}^{6}M_{nj}A_{nj}\xi_{nj}r^{\xi_{nj}-1})e^{in\theta} \end{array} \right]
$$

(59)

$$
D_\theta = g_{31}^0 r^{\beta-1}\left[\sum_{n=-\infty,n\neq0}^{\infty}(\sum_{j=1}^{6}A_{nj}inr^{\xi_{nj}})e^{in\theta} \right] + g_{31}^0 r^\beta \left[\begin{array}{l} \sum_{j=5}^{6} \beta_{0j}\xi_{0j}r^{\xi_{0j}-1} + \\ \sum_{n=-\infty,n\neq0}^{\infty}(\sum_{j=1}^{6}N_{nj}A_{nj}\xi_{nj}r^{\xi_{nj}-1})e^{in\theta} \end{array} \right]
$$

$$
-g_{31}^0 r^{\beta-1}\left[\begin{array}{l} \sum_{j=5}^{6} \beta_{0j}r^{\xi_{0j}} + \\ \sum_{n=-\infty,n\neq0}^{\infty}(\sum_{j=1}^{6}N_{nj}A_{nj}r^{\xi_{nj}})e^{in\theta} \end{array} \right] - h_{51}^0 r^{\beta-1}\left[\sum_{n=-\infty,n\neq0}^{\infty}(\sum_{j=1}^{6}M_{nj}A_{nj}inr^{\xi_{nj}})e^{in\theta} \right]
$$

(60)

where the unknown constant A_{nj} $(j=1,...,6)$ are specified from boundary conditions.

6 RESULTS AND DISCUSSION

To examine the proposed solution, three different examples are considered. In the first example, to validate our solutions, a thick piezoelectric hollow cylinder under electro thermo mechanical load is considered. The inner and outer radiuses of cylinder are *a=0.25m* and *b=1 m* respectively and the material is considered to be PZT-4. Also the boundary conditions are considered similar to those proposed in reference [21] and are obtained as:

$$
\sigma_{rr}(a,\theta)=-1, \quad \sigma_{rr}(b,\theta)=0, \quad T(a,\theta)=0, \quad T(b,\theta)=100, \quad \varphi(a,\theta)=1, \quad \varphi(b,\theta)=0. \tag{61}
$$

In reference [21], an analytical solution for electro thermo piezoelectric hollow cylinder is presented. All fields are considered to be symmetric along the circumference of the cylinder and the stresses, temperature and electric field are varied across the thickness. The results in that paper are presented for two materials, PZT-4 and Batio3. Also the results are presented for three ratio of outer and inner radius of cylinder as R= 0.25, 0.5, 0.75.

Fig.2 shows the stresses and electric field across the thickness of the piezoelectric hollow cylinder with the material and boundary conditions proposed in [21]. Fig. 3 shows the results reported in reference [21]. From these two figures, by considering the red line in Fig. 3 and for R=0.25 (0.25<R<1), it can be seen that the boundary conditions are satisfied in Fig. 2. Considering the boundary conditions and material properties, in addition, the curves in Figs. 2 and 3 also show very good agreement between stress and electric field with those reported in [21].

(a)

(b)
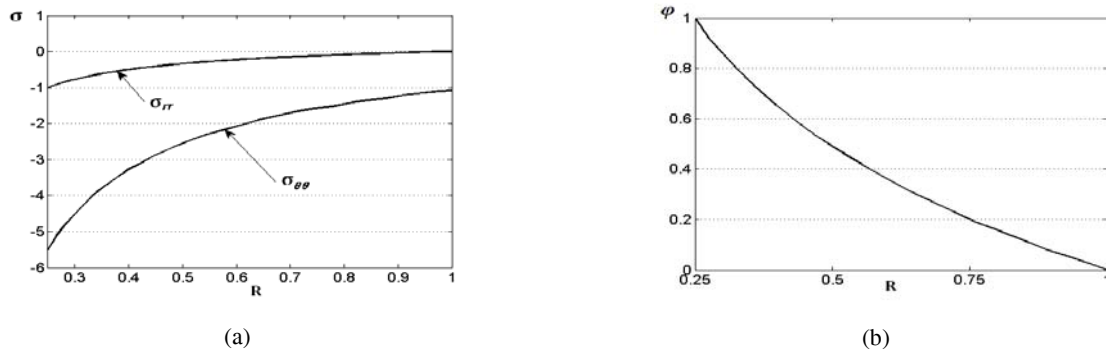
Fig. 2
Stresses and electric field distributions due to electro thermo mechanical load.


(a)

(b)

(c)
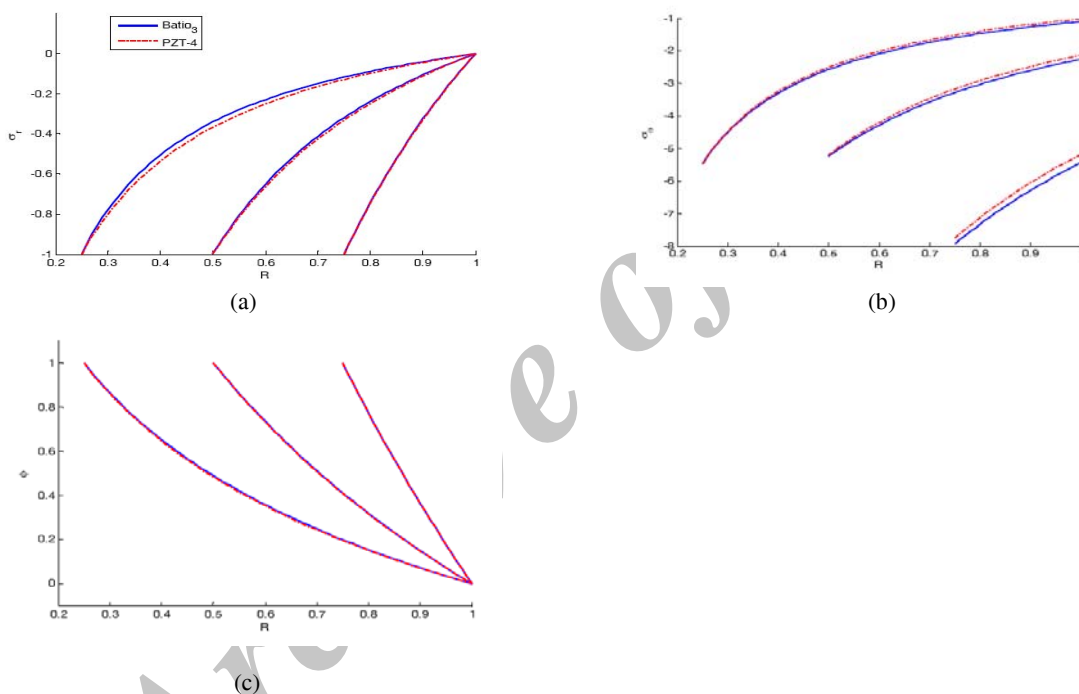
Fig. 3
Stresses and electric field distributions due to electro thermo mechanical load [21].

In the second example, the behavior of hollow cylinder subjected to two dimensional electro mechanical loads is considered. The results from analytical method are compared to the results from FEM software. The mechanical and electrical boundary conditions are shown in Fig. 4 and considered as:

$$
\begin{aligned}
&\sigma_{rr}(a,\theta) = 1\cos 2\theta \; MPa, &&u(b,\theta) = 0, \\
&\sigma_{r\theta}(a,\theta) = 0 &&v(b,\theta) = 0, \\
&\varphi(a,\theta) = 10\sin 2\theta \; W/A, &&\varphi(b,\theta) = 0.
\end{aligned}
\tag{62}
$$

The material chosen for this example is PZT-4 and the material constants are [22]:

$$c_{11} = c_{22} = 115 \ GPa, \quad c_{12} = c_{21} = 74 \ GPa, \quad c_{31} = 25.6 \ GPa, \quad g_{11} = 15.1 \ \text{C/m}^2,$$

$$g_{21} = -5.2 \ \text{C/m}^2, \quad g_{31} = 12.7 \ \text{C/m}^2, \quad h_{11} = h_{21} = 6.5 \times 10^{-9} \ \text{C}^2/\text{Nm}^2, \quad (63)$$

$$d_{11} = 2.65 \times 10^5 \ \text{N/m}^2\text{K}, \quad d_{21} = 1.97 \times 10^5 \ \text{N/m}^2\text{K}, \quad b_{11} = b_{21} = 5.4 \times 10^{-5} \ \text{C/m}^2 K.$$
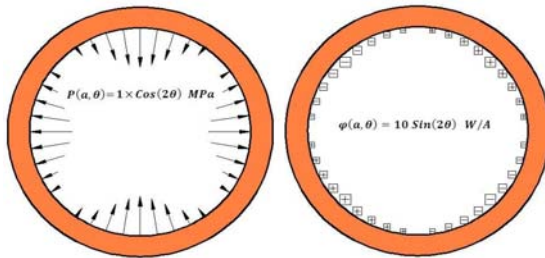

Fig. 4
Piezoelectric hollow cylinder subjected to two-dimensional electro mechanical fields.

Fig.5 shows the radial and shear stress in the thickness of piezoelectric hollow cylinder due to the given boundary conditions. To compare the results, Fig.6 shows the results in the radius in midpoint of the thickness of cylinder (R= 1.1 m) for various degrees. It can be seen that the results show good agreement between the analytical and FEM results.
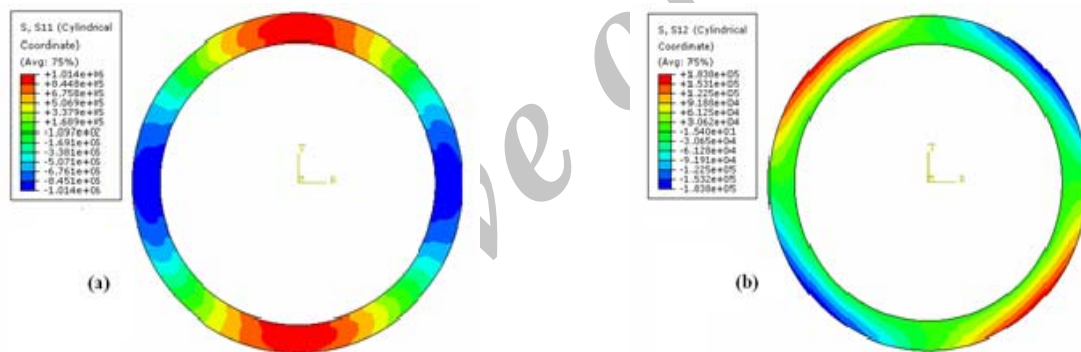
Fig. 5
Radial (a) and shear (b) stresses in the thickness of piezoelectric hollow cylinder.
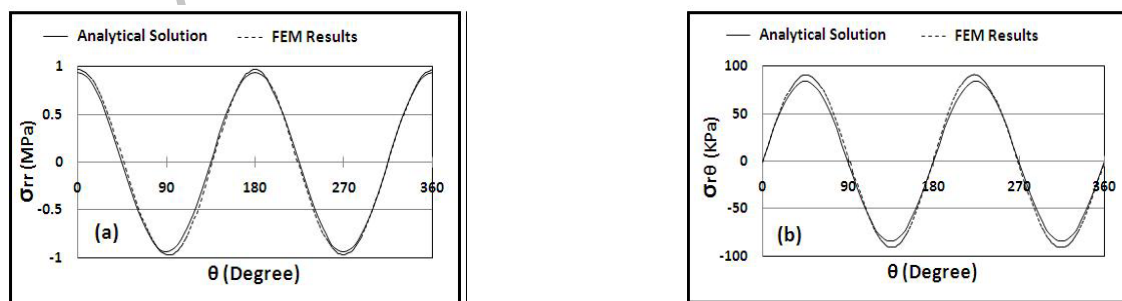
Fig. 6
Radial and shear stresses from FEM and analytical solution in R=1.1.

In the third example, the same thick-walled cylinder is subjected to two-dimensional three phase electric potential, thermal and mechanical loads. To investigate the effects of combining the in-phase and opposite-phase of electro thermo mechanical loads to control the electrical and mechanical parameters, four various boundary conditions are considered as:

Case 1: All boundary conditions are in-phase

$$
\begin{aligned}
T(a,\theta) &= \cos 2\theta \ ^\circ C, & \varphi(a,\theta) &= 50 \cos 2\theta \ W/A, \\
T(b,\theta) &= 0, & \varphi(b,\theta) &= 0, \\
\sigma_{rr}(a,\theta) &= 10 \cos 2\theta \ MPa, & u(b,\theta) &= 0, \\
\sigma_{r\theta}(a,\theta) &= 0 & v(b,\theta) &= 0.
\end{aligned}
\tag{64}
$$

Case 2: Electric potential is in opposite-phase

$$
\begin{aligned}
T(a,\theta) &= \cos 2\theta \ ^\circ C, & \varphi(a,\theta) &= 50 \sin 2\theta \ W/A, \\
T(b,\theta) &= 0, & \varphi(b,\theta) &= 0, \\
\sigma_{rr}(a,\theta) &= 10 \cos 2\theta \ MPa, & u(b,\theta) &= 0, \\
\sigma_{r\theta}(a,\theta) &= 0 & v(b,\theta) &= 0.
\end{aligned}
\tag{65}
$$

Case 3: Pressure is in opposite-phase

$$
\begin{aligned}
T(a,\theta) &= \cos 2\theta \ ^\circ C, & \varphi(a,\theta) &= 50 \cos 2\theta \ W/A, \\
T(b,\theta) &= 0, & \varphi(b,\theta) &= 0, \\
\sigma_{rr}(a,\theta) &= 10 \sin 2\theta \ MPa, & u(b,\theta) &= 0, \\
\sigma_{r\theta}(a,\theta) &= 0 & v(b,\theta) &= 0.
\end{aligned}
\tag{66}
$$

Case 4: Temperature is in opposite-phase

$$
\begin{aligned}
T(a,\theta) &= \sin 2\theta \ ^\circ C, & \varphi(a,\theta) &= 50 \cos 2\theta \ W/A, \\
T(b,\theta) &= 0, & \varphi(b,\theta) &= 0, \\
\sigma_{rr}(a,\theta) &= 10 \cos 2\theta \ MPa, & u(b,\theta) &= 0, \\
\sigma_{r\theta}(a,\theta) &= 0 & v(b,\theta) &= 0.
\end{aligned}
\tag{67}
$$

Fig.7 shows the temperature distribution across the wall thickness of cylinder along the radius and circumferential directions for the proposed cases. The temperature at the inner and outer radius of cylinder satisfies the given thermal boundary conditions. Since the thermal equation is solved separately, the electro mechanical loads have no effect on temperature distributions across the thickness of cylinder and only in case 4, due to different thermal boundary conditions from other cases, the temperature has different shape from the others.
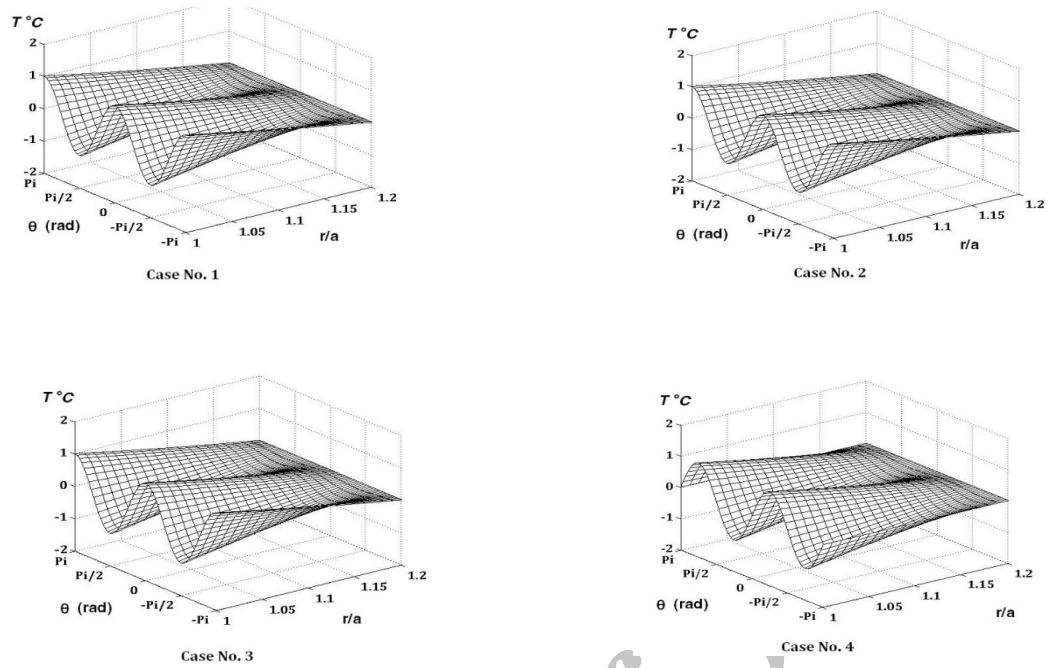
Fig. 7
Temperature distributions due to electro thermo mechanical load.
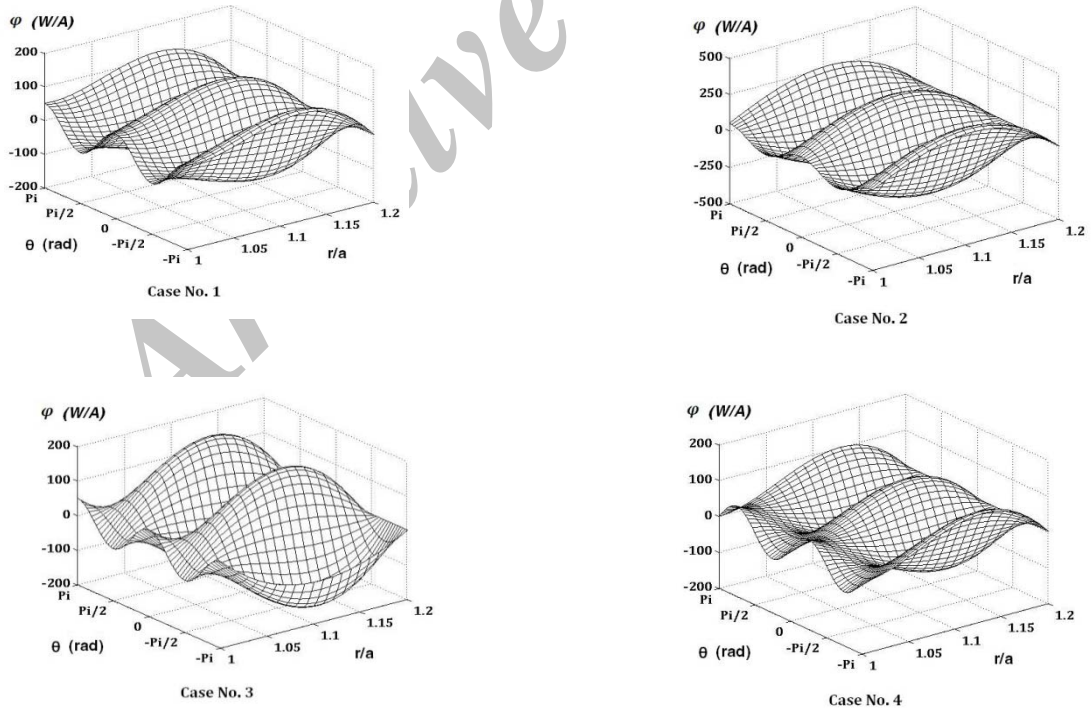

Fig.8
Electric potential distributions due to electro thermo mechanical load.

Fig.8 shows the electric potential distribution across the cylinder wall thickness. It can be seen that, due to the given boundary conditions, the electric potential follow the pattern of the electric potential distribution at the internal surface of the cylinder. It is also perceived that the variation between the phases of electro thermo mechanical loads effectively alters the electric potential along the radial and circumferential direction. From figure 5 one can see that if the electric potential at inner boundary be in opposite-phase with the other boundary loads, the maximum value of electric potential occurs in the middle of thickness. If the mechanical loads be in opposite phase with other boundary conditions, the electric potential vary sharply along the circumferential direction but the values of pick points have inconsiderable difference from the values of pick point in other cases. The radial, shear and circumferential stresses are shown in Figs. 9 to 11 respectively. Obviously the stresses in these figures satisfy the proposed boundary conditions for each case. It can be seen from Figs. 9 and 11 that the various phases of loads have inconsiderable effects on values of radial and circumferential stresses. But in case 3 from Figs. 9 and 11, we can find that the shape of radial and circumferential stresses across the thickness of cylinder are different from other cases due to different boundary conditions. As one can see in Fig. 10, if the pressure be in opposite-phase with other conditions, the shape and values of shear stress changes.
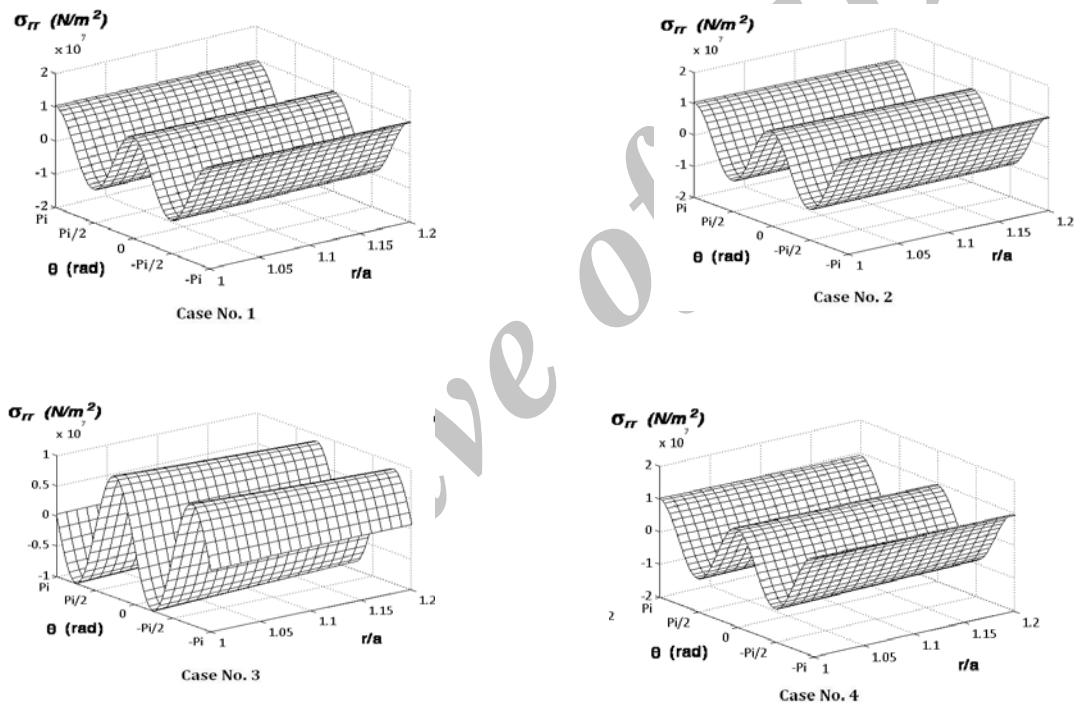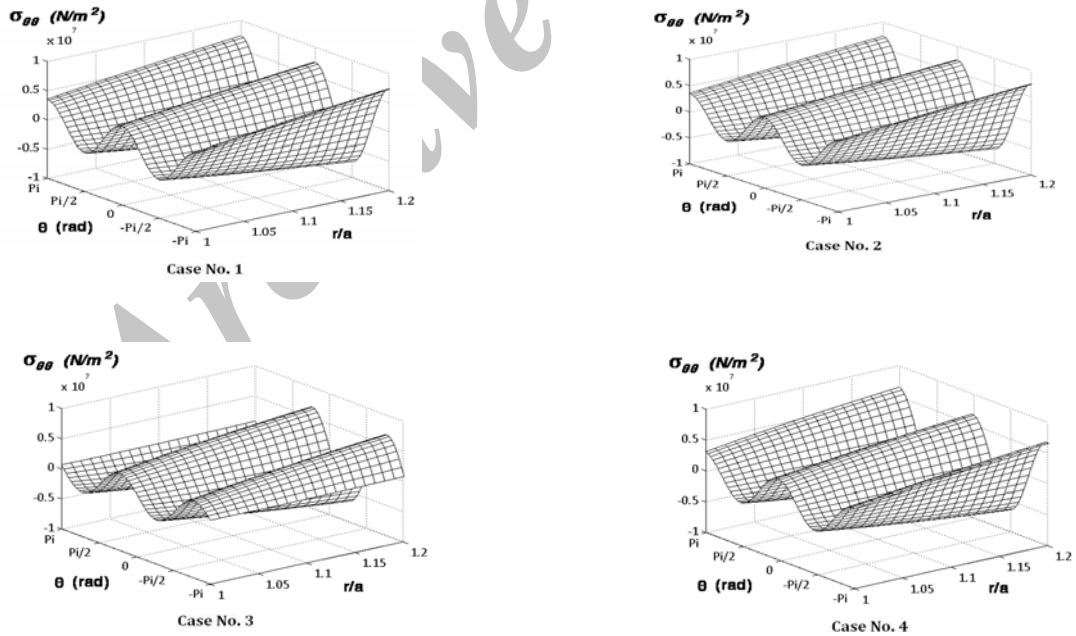

Fig. 9
Radial stress distributions due to electro thermo mechanical load.
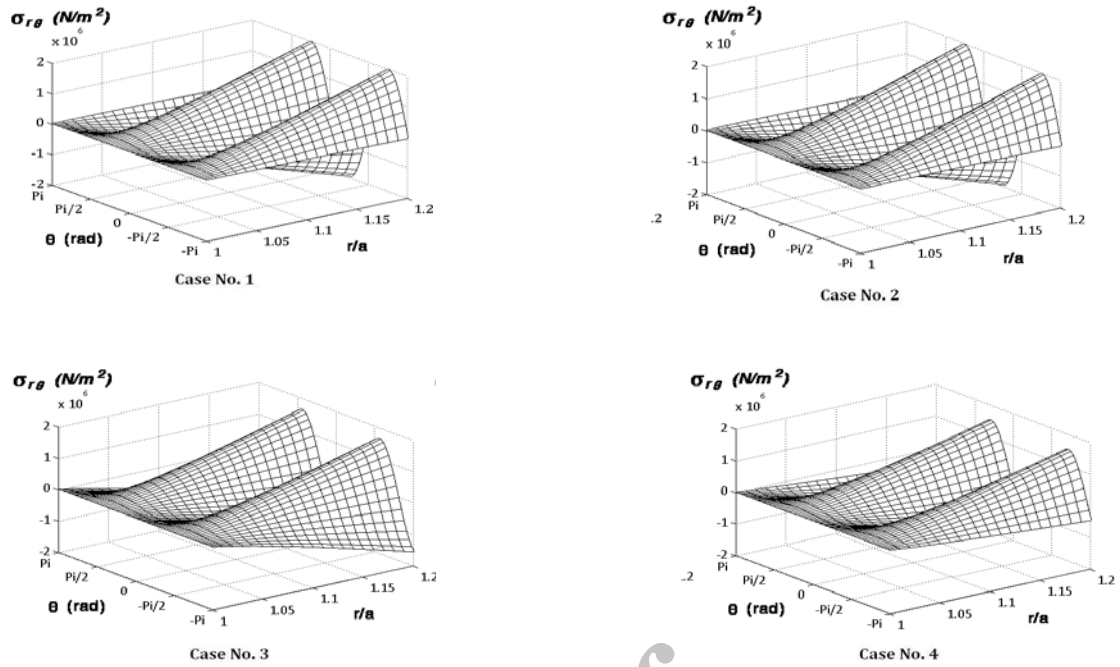
Fig. 10
Shear stress distributions due to electro thermo mechanical load.

Fig. 11
Circumferential stress distributions due to electro thermo mechanical load.
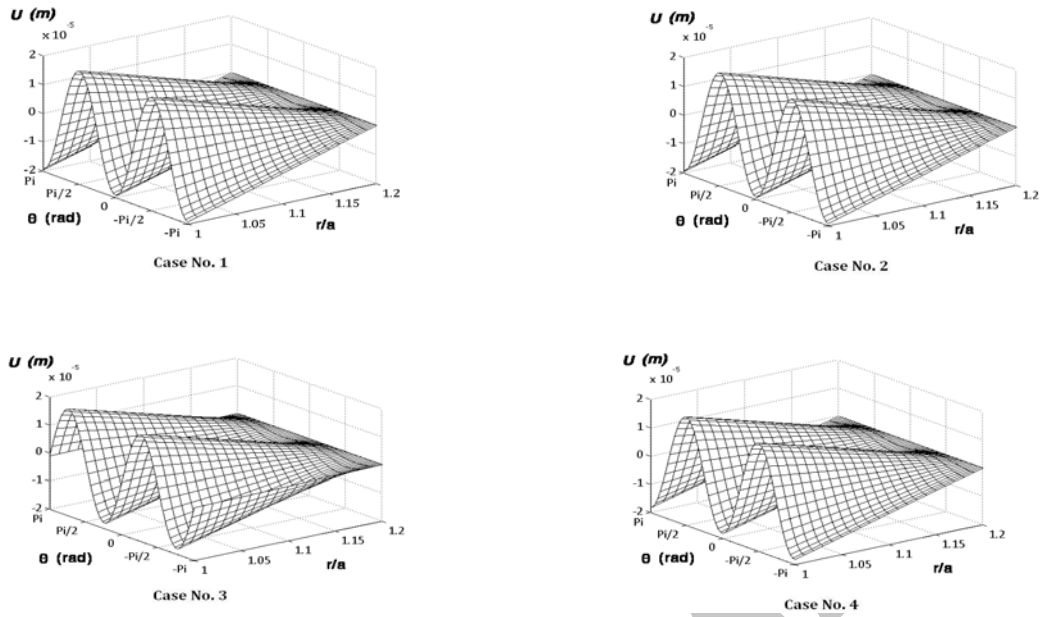
Fig. 12
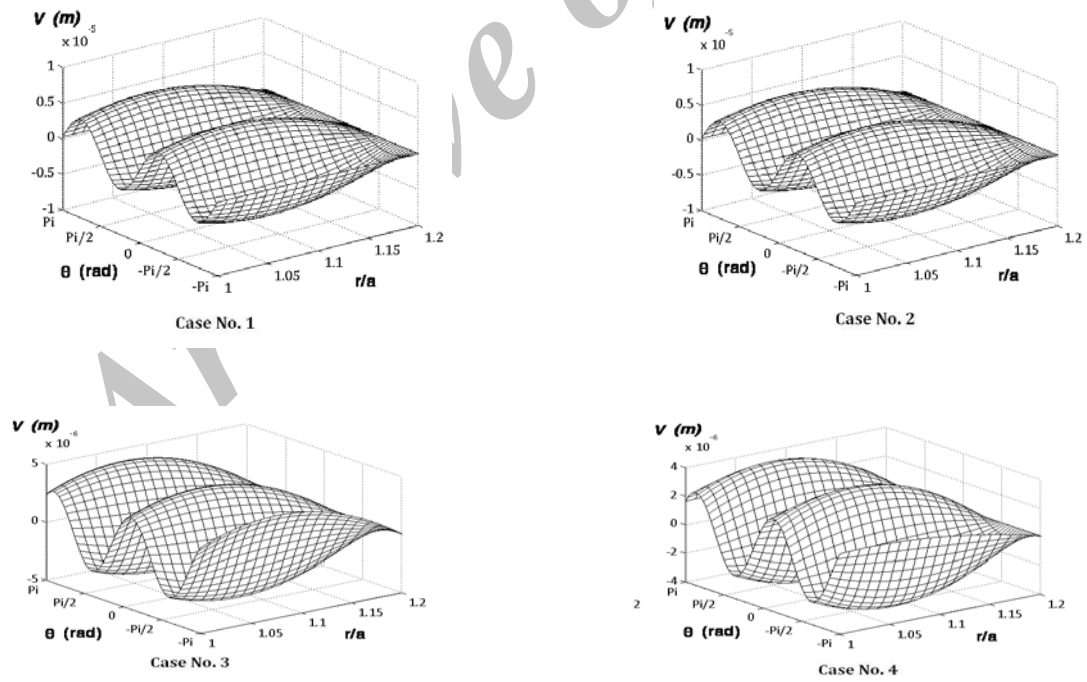Radial displacement distributions due to electro thermo mechanical load.

Fig.13
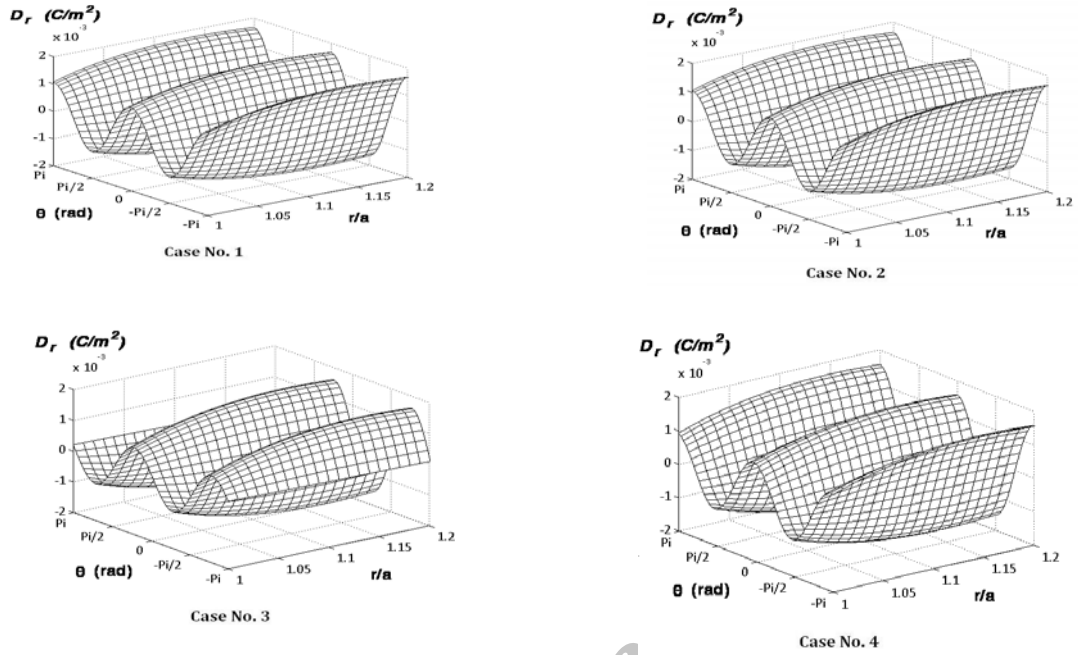Circumferential displacement distributions due to electro thermo mechanical load.

Fig.14
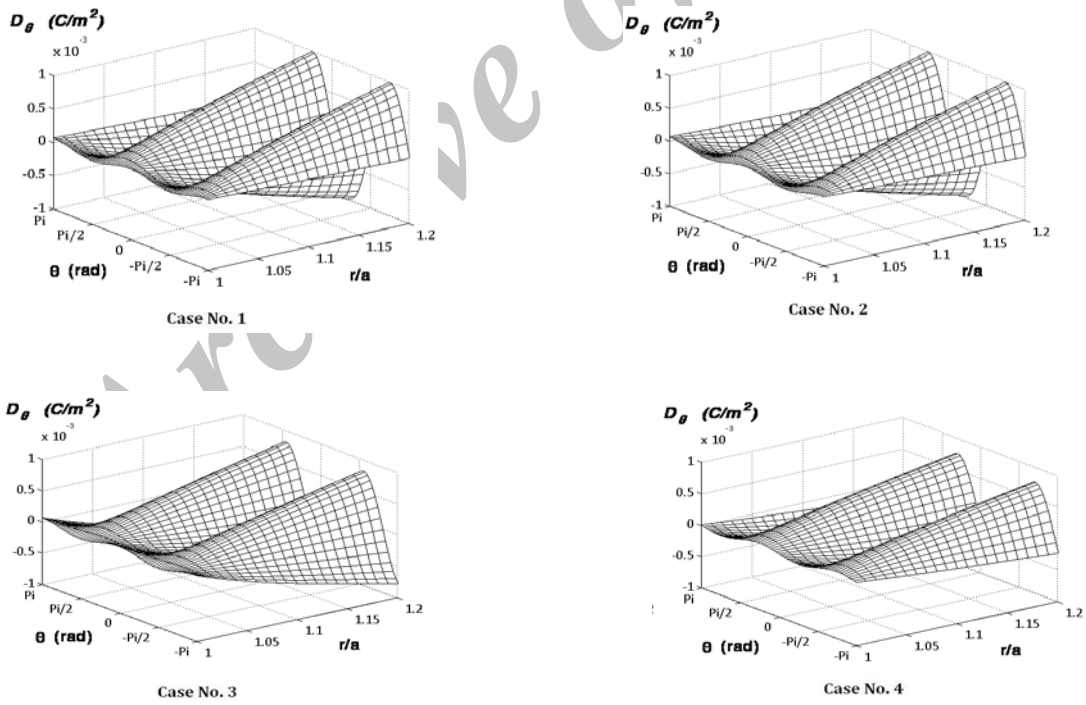Radial electric displacements due to electro thermo mechanical load.

Fig.15
Circumferential electric displacements due to electro thermo mechanical load.

Fig. 12 and 13 show the radial and circumferential displacements across the thickness of cylinder respectively. It can be seen that except case 3, that the pressure is in the opposite phase, the various phases have no great effect on radial displacement but Fig. 13 shows that the opposite phase of pressure and thermal condition can increase the values and changes the shapes of the circumferential displacement along the thickness of cylinder. Figs. 14 and 15 demonstrate the electric radial and circumferential displacements respectively. It can be seen from these figures that in any cases, the electric radial displacements have no effective variations except in case 3, that due to presented boundary conditions, the shape of electric radial displacements changes. Also changes in the phase of pressure can change that value and shapes of electric circumferential displacements. From the above discussions, it can be concluded that by considering the special boundary conditions and combination of them, the mechanical and electrical displacements and stresses can be changed, controlled and optimized along the thickness of cylinder for better design while using this kind of structures.

7 CONCLUSIONS

This paper presents the analytical solution for two-dimensional electro thermo mechanical behavior of a piezoelectric hollow cylinder as a sensor or actuator. The method of solution is based on the direct method and by using the complex Fourier series, the Navier equations were solved. The advantage of this method is its generality and from mathematical point of view, any type of the thermo mechanical and electrical boundary conditions can be considered without any restrictions. Three types of examples are considered to validate our solution and investigate the effect of two-dimensional electro thermo mechanical behavior of cylinder under the effect of combination boundary conditions in the in-phase and opposite-phase of electro thermo mechanical loads. The numerical examples have revealed that by using this method and considering the special boundary conditions, the mechanical and electrical displacements and stresses in a piezoelectric hollow cylinder can be controlled and optimized to design and work efficiency under the various loads.

APPENDIX

$$w_1 = \frac{c_{31}}{c_{11}}, \quad w_2 = \frac{-c_{22}}{c_{11}}, \quad w_3 = \frac{c_{12} + c_{31}}{c_{11}}, \quad w_4 = \frac{-c_{31} - c_{22}}{c_{11}}, \quad w_5 = \frac{g_{11}}{c_{11}}, \quad w_6 = \frac{g_{31}}{c_{11}}, \quad w_7 = \frac{g_{11} - g_{21}}{c_{11}},$$

$$w_8 = \frac{d_{11}}{c_{11}}, \quad w_9 = \frac{-d_{21} + d_{11}}{c_{11}}, \quad w_{10} = \frac{c_{22}}{c_{31}}, \quad w_{11} = \frac{-c_{31}}{c_{31}}, \quad w_{12} = \frac{c_{31} + c_{21}}{c_{31}}, \quad w_{13} = \frac{c_{31} + c_{22}}{c_{31}},$$

$$w_{14} = \frac{g_{31} + g_{21}}{c_{31}}, \quad w_{15} = \frac{g_{31}}{c_{31}}, \quad w_{16} = \frac{d_{21}}{c_{31}}, \quad w_{17} = \frac{h_{21}}{h_{11}}, \quad w_{18} = \frac{g_{11}}{-h_{11}}, \quad w_{19} = \frac{g_{11} + g_{21}}{-h_{11}},$$

$$w_{20} = \frac{g_{31}}{-h_{11}}, \quad w_{21} = \frac{g_{21} + g_{31}}{-h_{11}}, \quad w_{22} = \frac{g_{31}}{h_{11}}, \quad w_{23} = \frac{b_{21}}{h_{11}}, \quad w_{24} = \frac{b_{11}}{h_{11}}$$

$$(A.1)$$

$$X_1 = (\beta_{n1} + 1).\beta_{n1} + \beta_{n1} + 1 - w_1 n^2 + w_2, \qquad X_2 = w_3 in(\beta_{n1} + 1) + w_4 in + w_7(\beta_{n1} + 1)$$

$$X_3 = w_5(\beta_{n1} + 1).\beta_{n1} - w_6 n^2, \qquad X_4 = (\beta_{n2} + 1).\beta_{n2} + \beta_{n2} + 1 - w_1 n^2 + w_2$$

$$X_5 = w_3 in(\beta_{n2} + 1) + w_4 in + w_7(\beta_{n2} + 1), \qquad X_6 = w_5(\beta_{n2} + 1).\beta_{n2} - w_6 n^2$$

$$X_7 = w_{12} in(\beta_{n1} + 1) + w_{13} in, \qquad X_8 = (\beta_{n1} + 1).\beta_{n1} + \beta_{n1} + 1 - w_{10} n^2 + w_{11}$$

$$X_9 = w_{14} in(\beta_{n1} + 1) + w_{15} in, \qquad X_{10} = w_{12} in(\beta_{n2} + 1) + w_{13} in$$

$$X_{11} = (\beta_{n2} + 1).\beta_{n2} + \beta_{n2} + 1 - w_{10} n^2 + w_{11}, \qquad X_{12} = w_{14} in(\beta_{n2} + 1) + w_{15} in$$

$$X_{13} = w_{18}(\beta_{n1} + 1).\beta_{n1} + w_{19}(\beta_{n1} + 1) - w_{20} n^2, \quad X_{14} = w_{21} in(\beta_{n1} + 1) + w_{22} in$$

$$X_{15} = (\beta_{n1} + 1).\beta_{n1} + \beta_{n1} + 1 - w_{17} n^2, \qquad X_{16} = w_{18}(\beta_{n2} + 1).\beta_{n2} + w_{19}(\beta_{n2} + 1) - w_{20} n^2$$

$$X_{17} = w_{21} in(\beta_{n2} + 1) + w_{22} in, \qquad X_{18} = (\beta_{n2} + 1).\beta_{n2} + \beta_{n2} + 1 - w_{17} n^2$$

$$X_{19} = (\beta_{01}+1).\beta_{01}+\beta_{01}+w_2, \qquad\qquad X_{20} = w_5(\beta_{01}+1).\beta_{01}+w_7(\beta_{01}+1)$$
$$X_{21} = (\beta_{02}+1).\beta_{02}+\beta_{02}+1+w_2, \qquad\qquad X_{22} = w_5.(\beta_{02}+1).\beta_{02}+w_7(\beta_{02}+1)$$
$$X_{23} = w_{18}.(\beta_{01}+1).\beta_{01}+w_{19}.(\beta_{01}+1), \qquad X_{24} = (\beta_{01}+1).\beta_{01}+\beta_{01}+1$$
$$X_{25} = w_{18}.(\beta_{02}+1).\beta_{02}+w_{19}.(\beta_{02}+1), \qquad X_{26} = (\beta_{02}+1).\beta_{02}+\beta_{02}+1$$
$$Y_1 = w_8\beta_{n1}+w_9, \qquad Y_2 = w_8\beta_{n2}+w_9, \qquad Y_3 = Y_4 = w_{16}in,$$
$$Y_5 = w_{23}in+w_{24}(\beta_{n1}+1)+w_{24}, \qquad Y_6 = w_{23}in+w_{24}(\beta_{n2}+1)+w_{24}$$
$$Y_7 = w_8\beta_{01}+w_9, \quad Y_8 = w_8\beta_{02}+w_9, \quad Y_9 = w_{24}\beta_{01}+w_{24}, \quad Y_{10} = w_{24}\beta_{n2}+w_{24}$$

(A.2)

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