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# Isogeometric-Based Modeling and Analysis of Laminated Composite Plates Under Transverse Loading

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**ABSTRACT**

Analysis of the laminated composite plates under transverse loading is considered using the new method of Isogeometric Analysis (IGA). Non-Uniform Rational B-Splines(NURBS)are used as shape functions for modeling the geometry of the structure and also are used as shape functions in the analysis process. To show robustness of the new technique, some examples are represented and are compared with the theoretical method at the end. The obtained results show the efficiency of the method.



**Keywords:** Isogeometric analysis; Nun-Uniform rational B-Splines ; Composite plates; Genetic algorithm



## 1 INTRODUCTION

ISOGEOMETRIC analysis has been introduced by T.J.R. Hughes and co-authors in [1] as a novel technique for the discretization of partial differential equations in 2005. Isogeometric analysis is a computational mechanics technology based on functions used to represent geometry. The idea is to build a geometric model, e.g. through a computer aided design (CAD) system, and to directly use in the analysis, the functions describing the geometry, rather than approximating it through a finite element mesh. In CAD systems, non-uniform rational B-splines are the dominant technology. When a NURBS model is constructed, the basis functions used to define the geometry can be systematically enriched by h-, p-, or k-refinement (i.e., smooth order elevation) without altering the geometry or its parameterization. This means that mesh refinement techniques can be utilized without a link to the CAD database, in contrast with finite element methods. This appears to be a distinct advantage of isogeometric analysis over finite element analysis. In addition, on a per degree-of-freedom basis, isogeometric analysis has exhibited superior accuracy and robustness compared with finite element analysis. This is particularly true when a k-refined basis is adopted. In this case, the upper part of the discrete spectrum shows a much better behavior, resulting in better conditioned discrete systems. It appears that isogeometric analysis offers several important advantages over classical finite element analysis. In isogeometric analysis geometry is modeled exactly, in contrast with finite element method in which, geometry is approximated using polynomial functions. It is apparent that the way to break down the barriers between engineering design and analysis is to reconstitute the entire process, but at the same time maintain compatibility with existing practices. A fundamental step is to focus on one, and only one, geometric model, which can be utilized directly as an analysis model, or from which geometrically precise analysis models can be automatically built. This will require a change from classical finite element analysis to an analysis procedure based on CAD representations. The first and most important difference is that isogeometric analysis employs the exact geometry at all levels of discretization, whereas FEA uses piecewise polynomial approximations, even for such common objects as conic sections. This geometric exactness not only affects the accuracy of computed solutions,

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but even the analysis process as a whole as refinement requires no external description of the geometry, unlike in FEA. The second striking difference between the methods is that neither the control points nor the control variables of isogeometric analysis are interpolated, unlike nodal points and nodal variables. This means that we cannot strictly interpret these entities by themselves, but only in conjunction with the basis functions. Solutions in both cases, however, are linear combinations of coefficients and basis functions, and so nothing about the mathematical structure of the Galerkin method or its implementation differs between the methods. Other major differences relate to the properties possessed by the bases. Isogeometric analysis and classical FEA have many similarities as well. They are both isoparametric implementations of Galerkin's method, and as such they have a very similar code architecture. Both methods use compactly supported basis functions, and the bandwidth of matrices corresponding to a given polynomial order are the same for the two methods. Both bases obey the partition of unity property and affine transformations are achieved by applying them directly to the vector valued coefficients that define the geometry. Isogeometric is a novel technique, and there are lots of works to do with it. We can not use this method to analyze structures like trusses and frames. Maybe in the near future, it would be possible. This method hasn't been expanded for discreet systems widely.

IGA is having a growing impact on several fields, from fluid dynamics [2, 3], to structural mechanics [4, 5, 6, 7] and electromagnetics [8, 9]. A comprehensive reference for IGA is given elsewhere [10]. IGA methodologies are designed with the aim of improving the interoperability between numerical simulation of physical phenomena and the Computer Aided Design (CAD) systems. Indeed, the ultimate goal is to drastically reduce the error in the representation of the computational domain and the re-meshing by the use of the "exact" CAD geometry directly at the coarsest level of discretization. This is achieved by using B-Splines or Nun-Uniform Rational B-Splines (NURBS) for the geometry description as well as for the representation of the unknown fields. The use of Spline or NURBS functions, together with isoparametric concepts, results in an extremely successful idea and paves the way to many new numerical schemes enjoying features that would be extremely hard to achieve within a standard Finite Element Method (FEM). Splines and NURBS offer a flexible set of basis functions for which refinement, de-refinement, degree elevation and mesh deformation are very efficient (e.g., [6]).

Laminated composites plates exhibit a considerable variation in their material properties due to involvement of number of parameters that cannot be controlled effectively during fabrication. Randomness in several factors such as fiber orientation, volume fraction, fiber-matrix interface, and curing parameters is inherent in such laminated composites plates. Despite these uncertainties, its use in engineering applications has gained increasing popularity in recent years due to advantages like, weight reduction (high strength/stiffness to weight ratio), longer life (no corrosion, low wear), fatigue endurance, and inherent damping. These plates are commonly employed in engineering applications as thin plates.

## 2 ISOGEOMETRIC ANALYSIS FORMULATION

NURBS are a standard tool for describing and modeling curves and surfaces in computer aided design and computer graphics (see Piegl and Tiller [11] and Rogers [12] for an extensive description of these functions and of their properties). In IGA method, NURBS are used as basis functions in the analysis process. In the following, we present a summary of the main features of isogeometric analysis:

- A mesh for a NURBS patch is defined by the product of knot vectors.
- Knot spans subdivide the domain into "elements".
- The support of each basis function consists of a small number of elements.
- The control points associated with the basis functions define the geometry.
- The isoparametric concept is invoked, that is, the unknown variables are represented in terms of the basis functions which define the geometry. The coefficients of the basis functions are the degrees-of-freedom, or control variables.
- Three different mesh refinement strategies are possible: analogues of classical *h*-refinement (by knot insertion) and *p*-refinement (by order elevation of the basis functions), and a new possibility referred to as *k*-refinement, which increases smoothness in addition to order.
- The element arrays constructed from isoparametric NURBS can be assembled into global arrays in the same way as finite elements.
- Dirichlet boundary conditions are applied to the control variables, in the same way as in finite elements.Neumann boundary conditions are satisfied naturally as in standard finite element formulations.

Generally, a NURBS curve of order $p$ is defined as:

$$X(\xi)=\sum_{i=1}^{n} R_{i,p}(\xi)B_i \tag{1}$$

$$R_{i,p}(\xi)=\frac{N_{i,p}(\xi)w_i}{\sum_{i=1}^{n} N_{i,p}(\xi)w_i} \tag{2}$$

$R_{i,p}$ stands for the univariate NURBS basis functions, $B_i$ are a set of n control points, $w_i$ are a set of $n$ weights corresponding to the control points that must be non-negative and $N_{i,p}$ represents the B-spline basis function of order $p$. To construct a set of $n$ B-spline basis functions of order $p$, a knot vector is defined in a parametric space as follow;

$$\Xi=\left\{\xi_1 \quad \xi_2 \quad \ldots \quad \xi_{n+p+1}\right\} \quad \xi_i \le \xi_{i+1} \qquad i=1,2,\ldots,n+p \tag{3}$$

The knot vectors used for analysis purposes are generally open knot vectors to satisfy the Kronecker-delta property at boundary points. In an open knot vector, the first and last knots are repeated $p+1$ times. Given a knot vector, the univariate B-spline basis function $N_{i,p}$ can be constructed by the following Cox-de Boor recursive formula.

$$\begin{aligned} N_{i,0}(\xi)&=\begin{cases}1 & if\ \xi_i \le \xi \le \xi_{i+1}\\ 0 & otherwise\end{cases}\\ N_{i,p}(\xi)&=\frac{\xi-\xi_i}{\xi_{i+p}-\xi_i}N_{i,p-1}(\xi)+\frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}N_{i+1,p-1}(\xi)\end{aligned} \tag{4}$$

Generally, a NURBS surface of order $p$ in $\xi$ direction and order $q$ in $\eta$ direction can be expressed as:

$$\begin{aligned} S(\xi,\eta)&=\sum_{i=1}^{n}\sum_{j=1}^{m} R_{i,j}^{p,q}(\xi,\eta)B_{i,j}\\ R_{i,j}^{p,q}(\xi,\eta)&=\frac{N_{i,p}(\xi)M_{j,q}(\eta)w_{i,j}}{\sum_{i=1}^{n}\sum_{j=1}^{m} N_{i,p}(\xi)M_{j,q}(\eta)w_{i,j}}\end{aligned} \tag{5}$$

$R_{i,j}^{p,q}$ stand for the bivariate NURBS basis functions. Quadratic B-spline basis functions are shown in Fig. 1.

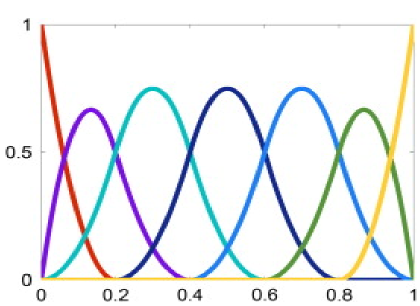



**Fig. 1**
Quadratic basis functions for an open knot vector $\Xi$ ={0, 0, 0, 0.2, 0.4, 0.6, 0.8, 1, 1, 1}.

Based on the classical thin plate theory, only the deflection of the plate $w$ is chosen as the independent variable, while the other two displacement components $u$ and $v$ can be obtained from $w$. Therefore, the NURBS basis is employed for both the parameterization of the geometry and the approximation of the deflection field $w$ as follows:

$$w(x(\xi)) = \sum_{i=1}^{n\times m} \phi_i(\xi) w_i \tag{6}$$

$$x(\xi) = \sum_{i=1}^{n\times m} \phi_i(\xi) B_i \tag{7}$$

$\xi = (\xi, \eta)$ is the parametric coordinates vector, $x = (x, y)$ is the physical coordinates vector, $B_i$ represents the control points of a $n \times m$ control mesh, $w_i$ represents the deflection field at each control point, and $\phi_i(\xi)$ are the bivariate basis functions of order $p$ and $q$ respectively.

A composite laminate with thickness $h$ under transversally concentrated loading is shown in Fig.1. The reference plane, $z = 0$, is located at the undeformed neutral plane of the laminated plate. The direction of fibers in a layer is indicated by α, Fig. 2.

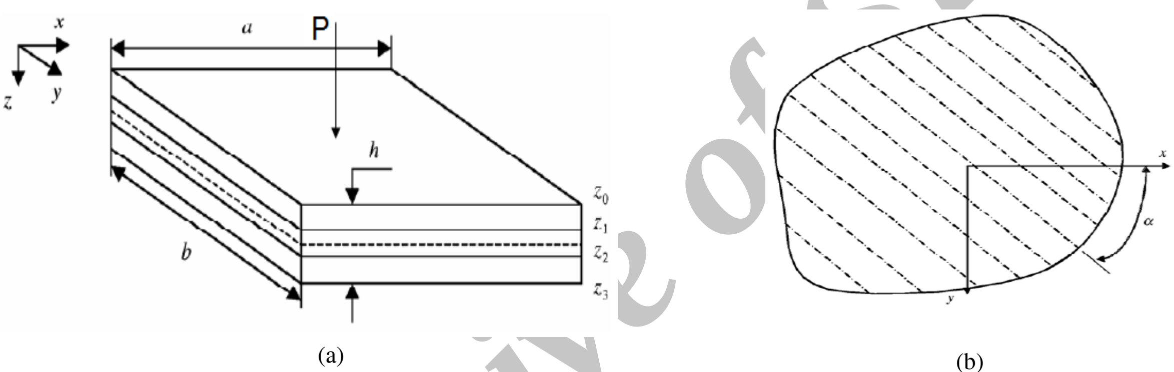



**Fig.2**
(a) A laminated plate under transverse loading. (b) Fiber orientation in a laminate.

The bending strain energy of the laminates is written as:

$$\Pi_b = 1/2 \int_A \varepsilon_p^T \sigma dA, \tag{8}$$

where $A$ stands for the area of the plate, $\varepsilon_p$ is the pseudo-strain, and $\sigma_p$ is the pseudo-stress. For analyzing thin composite laminates, the Classical Lamination Theory (CLT) has been used. CLT assumes that normal to the neutral surface of undeformed plate remains straight and normal to the neutral surface during deformation, which results in [13]:

$$\varepsilon_{xz} = 0 \quad , \quad \varepsilon_{yz} = 0 \tag{9}$$

So, only $\varepsilon_x$, $\varepsilon_y$ and $\varepsilon_{xy}$ are computed, and thus, the equations of plain stress are applied. Displacements in the $X$ and $Y$ directions, $u$ and, $v$ at a distance $z$, from the neutral surface, can be expressed by:

$$u = -z\frac{\partial w}{\partial x}, v = -z\frac{\partial w}{\partial y} \tag{10}$$

Here, $w$ is the deflection of the middle plane of the plate in the $Z$ direction. Using the above equations, one can write:

$$u = \begin{bmatrix} u & v & w \end{bmatrix}^T = \begin{bmatrix} -z\frac{\partial}{\partial x} & -z\frac{\partial}{\partial y} & 1 \end{bmatrix}^T w = L_u, \tag{11}$$

The relationship between the three components of strain and the deflection can be given by:

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}, \varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}, \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{12}$$

or, in matrix form,

$$\varepsilon = zLw, \tag{13}$$

where $\varepsilon$ is the vector of in-plane strains, and $L$ is the differential operator vector given by:

$$L = \begin{bmatrix} -\frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial y^2} & -\frac{\partial^2}{\partial x \partial y} \end{bmatrix}^T \tag{14}$$

Pseudo-strains and pseudo-stresses are defined by:

$$\varepsilon_p = Lw, \sigma_p = D\varepsilon_p = DLw \tag{15}$$

In which, $D$ is the property matrix of materials. Based on the Hamilton principle, the equilibrium equation for a 2D solid mechanics can be written in the form [14]:

$$L^T \sigma + b = 0 \tag{16}$$

where $L$ is the differential operator, $\sigma$ is the stress tensor and, $b$ is the body force vector. The weak form can be given by:

$$\int_{\Omega} \delta\varepsilon^T \sigma d\Omega - \int_{\Omega} \delta w^T b d\Omega - \int_{\Gamma_t} \delta w^T t d\Gamma = 0, \tag{17}$$

$$\int_{\Omega} \delta(Lw)^T (DLw) d\Omega - \int_{\Omega} \delta w^T b d\Omega - \int_{\Gamma_t} \delta w^T t d\Gamma = 0, \tag{18}$$

In which $b$ is the body force, $t$ is the traction force, $\Omega$ is the problem domain, and $\Gamma_t$ stands for the boundary of the solids on which traction forces are prescribed. The first term in Eq. (17) is the strain energy, the second and third terms are used to define works done by the external forces. As mentioned before, for displacement $w$ one can write:

$$w^h(x(\xi)) = \sum_{i=1}^{n \times m} \phi_i(\xi) w_i \tag{19}$$

By using Eq. (19), the product of $Lw$ (which gives the strain) becomes:

$$Lw \approx Lw^h = L\sum_{i=1}^{n\times m}\varphi_i(\xi)w_i = \sum_{i=1}^{n\times m}\underbrace{L\varphi_i(\xi)}_{B_i}w_i = \sum_{i=1}^{n\times m}B_i w_i \ . \tag{20}$$

Substituting Eqs. (19) and (20) into Eq. (18) results in:

$$\int_\Omega \delta(\sum_{i=1}^{n\times m}B_i w_i)^T(D\sum_{j=1}^{n\times m}B_j w_j)d\Omega - \int_\Omega \delta(\sum_{i=1}^{n\times m}\varphi_i w_i)^T b d\Omega - \int_{\Gamma_t}\delta(\sum_{i=1}^{n\times m}\varphi_i w_i)^T t d\Gamma = 0 \ . \tag{21}$$

Let us look at the first term in the above equation:

$$\int_\Omega \delta(\sum_{i=1}^{n\times m}B_i w_i)^T(D\sum_{j=1}^{n\times m}B_j w_j)d\Omega = \int_\Omega \delta(\sum_{i=1}^{n\times m}w_i^T B_i^T)(D\sum_{j=1}^{n\times m}B_j w_j)d\Omega \tag{22}$$

It should be noted that the summation, integration, and variation are all linear operators and therefore, they are exchangeable. Hence,

$$\int_\Omega \delta(\sum_{i=1}^{n\times m}w_i^T B_i^T)(D\sum_{j=1}^{n\times m}B_j w_j)d\Omega = \sum_{i=1}^{n}\sum_{j=1}^{m}\delta w_i^T \underbrace{\int_\Omega B_i^T D B_j d\Omega}_{k_{ij}} w_i = \delta W^T K W. \tag{23}$$

The above equation, $k_{ij}$ is the element stiffness matrix, and $K$ is the global stiffness matrix assembled using the element stiffness matrices. Next, let us examine the second term in Eq. (21).

$$\int_\Omega \delta\left(\sum_{I}^{n\times m}\Phi_I w_I\right)^T b d\Omega = \sum_{I}^{n\times m}\delta w_I^T \underbrace{\int_\Omega \Phi_I^T b d\Omega}_{f_I} = \sum_{I}^{n\times m}\delta w_I^T f_I = \delta W^T F \ , \tag{24}$$

where, $f_i$ is the force vector, and $F$ is the global force. The treatment for the third term in Eq.(21) is exactly the same as that for the second term, except that the body force vector is replaced by the traction vector. Therefore, the additional force vector can be given as:

$$t_i = \int_{\Gamma_t}\varphi_i^T \bar{t} d\Gamma \tag{25}$$

Finally, summarizing Eqs. (22), (23), (24) and (25), gives:

$$\underbrace{\int_\Omega \delta(\sum_{i=1}^{n\times m}B_i w_i)^T(D\sum_{j=1}^{n\times m}B_j w_j)d\Omega}_{\delta W^T K W} - \underbrace{\int_\Omega \delta(\sum_{i=1}^{n\times m}\varphi_i w_i)^T b d\Omega - \int_{\Gamma_t}\delta(\sum_{i=1}^{n\times m}\varphi_i w_i)^T t d\Gamma}_{\delta W^T F} = 0 \tag{26}$$

which, results in:

$$\delta W^T(KW - F) = 0. \tag{27}$$

Because $\delta W$ is arbitrary, the above equation can be satisfied only if:

$$KW - F = 0 \Rightarrow KW = F\ . \tag{28}$$

This is the final discrete system equation for the entire problem domain. Stiffness matrix for laminated composite plates is defined as:

$$\begin{aligned} k_{ij} = \int_{\Omega} [D_{11} \frac{\partial^2 \varphi_i}{\partial x^2}\frac{\partial^2 \varphi_j}{\partial x^2} + D_{12}(\frac{\partial^2 \varphi_i}{\partial x^2}\frac{\partial^2 \varphi_j}{\partial y^2} + \frac{\partial^2 \varphi_j}{\partial x^2}\frac{\partial^2 \varphi_i}{\partial y^2}) + D_{22}\frac{\partial^2 \varphi_i}{\partial y^2}\frac{\partial^2 \varphi_j}{\partial y^2}, \\ 4D_{66}\frac{\partial^2 \varphi_i}{\partial x \partial y}\frac{\partial^2 \varphi_j}{\partial x \partial y} + 2D_{16}(\frac{\partial^2 \varphi_i}{\partial x^2}\frac{\partial^2 \varphi_j}{\partial x \partial y} + \frac{\partial^2 \varphi_j}{\partial y^2}\frac{\partial^2 \varphi_i}{\partial x \partial y}) + 2D_{26}(\frac{\partial^2 \varphi_i}{\partial y^2}\frac{\partial^2 \varphi_j}{\partial x \partial y} + \frac{\partial^2 \varphi_j}{\partial y^2}\frac{\partial^2 \varphi_i}{\partial x \partial y})]d\Omega. \end{aligned} \tag{29}$$

For isotropic plates, stiffness matrix has a simpler form:

$$K_{ij} = \frac{D}{2}\int_{\Omega}[2\left(\frac{\partial^2 \varphi_i}{\partial x^2} + \frac{\partial^2 \varphi_i}{\partial y^2}\right)\left(\frac{\partial^2 \varphi_j}{\partial x^2} + \frac{\partial^2 \varphi_j}{\partial y^2}\right) - 2(1-\nu)[\frac{\partial^2 \varphi_i}{\partial x^2}\frac{\partial^2 \varphi_j}{\partial y^2} + \frac{\partial^2 \varphi_j}{\partial x^2}\frac{\partial^2 \varphi_j}{\partial y^2} - 2\frac{\partial^2 \varphi_i}{\partial x \partial y}\frac{\partial^2 \varphi_j}{\partial x \partial y}]d\Omega. \tag{30}$$

## 3 ENFORCING BOUNDARY CONDITIONS

The application of boundary conditions has been a difficult case, the details of which may be found in [15,16]. One simple approach is a collocation method by which conditions are enforced exactly at a discrete set of boundary points. This is usually accomplished by replacing rows of the matrix equations resulted from discretization of the weak form with equations, which ensure the enforcement of boundary conditions.

Degrees Of Freedom (DOF) of the structure can be categorized into two groups; constrained DOF, $D_c$, and unconstrained DOF, $D_f$. The partition of displacement and force vectors can be written as:

$$W = \begin{Bmatrix} w_f \\ w_c \end{Bmatrix}, F = \begin{Bmatrix} f_f \\ f_c \end{Bmatrix} \tag{31}$$

And, the stiffness matrix $K$ and the shape function matrix $\Phi$ are partitioned as:

$$K = \begin{bmatrix} k_{ff} & k_{fc} \\ k_{cf} & k_{cc} \end{bmatrix}, \Phi = \begin{bmatrix} \phi_{ff} & \phi_{fc} \\ \phi_{cf} & \phi_{cc} \end{bmatrix}. \tag{32}$$

Then Eq. (28) can be written in the following form:

$$\begin{bmatrix} k_{ff} & k_{fc} \\ k_{cf} & k_{cc} \end{bmatrix} \begin{Bmatrix} w_f \\ w_c \end{Bmatrix} = \begin{Bmatrix} f_f \\ f_c \end{Bmatrix}. \tag{33}$$

$$\begin{Bmatrix} w_f^h \\ w_c^h \end{Bmatrix} = \begin{bmatrix} \varphi_{ff} & \varphi_{fc} \\ \varphi_{cf} & \varphi_{cc} \end{bmatrix} \begin{Bmatrix} w_f \\ w_c \end{Bmatrix}. \tag{34}$$

To enforce boundary conditions, second rows of (33) which are related to boundary points, are replaced by second rows of (34) in the following form:

$$\begin{bmatrix} k_{ff} & k_{fc} \\ \phi_{cf} & \phi_{cc} \end{bmatrix} \begin{Bmatrix} w_f \\ w_c \end{Bmatrix} = \begin{Bmatrix} f_f \\ w_c^h \end{Bmatrix}. \tag{35}$$

*Example1*
In this example, an exponential function, Eq. (36), is approximated using the NURBS shape functions, and the moving least square (MLS)shape functions which are used in the element free Galerkin method.

$$f(x) = \exp(-16\times(\text{x-4})^2 + 3) \tag{36}$$

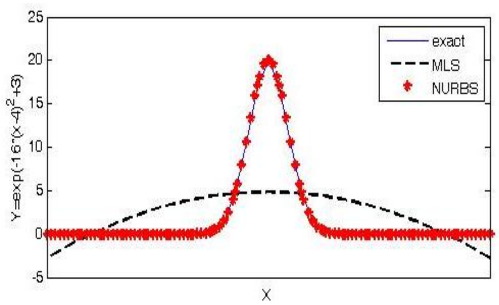
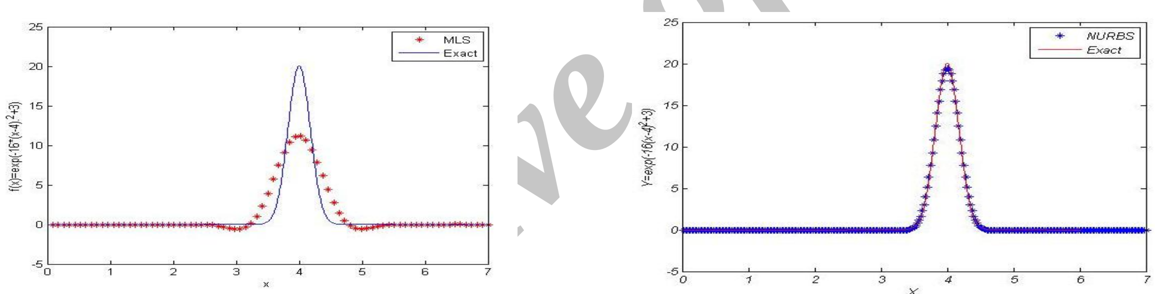



**Fig. 3**
Function approximation with two methods of NURBS and MLS.

As we can see, robustness of NURBS shape functions which are used in the IGA method, are more powerful than MLS shape functions of the element free Galerkin (EFG) method.

In the following figures, at first, the exponential function is approximated using MLS shape functions, then the same function is approximated using NURBS shape functions. We can see robustness of the NURBS functions in comparison with MLS shape functions of the element free Galerkin method.




(a) (b)

**Fig.4**
(a) Function approximation by the MLS method. (b)Function approximation by the NURBS method.

*Example2*
This example is an isotropic rectangular plate, as in Fig. 5, with simply supported boundary conditions and uniform thickness $h = 0.001m$, a length of $0.5m$, and a width equal to $0.25m$ subjected to two laterally concentrated loads, $f_1 = f_2 = 650kgf$. The first load is applied at the point $x = 0.250m, y = 0.083m$, and the second load is applied at the point $x = 0.250m, y = 0.166m$. Young's modulus and Poisson's ratio are assumed to be $E = 2\times10^9 kg/m^2$ and $\nu = 0.3$, respectively. The IGA method is used for the analysis process. The results are compared with those of the exact method to show the IGA robustness.

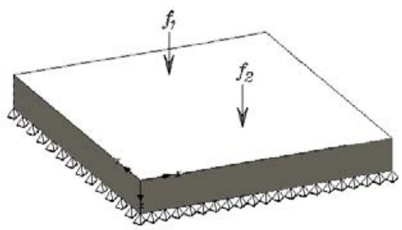



**Fig. 5**
Plate related to example 2.

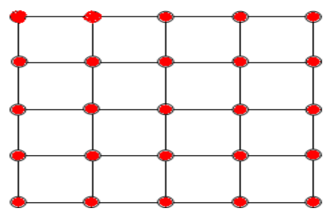


**Fig. 6**
Control net used in example 2.

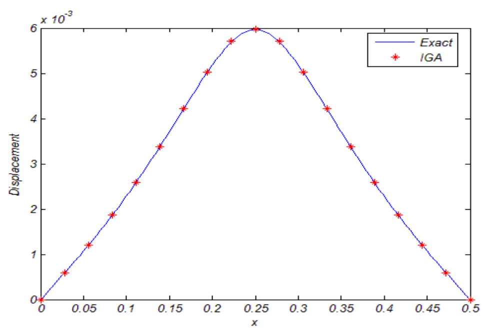



**Fig. 7**
Deflection computed with the IGA and exact method.

The plate is modeled using NURBS shape functions. Deflections, pseudo strains, and pseudo stresses are calculated by the IGA method. The control points used to model the geometry, before refinement, are displayed in Fig. 6. The degrees of NURBS shape functions in each direction are equal to 2. The knot vector in each direction is: $\Xi = \{0,0,0,0.5,1,1,1\}$ . The results of deflection for $x \in [0,0.5]$ and, $y = 0.125m$ are compared with exact method in Fig. 7. Compared with the exact results, good agreement has been achieved. Energy norm has been calculated using Eq. (37). For this example, the energy norm was equal to; $e_e = 0.033211452$ .

$$e_e = \sqrt{\left(\frac{1}{2}\int_{\Omega} (\varepsilon^{Num} - \varepsilon^{Exact})^T D(\varepsilon^{Num} - \varepsilon^{Exact}) d\Omega\right.} \ . \tag{37}$$

*Example3*
This example is a laminated plate with 8 layers, four of which are symmetrical as shown in table (1). The laminated plate is under five transversally concentrated loads, Fig. 8. The laminate is simply supported at all edges. The loads are applied at the points as following:

$$x_2 = a/3, y_2 = b/5; x_3 = 3a/4, y_3 = b/3; x_4 = a/3, y_4 = 2b/3; x_5 = 3a/5, y_5 = 4b/5; x_1 = a/2, y_1 = b/2$$
$$f_1 = 2000kg.f, f_2 = 4500kg.f, f_3 = 2000kg.f, f_4 = 1000kg.f, f_5 = 500kg.f \ .$$

Ply angles, thicknesses and material types used for the layers are indicated in Table .1. IGA method is used for the analysis process. Moments computed by the IGA method, are shown in Table 1.

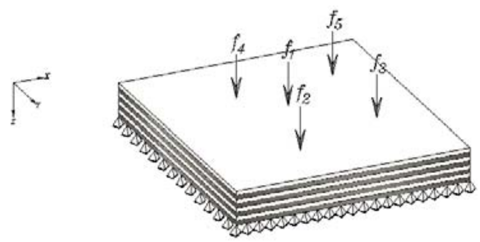



**Fig. 8**
The laminate with 8 layers related to example 3.

**Table1**
Parameters of laminates for example 3

| layer no. | Angle(deg) | Thickness(m) | Material |
|---|---|---|---|
| 1(8) | -75 | 0.001 | T300/5208 |
| 2(7) | 60 | 0.0008 | AS/3501 |
| 3(6) | 45 | 0.0015 | IM6/APC2 |
| 4(5) | 90 | 0.0008 | S2-449/SP |
| $M_x^*$ | -256.54321 $kgf.m$ | | |
| $M_y^*$ | -1163.67876 $kgf.m$ | | |
| $M_{xy}^*$ | 164.98732 $kgf.m$ | | |

## 4 CONCLUSIONS

In this article, isogeometric analysis method is used to analyze laminated composite plates. The obtained results are compared with those of the exact method. Robustness and accuracy of the IGA method is shown perfectly. In the first example, we can see NURBS shape functions in comparison with moving least square technique, which is used as shape function in the element free Galerkin method. At the third example, a laminated composite plate under transverse loading is analyzed by the method of isogeometric analysis. The obtained results show the robustness and accuracy of the IGA method.

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