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Vibration Analysis of a Nonlinear Beam Under Axial Force by Homotopy Analysis Method

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ABSTRACT

In this paper, Homotopy Analysis Method is used to analyze free non-linear vibrations of a beam simply supported by pinned ends under axial force. Mid-plane stretching is also considered for dynamic equation extracted for the beam. Galerkin decomposition technique is used to transform a partial dimensionless nonlinear differential equation of dynamic motion into an ordinary nonlinear differential equation. Then Homotopy Analysis Method is employed to obtain an analytic expression for nonlinear natural frequencies. Effects of design parameters including axial force and slenderness ratio on nonlinear natural frequencies are studied. Moreover, effects of factors of nonlinear terms on the general shape of the time response are taken into account. Combined Homotopy-Pade technique is used to reduce the number of approximation orders without affecting final accuracy. The results indicate increased speed of convergence as Homotopy and Pade are combined. The obtained analytic expressions can be used for a vast range of data. Comparison of the results with numerical data indicated a good conformance. Having compared accuracy of this method with that of the Homotopy perturbation analytic method, it is concluded that Homotopy Analysis Method is a very strong method for analytic and vibration analysis of structures.

Keywords: Nonlinear vibration; Homotopy analysis method;Beam axial force; Stretching effect

1 INTRODUCTION

BEAMS, which have a variety of applications in different sizes from micro/nano size structures, such as micro oscillators, to macro size airplane wings, flexible satellites and long span bridges, are among the most important engineering structures. Large amplitude vibrations usually lead to material fatigue and structure breakdown. These effects are more important on natural frequencies of structures [1]. Large amplitude vibration can lead to nonlinear effects in these systems. Nonlinear sources may be geometrical, inertial or material. Geometrical nonlinearity may stem from nonlinear stretching or large curvatures. Nonlinear effects are created by concentrated or distributed masses. Material nonlinearity breaks out when stress is a nonlinear function of strain [2]. Euler-Bernoulli's theorem of beam considers that cross-sectional planes which are prior to deformation, perpendicular to the main axis remain constantly perpendicular to the main axis and undergo no strain over the plane [3]. In effect it is assumed that effects of transverse shear deformation and transverse normal strains are negligible so they are not taken into account [4].

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Pillai- Rao studied the problem of free large amplitude vibrations in a simply supported uniform beam and obtained frequency response of the system through some methods including elliptic functions, harmonic balance and simple harmonic oscillations [5]. Foda employed multiple time-scale (MTS) method in order to analyze nonlinear oscillations of a beam with pinned ends in which effect of shear deformation and rotary inertia are considered [6]. Ramezani and et al utilized a uniform method for the same problem with doubly clamped boundary conditions and concluded that when beams theorem is used to study micro- and nano-sized electromechanical structures, effects of shear deformation and rotary inertia shall be taken into consideration for a precise dynamic analysis [7]. Generally it is very difficult to find a precise solution for nonlinear oscillations of beams. Hence, approximate analytic and numerical methods are used. Although numerical methods have some advantages, analytic solutions seem to be more attractive for parametric study of the problem and investigation of the problem physics. Analytic solutions also present a reference framework for validation and inspection of numerical methods. Even though it is not easy to use an analytic method for analysis of nonlinear oscillations of the beams, there can be certain analytic methods such as perturbation techniques for this problem. On the whole, analytic methods have their own constraints. For example, perturbation method, which enjoys the widest application among from analytic methods, is limited to weak non-linearity and implemented on the basis of a single small parameter in the equation. Most of the nonlinear problems, particularly those with strong nonlinearity, lack a small parameter. Hence, Liao proposed the HAM for analytic solution of the complex nonlinear problems [8]. In this method , there is no need for a small parameter in the problem as a result of which researchers have resorted to this method for solving different problems and have also obtained acceptable responses. Seddighi and et al employed this method to arrive at precise resolution of vibrations of a beam under strong nonlinear damping and reached an analytic statement for it in the time range [9]. Hosseini and et al introduced a precise analytic solution for free nonlinear vibrations of a durable oscillator with inertia and third-order static nonlinearities [10]. Works of different authors, mentioned here, indicate this method has been able to overcome limitations and constraints of the traditional perturbation methods and predict nonlinear systems' behavior precisely.

Hence, this paper has considered mid-plane stretching and utilized HAM for analysis of free nonlinear vibrations of a beam with pinned ends. To do so, first partial differential equation of the problem is transformed into a typical differential equation via Galerkin decomposition technique. Then HAM is used to solve the problem, the response of which is compared with that of the fourth-order Runge–Kutta numerical method. After that effect of design factors and nonlinear terms' coefficients on time response is studied. Finally HPM is utilized to compare precision of the said method with that of the other analytic methods. To increase the speed of convergence, this method is combined with Pade mathematical method.

2 PROBLEM FORMULATION

Partial nonlinear differential equation of the beam will be as follows if mid-plane stretching is not ignored [11].

$$EI\frac{\partial^4 \widehat{w}}{\partial \widehat{x}^4}+m\frac{\partial^2 \widehat{w}}{\partial \widehat{t}^2}+\left(-N_0-\frac{EA}{2L}\int_0^L\left(\frac{\partial \widehat{w}}{\partial \widehat{x}}\right)^2 dx+\frac{EA}{8L}\int_0^L\left(\frac{\partial \widehat{w}}{\partial \widehat{x}}\right)^4 dx\right)\frac{\partial^2 \widehat{w}}{\partial \widehat{x}^2}=0 \quad (1)$$

where E , elasticity module of the beam material, I , second moment of the beam cross-section with relation to bending axis, $\widehat{w}$,beam transverse deformation , M , beam linear density , $\widehat{t}$, time, A, beam transverse cross-section, N_o, axial force applied to beam and L beam length. It is assumed that beam starts to oscillate with a natural frequency and dimensionless variables t, x ,w and t are introduced which are defined in Eqs. (2) to (4). Eq.(1) can be rewritten as dimensionless Eq.(6).

$$t=\frac{\widehat{t}}{T} \quad (2)$$

$$x=\frac{\widehat{x}}{L} \quad (3)$$

$$w=\frac{\hat{w}}{L} \tag{4}$$

$$T=\frac{1}{\beta^2}\sqrt{\frac{mL^4}{EI}} \tag{5}$$

$$\frac{\partial^4 w}{\partial x^4}+\beta^4\frac{\partial^2 w}{\partial t^2}-\left(\frac{N_0}{EA}+\frac{1}{2}\int_0^1\left(\frac{\partial w}{\partial x}\right)^2 dx-\frac{1}{8}\int_0^1\left(\frac{\partial w}{\partial x}\right)^4 dx\right)\left(\frac{L}{r}\right)^2\frac{\partial^2 w}{\partial x^2}=0 \tag{6}$$

where in the above equation

$$r^2=\frac{I}{A} \tag{7}$$

Now the solution of the Eq. (6) can be assumed as:

$$w(x,t)=\phi(x)q(t) \tag{8}$$

To solve Eq.(6) , $\phi(x)$ is assumed to be the dimensionless vibration model of the beam. Considering the simply supported boundary conditions, β_n is taken as follow:

$$\beta_n=n\pi \tag{9}$$

As a result , $\phi(x)$ can be expressed as Eq.(10).

$$\phi(x)=\sin(n\pi x) \tag{10}$$

In accordance with Galerkin decomposition technique , Eq.(8) is placed in Eq.(6) and the remainder is integrated with $\phi(x)$ over the entire domain of the problem, as a result of which a typical differential equation can be achieved as follows:

$$\frac{d^2 q}{dt^2}+Kq+\alpha q^3-\beta_0 q^5=0 \tag{11}$$

$$q(0)=a_0\quad,\quad\frac{dq}{dt}(0)=0 \tag{12}$$

$$K=\frac{1}{\beta^4 A_1}\left(A_3+\frac{N_0}{EA}\left(\frac{L}{r}\right)^2 A_2\right) \tag{13}$$

$$\alpha=\frac{1}{2\beta^4 A_1}\left(\frac{L}{r}\right)^2 A_2^2 \tag{14}$$

$$\beta_0=\frac{1}{8\beta^4 A_1}\left(\frac{L}{r}\right)^2 A_2 A_4 \tag{15}$$

In Eqs. (13) and (15) , (Ai)s ,($1\le i\le 4$) can be achieved as follow:

$$A_1=\int_0^1\phi^2 dx\ ,\quad A_2=\int_0^1\phi'^2 dx\ ,\quad A_3=\int_0^1\phi''^2 dx\ ,\quad A_4=\int_0^1\phi'^4 dx \tag{16}$$

In the above-mentioned formulae, the sign of Prim indicates a differentiation with relation to the independent variable x .

3 HOMOTOPY ANALYTIC METHOD

3.1 Original idea

Homotopy analysis is a general analytic method for solving nonlinear differential equations [12, 13]. This method transforms a nonlinear differential equation into indefinite number of linear differential equations with auxiliary parameter P varying between 0 and 1 [12]. As the value of P increases from 0 to 1, solution of the problem moves from initial conjecture to precise solution. In order to show initial idea of Homotopy, a nonlinear differential equation is considered which follows:

$$N[q(t)] = 0 \tag{17}$$

where N, a nonlinear differential operator and $q(t)$, an unknown function of variable t. Homotopy is created as follow:

$$\overline{H}(\varphi, q, h, H(t)) = (1-p)L[\varphi(t,p) - q_0(t)] - phH(t)N[\varphi(t,p)] \tag{18}$$

where φ, h and $H(t)$ are a function of t, p, auxiliary parameter and auxiliary non-zero function respectively. Auxiliary parameter and auxiliary function set convergence area of the solution. Parameter L indicates a linear auxiliary operator. As value of P increases from Zero to One, $\varphi(t,0)$ moves from initial approximation to precise solution. In other words, $\varphi(t,0)=q_0(t)$ which is solution of this Homotopy "$\overline{H}(\varphi,p,h,H(t))\big|_{p=0}=0$" moves to $\varphi(t,1)=q(t)$ which is solution of this Homotopy "$\overline{H}(\varphi,p,h,H(t))\big|_{p=1}=0$". Application of $\overline{H}(\varphi,p,h,H(t))=0$ will bring about Zero transformation as follow:

$$(1-p)L[\varphi(t,p) - q_0(t)] = phH(t)N[\varphi(t,p)] \tag{19}$$

Considering boundary conditions as follow:

$$\varphi(0,p) = a \;,\; \frac{d\varphi(0,p)}{dt} = 0 \tag{20}$$

Functions $\varphi(t,p)$ and $\omega(p)$ could be extracted as exponent series of P by means of Taylor theory,

$$\varphi(t,p) = \varphi(t,0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \varphi(t,p)}{\partial p^m}\bigg|_{p=0} p^m = q_0(t) + \sum_{m=1}^{\infty} q_m(t) p^m \tag{21}$$

$$\omega(p) = \omega_0 + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \omega(p)}{\partial p^m}\bigg|_{p=0} p^m = \omega_0 + \sum_{m=1}^{\infty} \omega_m p^m \tag{22}$$

where $q_m(t)$ and ω_m are M-order transformation. Differentiation of zero-order transformation equation towards P and setting values of P on zero will yield first-order transformation equation ($m=1$) as follow, which is first-order approximation of $q(t)$.

$$L[q_1(t)] = hH(t)\, N[q_0(t), \omega_0]\big|_{q=0} \tag{23}$$

Considering boundary conditions as follow:

$$q_1(0)=0\ ,\ \ \frac{dq_1}{dt}(t)=0 \tag{24}$$

Approximations beyond the order of solution could be reached through computation of *m*-order transformation equation (*m>1*) that is expressed as follow[14,15]:

$$L[q_m(t)-q_{m-1}(t)]=hH(t)R_m(q_{m-1},\vec{\omega}_{m-1}) \tag{25}$$

where q_{m-1}, ω_{m-1} and $R_m(q_{m-1},\omega_{m-1})$ are defined as follow:

$$R_m(\vec{q}_{m-1},\vec{\omega}_{m-1})=\frac{1}{(m-1)!}\frac{d^{m-1}N[\varphi(t,p),\omega(p)]}{dp^{m-1}}\bigg|_{p=0} \tag{26}$$

$$\vec{q}_{m-1}=\{q_0,q_1,q_2,.....,q_{m-1}\},\qquad \vec{\omega}_{m-1}=\{\omega_0,\omega_1,\omega_2,.....,\omega_{m-1}\} \tag{27}$$

Considering boundary conditions as follow:

$$q_m(0)=\frac{dq_m}{dt}(0)=0 \tag{28}$$

3.2 Application of the HAM

Consider the problem equation which transforms as follow when variable $\tau=\omega t$:

$$\omega^2\ddot{q}(\tau)+Kq(\tau)+\alpha q^3(\tau)-\beta_0 q^5(\tau)=0 \tag{29}$$

In order to solve this equation through Homotopy, the first conjecture of the problem solution , which satisfies initial boundary conditions ,can be stated as follow:

$$q_0(\tau)=a_0\cos(\tau)\ ,\ \ q_0(0)=a_0\ ,\ \ \dot{q}_0(0)=0 \tag{30}$$

Linear operator can be expressed as follow:

$$L[\varphi(\tau,p)]=\omega_0{}^2[\frac{\partial^2\varphi(\tau,p)}{\partial\tau^2}+\varphi(\tau,p)] \tag{31}$$

where $L[a_0\cos(\tau)]=0$ and nonlinear operator can be considered as follow:

$$N[\varphi(\tau,p)]=\Omega^2(p)\frac{\partial^2\varphi(\tau,p)}{\partial p^2}+K\varphi(\tau,p)+\alpha\varphi^3(\tau,p)-\beta_0\varphi^5(\tau,p) \tag{32}$$

Having assumed auxiliary function $H(\tau)=1$, first-order transformation equation can be achieved as follow with regard to Eq. (23).

$$\omega_0{}^2[\frac{\partial^2 q_1(\tau)}{\partial\tau^2}+q_1(\tau)]=h[-a_0\omega_0{}^2\cos(\tau)+Ka_0\cos(\tau)+\alpha a_0{}^3\cos^3(\tau)-\beta_0 a_0{}^5\cos^5(\tau)] \tag{33}$$

Since final response is achieved by algebraic summation of different steps, and in order for the expansion to be uniform and have identical terms, auxiliary function H is considered as a unit and then coefficients of the term $cos(\tau)$ are set to zero in order to prevent emergence of big terms in time response. As a result

$$\omega_0 = \sqrt{K + \frac{\alpha a_0^{\ 3}}{4} - \frac{5\beta_0 a_0^{\ 4}}{8}} \tag{34}$$

That conforms well to what Ahmadian and et al [1] have achieved. Similarly considering the conditions of Eq. (24), response of the first-order transformation equation will be as follows:

$$q_1 = D_1 \cos(\tau) + D_2 \cos(3\tau) + D_3 \cos(5\tau) \tag{35}$$

where

$$D_1 = -\left(D_2 + D_3\right) \qquad , D_2 = \frac{1}{128}\frac{h a_0^{\ 3}\left(5\beta_0 a_0^{\ 2} - 4\alpha\right)}{\omega_0^{\ 2}} \qquad , D_3 = \frac{1}{384}\frac{h\beta_0 a_0^{\ 5}}{\omega_0^{\ 2}} \tag{36}$$

Values of $q_2(\tau)$, $q_3(\tau)$ and …. could be gained similarly and equivalent terms for nonlinear natural frequencies could be more precise with higher approximations.

3.3 Homotopy-pade technique

Pade approximation is the best approximation of function for the same -order fractional functions [14, 15].Sometimes Pade method yields a better approximation of function compared to Taylor series. It is also possible to use Pade method in case Taylor series fail to converge. In order to calculate a Pade approximation of [m, n] type, let's assume f as an expanded function of exponential series as follow:

$$f(z) = \sum_{k=0}^{m+n+1} a_k z^k \tag{37}$$

where

$$a_k = \frac{f^{(k)}(0)}{k!}, k = 0,1,2,......, m+n+1 \tag{38}$$

Pade approximation [m, n] of the function f is shown with fractional function r that can be written as follow:

$$r(z) = \frac{b_0 + b_1 z + ... + b_m z^m}{1 + c_1 z + ... + c_n z^n} = \frac{p(z)}{q(z)} \tag{39}$$

Generally function f is first expanded up to $m+n+1$ order around $x=a$ point via Taylor or Lorant series, then fractional approximation of Pade is calculated as the above formula. Pade technique is used for accelerating convergence of the given series. Homotopy-Pade technique [16] is a combination of traditional Pade method and HAM. In order to calculate [m, n] type approximation of Homotopy-Pade function $\varphi(t, p)$ ، traditional Pade technique[m, n] is achieved around the applied parameter P as follow:

$$\varphi(t,p)=\frac{\sum_{k=0}^{m}A_{m,k}(t)p^{k}}{\sum_{k=0}^{m}B_{m,k}(t)p^{k}} \tag{40}$$

where coefficient of $A_{m,k}(t)$ and $B_{m,k}(t)$ are concluded from (m, n) order approximations of q(t). It should be noted that making use of Homotopy-Pade technique helps to reduce the number of required approximations for a precise solution [15]. Hence, approximation of the [1,1] order of Homotopy-Pade could be written as follow for functions q(t) and ω.

$$\omega[1,1]_{Pade}=\frac{\omega_1\omega_0+\omega_1^2-\omega_2\omega_0}{\omega_1-\omega_2} \tag{41}$$

$$q[1,1]_{Pade}=\frac{q_1q_0+q_1^2-q_2q_0}{q_1-q_2} \tag{42}$$

4 RESULTS STUDIED

In order to show the accuracy and precision of the method in question, a numerical example is given for a beam with pinned ends. However, it should be noted that axes are dimensionless in all the charts. Fig. 1 indicates system response over the time range. As it is evident from this picture, results obtained from HAM conform well to those of the fourth-order Runge–Kutta method which is a highly accurate method. In order to compare the level of accuracy of this method with that of the other analytic methods, time- response obtained from HAM and HPM are indicated in Fig. 2 and the difference between these two methods is presented through numerical solution. It is clear that both methods have almost the same accuracy over small ranges, but as the range increases, accuracy of the HAM exceeds that of the HPM.

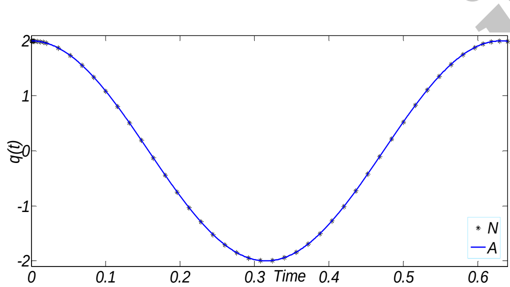
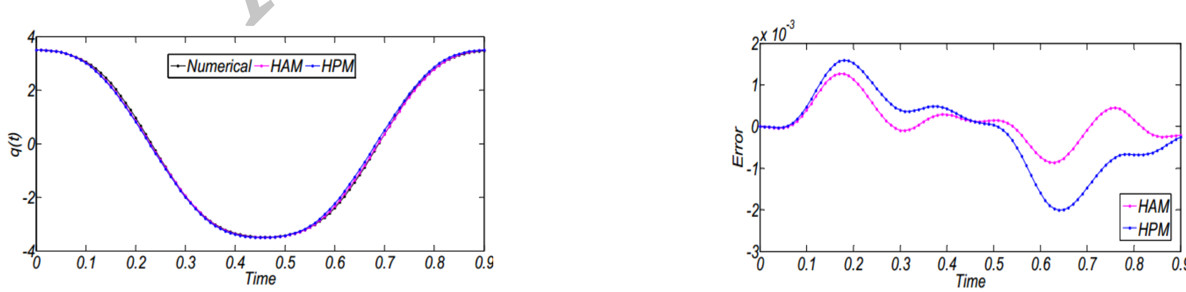


Fig. 1
Time-response curve of a beam with pinned ends; comparison of numerical solution and analytic solution of fourth-order approximation Homotopy.



(a) (b)

Fig. 2
Time-response curve of a beam with pinned ends; comparison of HAM and HPM solutions, and their rate of deviation from numerical solution.

Fig. 3 shows time-response chart of the beam in three different states in order to reflect the difference between linear and nonlinear assumption and also effect of ignoring fifth-order nonlinear term in equations of the beam oscillations. In Fig. 4 coefficient of third-order nonlinear term is changed, while other parameters are constant, and time-response curve is drawn for three different values. Fig. 5 shows the same process for changing coefficient of the fifth-order nonlinear term.

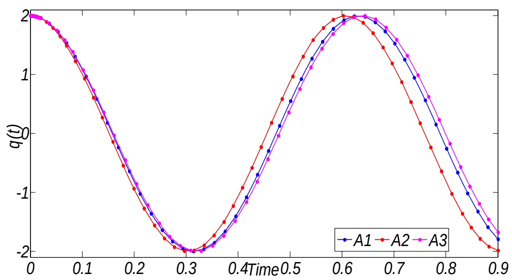


Fig. 3
Time-response curve of a beam with pinned ends; comparison of state a. nonlinear assumption, b. elimination of fifth-order nonlinear term, c. linear assumption.

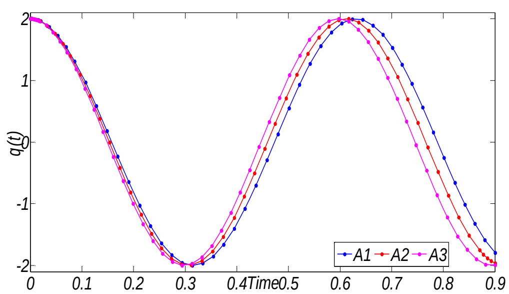


Fig. 4
Time-response curve of a beam with pinned ends; comparison of Homotopy analytic solution for three different values of third-order term, 3.5, 5.5 and 7.5 respectively.

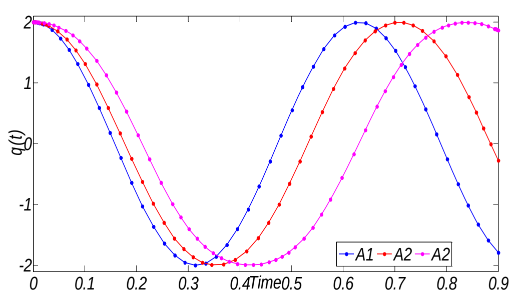


Fig. 5
Time-response curve of a beam with pinned ends; comparison of Homotopy analytic solution for three different values of fifth-order term, 0.8, 2.8 and 4.8 respectively.

In Fig.6, the number of approximation times is reduced via calculation of [1,1] order Homotopy-pade approximation that is highly accurate and conforms well to numerical solution similar to values in Fig. 1 (which are summation of the orders above HAM). Fig. 7 shows the curve of nonlinear frequencies based on auxiliary parameter *h*. As it is clear approximation of the fourth-order Homotopy and [1, 1] order Homotopy-pade conform over a larger range. In Fig. 8 analytic response of HAM is obtained for different values of the auxiliary parameter h in a given point and then compared with numerical solution of the same point. It shows that analytic solution for *h= -1* conforms completely to numerical solution. Figs. 9 and 10 indicate effect of the initial conditions on nonlinear natural frequencies in two states a. different axial force b. different slenderness ratio. It can be noticed that as initial displacement increases, nonlinear natural frequency rises which is indicative of good conformance between linear and nonlinear natural frequencies under small deformations. Noteworthy is that when stretching force or slenderness ratio increases, period of oscillation decreases and frequency increases. It should be noted that arrows on the figure indicate direction of the increase of axial force or slenderness ratio.

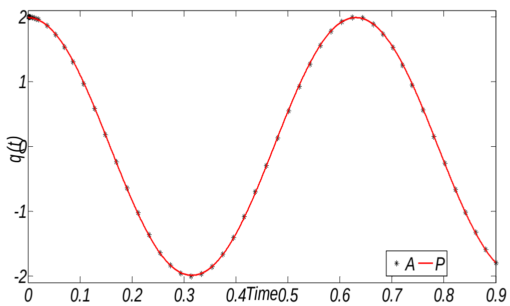


Fig. 6
Time-response curve using [1, 1] order Homotopy-pade approximation and its conformance with numerical solution.

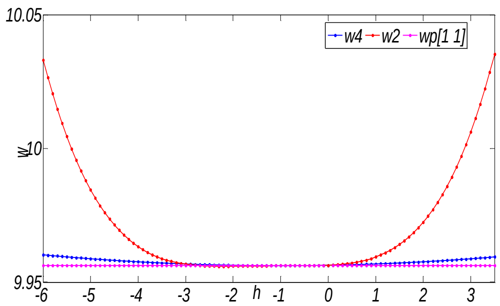


Fig. 7
Nonlinear frequencies curve based on auxiliary parameter *h*.

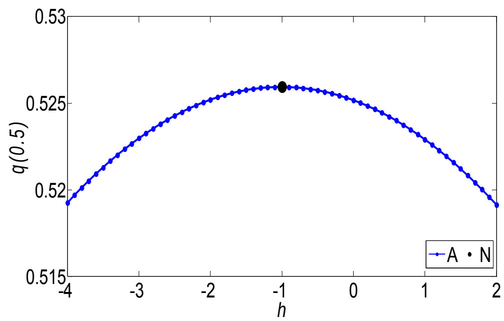


Fig. 8
Curve of the value of function in a point with Homotopy for different values of *h* compared to numerical solution.

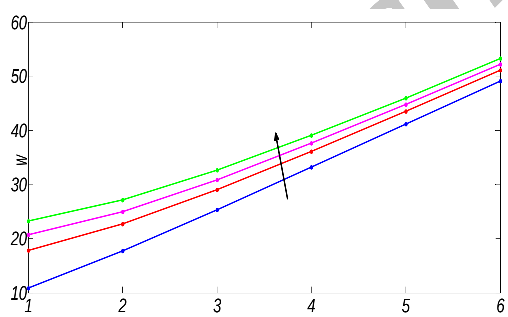


Fig. 9
Nonlinear natural frequencies curve based on initial displacement in different axial forces. Arrow shows direction of force increase linearly.

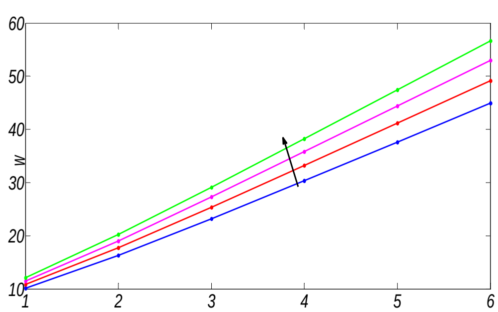


Fig. 10
Nonlinear natural frequencies curve based on initial displacement in different slenderness ratio. Arrow shows direction of slenderness ratio increase linearly.

5 CONCLUSIONS

In this study, mid-plane stretching was considered in Homotopy analytic method technique which was utilized in order to find nonlinear vibration behavior of a beam with pinned ends. Results are indicative of high accuracy compared to numerical solutions as well as other methods used in the previous works. Homotopy-Pade technique leads to reduced, number of approximations, while calculations accuracy remains similar to that of the upper orders of homotopy. In addition, effects of variations of the axial pre-stretching force and slenderness ratio on beam behavior were studied parametrically. It was noticed that as slenderness ratio and pre-stretching force of the beam increase, nonlinear natural frequencies of the beam increase too. Value of function was calculated for a certain amount based on the auxiliary parameter which concluded that for h= -1, function will conform fully to numerical value. As coefficient of the fifth-order nonlinear term increases, period of the time-response increases. However, increase of third-order nonlinear term coefficient leads to reduced period. It can be also concluded that compared to other current methods, Homotopy analytic method is a highly accurate and powerful method for solving nonlinear differential equations which makes it a good choice in solving a variety of engineering problems.

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