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# Frequency Analysis of Embedded Orthotropic Circular and Elliptical Micro/Nano-Plates Using Nonlocal Variational Principle

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**ABSTRACT**

In this paper, a continuum model based on the nonlocal elasticity theory is developed for vibration analysis of embedded orthotropic circular and elliptical micro/nano-plates. The nano-plate is bounded by a Pasternak foundation. Governing vibration equation of the nonlocal nano-plate is derived using Nonlocal Classical Plate Theory (NCPT). The weighted residual statement and the Galerkin method are applied to obtain a Quadratic Functional. The Ritz functions are used to form an assumed expression for transverse displacement which satisfies the kinematic boundary conditions. The Ritz functions eliminate the need for mesh generation and thus large degrees of freedom arising in discretization methods such as Finite Element Method (FEM). Effects of nonlocal parameter, lengths of nano-plate, aspect ratio, mode number, material properties and foundation parameters on the nano-plate natural frequencies are investigated. It is shown that the natural frequencies depend on the non-locality of the micro/nano-plate, especially at small dimensions.



**Keywords:** Nonlocal elasticity theory; Frequency analysis; Elliptical nano-plate; Variational principle



## 1 INTRODUCTION

SUPERIOR mechanical, chemical and electrical properties of nanostructures cause their wide usage in different nano-devices such as nano-sensors, nano-actuators and nano-composites. For this reason, proper physical and mechanical analysis of these structures encompasses numerous advantages. After the invention of Carbon Nano-tubes (CNTs) [1] and graphene sheets [2], both experimental and theoretical studies on micro- and nano-engineering of nanostructures have been accelerated. However, experimental measurements are hard to develop and depend on the development of devices for manipulation of nano-sized objects. Numerical techniques based on semi-empirical approaches [3, 4] such as molecular dynamic simulation, density functional theory, etc, provide a balance between accuracy and efficiency.

Although these numerical methods produce results which are in good agreement with experimental results, a large amount of computational capacities are needed, especially when the dimensions of the considered structure increase. Therefore, developing an appropriate mathematical model for analysis of nanostructures is an important issue. Recently, continuum modeling of nanostructures has been the subject of much attention [5-10]. As dimensions

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of a system reduce to the micro/nano scale, they become comparable to the inter-atomic or inter-molecular spacing of the system, and therefore, the material can no longer be modeled as a continuum. Moreover, at micro/nano scale, the influence of long-range inter-atomic and inter-molecular cohesive forces on the static and dynamic properties tends to be significant and cannot be neglected. These are known as “Quantum” or “Small scale” effects. Classical continuum models are unable to account for quantum effects. Therefore, there is a need to modify the classical continuum theories to account for these effects.

Modified continuum models that benefit from the computational efficiency of classical continuum models, at the same time, produce accurate results comparable to atomistic models [11]. These models can be effectively used to simulate very small to very large systems. A well-known class of modified continuum models is based on the concept of the nonlocal elasticity theory introduced by Eringen [11, 12]. In classical (local) continuum theories, it is assumed that stress state at a point in the continuum depends uniquely on strains at that point. In contrast, according to the nonlocal elasticity theory, it is assumed that stress state at a point depends on strains at all points of the continuum especially on those which are included in effective neighboring domains [12].

Vibration analysis of nanostructures is an important issue for proper design and usage of many NEMS devices such as oscillators, sensors and actuators. There exist numerous studies on the use of nonlocal continuum models for vibration analysis of CNTs and similar micro/nano-beams [13-19]. In spite of the importance of structural analysis of micro/nano-plates, few works have been reported on their theoretical modeling, as compared to the explorative studies on micro/nano-beams.

Circular and elliptical micro/nano-plates belong to a class of nano-plates which have special application in vibrating micro/nano-structures such as MEMS and NEMS [20]. Similar to CNTs, nano-plates possess superior mechanical properties [21-24]. Pradhan and Phadikar [25, 26] reported transverse vibration analysis of embedded single- and multi-layered graphene sheets. Aydogdu and Tolga [27] investigated the effect of different boundary conditions on buckling and vibration of nonlocal plates by Levy method. Jomehzadeh and Saidi [28, 29] studied three dimensional vibration of single-layered and large amplitude vibration of multi- layered graphene sheets. In all of these works it is suggested that for accurate prediction of dynamic response of micro/nano-plates, the Eringen nonlocal elasticity theory should be included in the continuum modeling.

In almost all of the studies conducted on frequency analysis of nano-plates, the shape of nano-plates has received less attention. Duan and Wang [30] reported exact solution for axisymmetric bending of circular graphene sheets based on the nonlocal elasticity theory. Farajpour et al. [31] reported axisymmetric buckling of circular graphene sheets using nonlocal continuum plate model. Further, Babaie and Shahidi [32] studied small scale effects on the buckling of quadrilateral nano-plates using the Galerkin method. Malekzadeh et al. [33, 34] investigated thermal buckling and free vibration of orthotropic arbitrary straight-sided quadrilateral nano-plates using the nonlocal classical and first order plate theories.

It is recognized that exact analytical solutions of plate vibration are only possible for plates with simple shapes like rectangle or circle, under certain boundary and loading conditions. For frequency analysis of plates with arbitrary shapes, numerical methods such as finite difference, finite element or finite strip method [35] are usually used. Although these methods provide a general framework for vibration analysis of nano-plates, they need large computational capacities. In FE analysis, for instance, due to the large number of discretization nodes, a large computational capacity is needed for an adequate approximation of the curved boundaries [36]. However, these inconveniences and approximations of boundary conditions via discretization do not arise when using the well-known Ritz method. Being a numerical approximate method, it eliminates the need for discretization by viewing the entire plate as a single super element. The Ritz method has been applied in different studies including static or dynamic, linear or nonlinear analysis of the plates with arbitrary geometry and edge conditions [37, 38]. Adali [39, 40] reported the variational principle for vibration of multi-layered graphene sheets based on the nonlocal elasticity theory without considering the foundation effects. Variational formulation for finite element analysis of nonlocal nano-plates was also reported by Phadikar and Pradhan [41]. Most recently, the nonlocal elasticity theory has been widely used for vibration analysis of micro/nano-plates and grapheme sheets [42-47].

In the present, work a nonlocal continuum model is developed for frequency analysis of embedded orthotropic circular and elliptical micro/nano-plates based on the Classical Plate Theory (CPT). A plate with elliptical shape is considered due to its relatively complicated shape. The effect of elastic foundation is also considered in the quadratic form for the micro/nano-plate. The principle of virtual work is used to derive the governing vibration equation and an approach similar to [41] is taken to obtain the weak form of the equation. Quadratic functional for vibration of the embedded nano-plate is obtained by the Galerkin method. Using the quadratic functional, natural frequencies can be found for nano-plates with an arbitrary shape. This procedure is similar to the Rayleigh-Ritz method in the local plate theory. Effects of the nonlocal parameter, nano-plate lengths, aspect ratio, mode number, material property and elastic foundation on the non-dimensional frequencies are illustrated. The results show noticeable effect of non-locality on the frequencies of small sized micro/nano-plates.

## 2 FORMULATION

The discrete and nonlocal continuum models of an embedded elliptical nano-plate are shown in Fig. 1 In order to obtain the governing equations, the Cartesian coordinate system is chosen with the origin fixed at the center of the mid-plane. The *x*, *y* and *z* coordinate axes are taken along the major axis, minor axis and thickness of the plate, respectively.

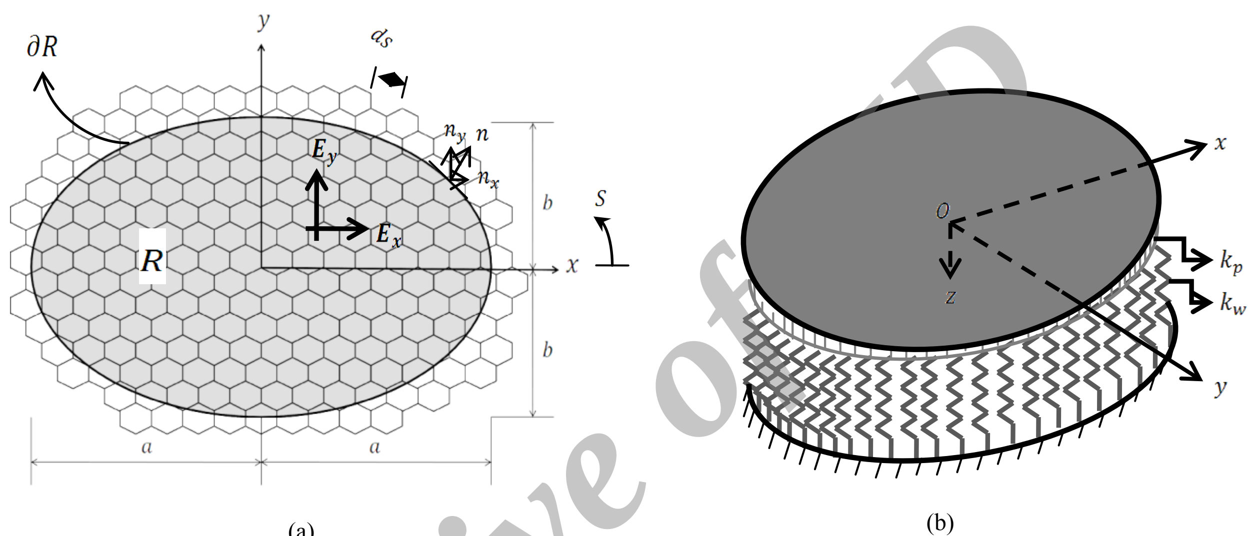


**Fig. 1**
Schematic of an embedded elliptical nano-plate: (a) Discrete model, (b) Nonlocal continuum model.

According to the CPT, the displacement fields at time *t* are as follow

$$u = u_0(x,y,t) - z\frac{\partial w}{\partial x}, \quad v = v_0(x,y,t) - z\frac{\partial w}{\partial y}, \quad w = w(x,y,t) \tag{1}$$

where $u_0, v_0$ and $w$ denote the displacements of the point $(x,y,0)$ along $x$ ,$y$ and $z$ directions, respectively.

The strain components are calculated as:

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{yy} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}\right) \quad \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \tag{2}$$

According to Eringen [11, 12], the nonlocal constitutive equation of a Hookean solid can be introduced by the following differential equation

$$\sigma^{(nl)} - \alpha l_e^2 \nabla^2 \sigma^{(nl)} = \sigma^{(l)} = S\varepsilon, \quad \alpha = \left(\frac{e_0 l_i}{l_e}\right)^2 \tag{3}$$

where $\sigma^{(l)}, \sigma, S$ and $\varepsilon$ denote local stress tensor, nonlocal stress tensor, elasticity tensor and strain tensor, respectively, defined as:

$$\sigma^{(l)} = \begin{Bmatrix} \sigma_{xx}^{(l)} \\ \sigma_{yy}^{(l)} \\ \sigma_{xy}^{(l)} \end{Bmatrix}, \quad \sigma^{(nl)} = \begin{Bmatrix} \sigma_{xx}^{(nl)} \\ \sigma_{yy}^{(nl)} \\ \sigma_{xy}^{(nl)} \end{Bmatrix}, \quad S = \begin{bmatrix} \dfrac{E_x}{1-\nu_x\nu_y} & \dfrac{\nu_x E_y}{1-\nu_x\nu_y} & 0 \\ \dfrac{\nu_y E_x}{1-\nu_x\nu_y} & \dfrac{E_y}{1-\nu_x\nu_y} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix}, \quad \varepsilon = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} \tag{4}$$

In Eq. (3) $\alpha$ represents the nonlocal effect that depends on a characteristic length ratio $l_i/l_e$ in which $l_i$ is an internal characteristic length (lattice parameter, size of grain, granular distance, distance between atomic bonds) and $l_e$ is an external characteristic length (wavelength, size or dimension of a sample of the system). The parameter $e_0$ is a constant that its value should be determined based on the material properties of the nano-plate [17]. It is obvious that for the well-known classical constitutive equation, $\alpha = 0$ . Moreover, the following stress resultants are used

$$N = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma^{(nl)} dz, \quad M = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma^{(nl)} dz, \qquad N = \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}, \quad M = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} \tag{5}$$

where *h* denotes the nano-plate thickness.

According to Eq. (3), the nonlocal effect enters through the constitutive relations. The principle of virtual work [48] which is independent of the constitutive relations can be applied to derive the equilibrium equations for nonlocal plates in the form of

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = m_0 \frac{\partial^2 u}{\partial t^2} \tag{6a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = m_0 \frac{\partial^2 v}{\partial t^2} \tag{6b}$$

$$\begin{aligned} &\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_{yy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{yx}\frac{\partial w}{\partial x}\right) \\ &+ q_0 - k_w w + k_p\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - m_0\frac{\partial^2 w}{\partial t^2} + m_2\left(\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial y^2 \partial t^2}\right) = 0 \end{aligned} \tag{6c}$$

where $k_w$ and $k_p$ denote Winkler and Pasternak coefficients of the elastic foundation, respectively. The Winkler modulus represents the foundation as a set of linear springs; however, the Pasternak coefficient represents the shear effect of the elastic foundation. In Eqs. (6), $m_0$ and $m_2$ are mass moments of inertia defined as:

$$m_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho dz, \quad m_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho z^2 dz \tag{7}$$

where $\rho$ denotes the nano-plate density.

Using Eqs. (2-5), the moment resultants can be expressed in terms of displacements as:

$$M-\mu\nabla^2 M=-Dk \tag{8}$$

Here, $\mu=(e_0 l_i)^2$ is the nonlocal parameter, $D$ is the bending rigidity matrix and $k$ is the curvature matrix of the nano-plate defined as follow

$$D=\frac{h^3}{12}S, \qquad k=\left\{\frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{\partial y^2} \quad 2\frac{\partial^2 w}{\partial x\partial y}\right\}^T \tag{9}$$

Using Eqs. (6c) and (8) and assuming a modal solution as $w(x,y,t)=W(x.y)e^{i\omega t}$, the following general governing equation will be obtained for a nonlocal plate in terms of displacements [27, 39, 40]

$$\begin{aligned}&D_{11}\frac{\partial^4 W}{\partial x^4}+2(D_{12}+2D_{33})\frac{\partial^4 W}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 W}{\partial y^4}-(1-\mu\nabla^2)\left[q_0-k_w W+k_p\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)+m_0\omega^2 W\right.\\&\left.-m_2\omega^2\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)+\frac{\partial}{\partial x}\left(N_{xx}\frac{\partial W}{\partial x}\right)+\frac{\partial}{\partial y}\left(N_{yy}\frac{\partial W}{\partial x}\right)+\frac{\partial}{\partial x}\left(N_{xy}\frac{\partial W}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{yx}\frac{\partial W}{\partial x}\right)\right]=0\end{aligned} \tag{10}$$

where $W$ is a continuous function which represents the deflection of the nano-plate middle surface and $\omega$ denotes the natural circular frequency.

## 3 SOLUTION PROCEDURE

### *3.1 Weak form formulation*

The weighted residual statement corresponding to Eq. (10) can be written as:

$$\begin{aligned}&\iint_R\left\{D_{11}\frac{\partial^4 W}{\partial x^4}+2(D_{12}+2D_{33})\frac{\partial^4 W}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 W}{\partial y^4}-(1-\mu\nabla^2)\left[q_0-k_w W+k_p\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)+m_0\omega^2 W\right.\right.\\&\left.\left.-m_2\omega^2\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)+\frac{\partial}{\partial x}\left(N_{xx}\frac{\partial W}{\partial x}\right)+\frac{\partial}{\partial y}\left(N_{yy}\frac{\partial W}{\partial y}\right)+\frac{\partial}{\partial x}\left(N_{xy}\frac{\partial W}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{yx}\frac{\partial W}{\partial x}\right)\right]\right\}\chi dxdy=0\end{aligned} \tag{11}$$

where $x$ denotes the weight function.

By breaking the integration in Eq. (11), the weak form will be obtained as:

$$\iint_R \dot{\Pi}^{(nl)}(x,y)dxdy+\oint_{\partial R}\Lambda(s)ds=0 \tag{12a}$$

$$\dot{\Pi}^{(nl)}(x,y)=\dot{U}(x,y)-\dot{T}(x,y)+\dot{V}(x,y)-\dot{Q}(x,y) \tag{12b}$$

where

$$\begin{aligned}\dot{U}(x,y)=&D_{11}\frac{\partial^2 W}{\partial x^2}\frac{\partial^2\chi}{\partial x^2}+D_{12}\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2\chi}{\partial y^2}+\frac{\partial^2 W}{\partial y^2}\frac{\partial^2\chi}{\partial x^2}\right]+D_{22}\frac{\partial^2 W}{\partial y^2}\frac{\partial^2\chi}{\partial y^2}+4D_{33}\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2\chi}{\partial x\partial y}\\&+k_w\left[W\chi+\mu\left(\frac{\partial W}{\partial x}\frac{\partial\chi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\chi}{\partial y}\right)\right]+k_p\left[\frac{\partial W}{\partial x}\frac{\partial\chi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\chi}{\partial y}+\mu\left(\frac{\partial^2 W}{\partial x^2}\frac{\partial^2\chi}{\partial x^2}+2\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2\chi}{\partial x\partial y}+\frac{\partial^2 W}{\partial y^2}\frac{\partial^2\chi}{\partial y^2}\right)\right]\end{aligned} \tag{12c}$$

$$\dot{T}(x,y)=\omega^2\times$$
$$\left\{m_0\left[W\chi+\mu\left(\frac{\partial W}{\partial x}\frac{\partial\chi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\chi}{\partial y}\right)\right]+m_2\left[\frac{\partial W}{\partial x}\frac{\partial\chi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\chi}{\partial y}+\mu\left(\frac{\partial^2 W}{\partial x^2}\frac{\partial^2\chi}{\partial x^2}+2\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2\chi}{\partial x\partial y}+\frac{\partial^2 W}{\partial y^2}\frac{\partial^2\chi}{\partial y^2}\right)\right]\right\} \tag{12d}$$

$$\dot{V}(x,y)=N_{xx}\left[\frac{\partial W}{\partial x}\frac{\partial\chi}{\partial x}+\mu\left(\frac{\partial^2 W}{\partial x^2}\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2\chi}{\partial x\partial y}\right)\right]+N_{yy}\left[\frac{\partial W}{\partial y}\frac{\partial\chi}{\partial y}+\mu\left(\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2\chi}{\partial x\partial y}+\frac{\partial^2 W}{\partial y^2}\frac{\partial^2\chi}{\partial y^2}\right)\right]$$
$$+N_{xy}\left[\frac{\partial W}{\partial x}\frac{\partial\chi}{\partial y}+\frac{\partial W}{\partial y}\frac{\partial\chi}{\partial x}+\mu\left(\frac{\partial^2 W}{\partial x^2}\frac{\partial^2\chi}{\partial x\partial y}+\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\frac{\partial^2\chi}{\partial x\partial y}+\frac{\partial^2 W}{\partial x\partial y}\frac{\partial^2\chi}{\partial y^2}\right)\right] \tag{12e}$$

$$\dot{Q}(x,y)=q_0\left[\chi-\mu\left(\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}\right)\right] \tag{12f}$$

$$\Lambda(s)=$$
$$\chi\left\{\frac{\partial^3 W}{\partial x^3}(D_{11}+\mu(k_p-m_2\omega^2+N_{xx}))n_x+\frac{\partial^3 W}{\partial x^2\partial y}(D_{12}+\mu(k_p-m_2\omega^2+N_{yy}+2N_{xy}\frac{n_x}{n_y}))n_y+\frac{\partial^3 W}{\partial x\partial y^2}(D_{12}\right.$$
$$+4D_{33}+\mu(k_p-m_2\omega^2+N_{xx}+2N_{xy}\frac{n_y}{n_x}))n_x+\frac{\partial^3 W}{\partial y^3}(D_{22}+\mu(k_p-m_2\omega^2+N_{yy}))n_y-\frac{\partial W}{\partial x}(k_p-m_2\omega^2+N_{xx}$$
$$\left.+N_{xy}\frac{n_y}{n_x}+\mu(k_w+m_0\omega^2))n_x-\frac{\partial W}{\partial y}(k_p-m_2\omega^2+N_{xy}\frac{n_x}{n_y}+N_{yy}+\mu(k_w+m_0\omega^2))n_y+\mu\left(\frac{\partial q_0}{\partial x}n_x+\frac{\partial q_0}{\partial y}n_y\right)\right\}$$
$$-\frac{\partial\chi}{\partial x}\left\{\frac{\partial^2 W}{\partial x^2}(D_{11}+\mu(k_p-m_2\omega^2+N_{xx}+N_{xy}\frac{n_y}{n_x}))n_x+\frac{\partial^2 W}{\partial x\partial y}(4D_{33}+\mu(k_p-m_2\omega^2+N_{xx}+N_{xy}\frac{n_x}{n_y}))n_y\right.$$
$$\left.\frac{\partial^2 W}{\partial y^2}(D_{12}n_x)+\mu q_0\right\}-\frac{\partial\chi}{\partial y}\left\{\frac{\partial^2 W}{\partial y^2}(D_{22}+\mu(k_p-m_2\omega^2+N_{xy}\frac{n_x}{n_y}+N_{yy}))n_y+\mu\frac{\partial^2 W}{\partial x\partial y}(k_p-m_2\omega^2+N_{xy}\frac{n_y}{n_x}\right.$$
$$\left.+N_{yy}))n_x+\frac{\partial^2 W}{\partial x^2}(D_{12}n_y)+\mu q_0\right\} \tag{13}$$

Here, $n_x, n_y$ represent the components of the unit normal vector on the boundary $\partial R$ of the nano-plate (see Fig. 1).

### *3.2 Ritz functions*

The Ritz functions are made from the product of a basic function and a set of orthogonal polynomials the degree of which may be increased until the desired accuracy is achieved. The basic function is defined by the product of equations of the specified boundary shape of the plate which are raised to the power of either 0, 1 or 2 corresponded respectively to the free, simply supported and clamped edges. The basic function ensures automatic satisfaction of the kinematic boundary conditions at the outset without the need to use Lagrangian multipliers (as in the Lagrangian multiplier method) [36]. In the present study the pb-2 type of Ritz functions is used which denotes polynomials, boundary expression and two dimensional. The pb-2 Ritz functions eliminates the tedious task of choosing the form of the infinite series or trigonometric or algebraic functions to suit the conditions of support along the edges. Accordingly, transverse deflection of the nano-plate mid-surface can be defined as:

$$W(x,y)=\sum_{q=0}^{p}\sum_{r=0}^{q}C_i\Phi_i(x,y) \tag{14}$$

In which, $C_i$ are the unknown coefficients to be varied and the Ritz functions, $\Phi_i$, are defined as:

$$\Phi_i=\left[1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right]^k\varphi_i(x,y) \tag{15}$$

Here, the term $1-(x/a)^2-(y/b)^2=0$ is the equation of the boundary which is an ellipse with $a$ and $b$ as its semi-major and semi-minor axis, respectively. In Eq. (15), $k$ is the power of the geometrical shape equation which is set to satisfy kinematical boundary conditions. Note that for clamped edges $k=2$. Also, $\varphi_i$ are polynomial functions in the form of

$$\varphi_i(x,y)=x^{q-r}y^r \tag{16}$$

where

$$i=\frac{(q+1)(q+2)}{2}-r \tag{17}$$

In Eq. (14), $p$ is the degree of polynomial set that may be increased until the desired accuracy is achieved.

*3.3 Quadratic Functional*

Principle of minimum total potential energy is valid under the assumption that stresses at a point can be uniquely defined in terms of the strains at that point [41]. However, it is found from differential form of the nonlocal elasticity that stresses at a point cannot be obtained unless stresses at the neighborhood of that point are known. Therefore, an explicit relation between stress and strain components cannot be found. In the case of integral form of the nonlocal elasticity theory, the variational principle is introduced by Eringen [12] for a homogeneous and anisotropic thermoelastic material. Furthermore, a variational formulation has been derived for FE analysis of the nano-plates by Phadikar and Pradhan [41]; but it lacked the effect of rotational vibration and elastic foundation. The pb-2 Ritz method has some advantages which were reported by Leiw and Wang [36]. Here, an inverse approach similar to the work of Phadikar and Pradhan [41] is taken to obtain a quasi quadratic form of the total potential or complementary energy based on the differential form of the nonlocal constitutive equation.

Using the Galerkin method, i.e. $x=\Phi_i$, it can be seen that all of the terms in the expression of $\dot{\Pi}^{(nl)}(x,y)$ are either bilinear and symmetric or linear. Therefore, the following relation can be concluded

$$\dot{\Pi}^{(nl)}(x,y)=\frac{\partial\Pi^{(nl)}(x,y)}{\partial C_i} \tag{18}$$

By integrating Eq. (18) and neglecting the boundary term $\Lambda(s)$ in Eq. (12a) for a clamped plate, the quasi expression for the total nonlocal "Quadratic Functional" of orthotropic nano-plates, according to CPT, can be found as:

$$\Pi_{total}^{(nl)}=\iint_R \Pi^{(nl)}(x,y)dxdy \tag{19a}$$

$$\Pi^{(nl)}(x,y)=U(x,y)-T(x,y)+V(x,y)-Q(x,y) \tag{19b}$$

in which

$$U(x,y)=\frac{1}{2}\left\{D_{11}\left(\frac{\partial^2 W}{\partial x^2}\right)^2+2D_{12}\left(\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right)+D_{22}\left(\frac{\partial^2 W}{\partial y^2}\right)^2+4D_{33}\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2+k_w\left[W^2+\mu\left(\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2\right)\right]\right.$$
$$\left.+k_p\left[\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2+\mu\left(\left(\frac{\partial^2 W}{\partial x^2}\right)^2+2\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 W}{\partial y^2}\right)^2\right)\right]\right\} \quad (20a)$$

$$T(x,y)=\frac{1}{2}m_0\omega^2\times$$
$$\left\{W^2+\mu\left(\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2\right)+\frac{m_2}{m_0}\left[\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2+\mu\left(\left(\frac{\partial^2 W}{\partial x^2}\right)^2+2\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 W}{\partial y^2}\right)^2\right)\right]\right\} \quad (20b)$$

$$V(x,y)=\frac{1}{2}N_{xx}\left\{\left(\frac{\partial W}{\partial x}\right)^2+\mu\left(\left(\frac{\partial^2 W}{\partial x^2}\right)^2+\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right)+\frac{N_{yy}}{N_{xx}}\left[\left(\frac{\partial W}{\partial y}\right)^2+\mu\left(\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 W}{\partial y^2}\right)^2\right)\right]\right.$$
$$\left.+2\frac{N_{xy}}{N_{xx}}\left[\frac{\partial W}{\partial x}\frac{\partial W}{\partial y}+\mu\left(\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial x\partial y}+\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W}{\partial x\partial y}\right)\right]\right\} \quad (20c)$$

$$Q(x,y)=q_0\left[W-\mu\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)\right] \quad (20d)$$

In the previous studies [15, 25, 41] , it has been shown that the results obtained through nonlocal analysis depend on the boundary conditions and in some cases, like bending analysis of clamped nano-beams and nano-plates, lead to unexpected values.

It can be found from Eqs. (12) and (13) that for the clamped boundary condition the variational principle is independent of the term related to the boundary of the nano-plate, i.e. similar to the local plate, it can be calculated only in the region of the plate. Also, it should be noted that for other constraints the boundary term in Eq. (13) should be considered [41].

Potential/complementary energy for the local orthotropic plates (Kim, 2003) can be found by putting $\mu=k_w=k_p=0$ in Eq. (19). Also, for $k_w=k_p=0$, the variational principle reported by Adali [40] will be obtained. Similar to the Rayleigh-Ritz method for classical (local) plates, the quasi potential energy in Eq. (19) can be used for the analysis of the clamped nano-plates with an arbitrary shape. The word "quasi" is employed here because the potential form in Eq. (19) is obtained from Eq. (18) and not directly through calculating the energy terms.

For generality and convenience, the coordinates are normalized by

$$\xi=\frac{x}{a},\qquad \eta=\frac{y}{b} \quad (21)$$

Substituting Eq. (14) into Eq. (19) and minimizing with respect to the coefficients $C_i$, i.e. $\frac{\partial\Pi_{tot}^{(nl)}}{\partial C_i}=0$, for vibration and buckling analysis the following set of simultaneous linear equations will be obtained

$$\left(\left[K\right]-\beta_1\lambda_\omega^2\left[M\right]-\beta_2\lambda_b\left[B\right]\right)\left\{C\right\}=\left\{0\right\} \quad (22)$$

where $[K],[M]$ and $[B]$ are stiffness, mass and buckling matrices of the nano-plate, respectively, defined in Appendix A. The scalar indicators $\beta$ take on $\beta_1 = 1, \beta_2 = 0$ for vibration analysis and $\beta_1 = 0, \beta_2 = 1$ for buckling analysis.Non-dimensional frequency and buckling parameters are expressed as $\lambda_\omega = \omega a^2 \sqrt{\rho h / D_{11}}$ and $\lambda_b = |N_{xx}| a^2 / D_{11}$ , respectively. Eq. (22) is a standard eigenproblem which can be solved for the fundamental frequencies or buckling loads of the orthotropic circular and elliptical nano-plates. The corresponding eigenvector $\{C\}$ represents the vibration or buckling mode shape of the nano-plate. The calculated frequencies are illustrated as modified non-dimensional frequency parameter $\bar{\lambda}_\omega = 4\sqrt{\lambda_\omega} / \pi^2$. It should be noted that the Gaussian quadrature technique is used for numerical integration of Eq. (22).

## 4 RESULTS AND DISCUTIONS

### *4.1 Validation and convergence studies*

In order to establish the validation of the current work, the natural frequencies of an orthotropic square graphene sheet are compared with those obtained by using DQ solution [27] and FE method [41] in Table 1. Material and geometrical parameters of the nano-plate are: Young's moduli $E_x = 1765\ Gpa$, $E_y = 1588\ Gpa$, shear modulus $G_{xy} = 678.85\ Gpa$, mass density $\rho = 2.3\, g / cm^3$, Poisson's ratios $\nu_x = 0.3$, $\nu_y = 0.27$, thickness $h = 0.34nm$ and length $L = 10.2nm$. It can be seen that the results calculated by using Eq. (22) for $p = 10$ are in good agreement with those obtained by DQ solution [27] and FE analysis [41]. To choose a sufficient degree for polynomial set, a convergence study is performed and the results are shown in Table 2. Here, an isotropic elliptical nano-plate with Young's moduli $E_x = E_y = 1.06\ Tpa$, mass density $\rho = 2300\, kg / m^3$, Poisson's ratios $\nu_x = \nu_y = 0.3$ and thickness $h = 0.34nm$ is considered. A desired agreement can be seen between the obtained results and those reported by Wang and Wang [48] and Lam et al. [49]. Results of another convergence study including the nonlocal effect are presented in Table 3 where the semi major axis is taken as 5 nm. From Tables 1, 2 and 3 it is found that a set of degree $p = 10$ is sufficient for the convergence of the results and this value will be used to generate all the results presented herein.

**Table 1**
Comparison of frequency parameter $\omega L^2 \sqrt{\rho / E_x h^2}$ for an orthotropic square graphene sheet

| Nonlocal Parameter | Present work | | | DQM [Aydogdu, Tolga, 2011] | FEM [Adali, 2009] |
|---|---|---|---|---|---|
| | P=4 | P=8 | P=10 | | |
| 0 | 10.5951 | 10.5941 | 10.5941 | 10.5941 | 10.5533 |
| 1 | 9.5466 | 9.5457 | 9.5457 | 9.5456 | 9.5125 |
| 2 | 8.7535 | 8.7526 | 8.7526 | 8.7526 | 8.7242 |
| 3 | 8.1275 | 8.1267 | 8.1267 | 8.1267 | 8.1016 |
| 4 | 7.6177 | 7.6169 | 7.6169 | 7.6169 | 7.5949 |

**Table 2**
Results of convergence study of $\bar{\lambda}_\omega$ for a local ( $\mu = 0$ ) elliptical nano-plate, $a / b = 2$

| Degree of polynomial set (p) | Mode numbers | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 11.096 | 16.047 | 22.829 | 28.379 | 35.906 | 35.934 |
| 5 | 11.096 | 16.008 | 22.829 | 28.314 | 31.620 | 35.934 |
| 6 | 11.096 | 16.008 | 22.690 | 28.314 | 31.620 | 35.695 |
| 7 | 11.096 | 16.008 | 22.690 | 28.312 | 31.222 | 35.695 |
| 8 | 11.096 | 16.008 | 22.686 | 28.312 | 31.222 | 35.684 |
| 10 | 11.096 | 16.008 | 22.686 | 28.312 | 31.206 | 35.684 |
| Wang and Wang, 1994 | 11.097 | 16.005 | 22.684 | 28.317 | 31.203 | 35.681 |
| Lam et al, 1992 | 11.136 | 16.008 | 22.687 | 28.312 | 31.225 | 35.684 |

**Table 3**
lts of convergence study of $\bar{\lambda}_{\omega}$ for different nonlocal parameters, $a = 5nm$ , $a/b = 2$

| Degree of polynomial set (p) | Nonlocal parameter | | | | |
|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 |
| 4 | 11.0958 | 8.5034 | 7.1427 | 6.2749 | 5.6608 |
| 5 | 11.0958 | 8.5034 | 7.1427 | 6.2749 | 5.6608 |
| 6 | 11.0957 | 8.5033 | 7.1425 | 6.2746 | 5.6605 |
| 7 | 11.0957 | 8.5033 | 7.1425 | 6.2746 | 5.6605 |
| 8 | 11.0957 | 8.5033 | 7.1425 | 6.2746 | 5.6605 |
| 10 | 11.0957 | 8.5033 | 7.1425 | 6.2746 | 5.6605 |

4.2 *Nonlocal effect*

In this section, the small scale effect for different nano-plate geometries and mode numbers is investigated. The semi-major axis of the nano-plate is varied between 5 *nm* and 30 *nm* and the aspect ratio is increased from 1 to 4. The nonlocal parameter of the nano-plate is assumed to vary between $\mu = 0nm^2$ and 4 $nm^2$.

The effects of the major axis length and the nonlocal parameter on the frequency parameter are shown in Fig. 2 for $a/b = 2$. From this figure, it is found that the natural frequencies of the nonlocal nano-plate are smaller than those of the local (classical) plate. In addition, for each value of the semi-major axis by increasing the nonlocal parameter the frequency decreases. The reason is that when the nonlocal parameter increases, the small scale effect increases as well and this leads to a reduction in the nano-plate stiffness [25-31]. Also, as the length of major axis increases, the frequency parameter increases as well. This implies that by increasing the external characteristic length of the nano-plate (here the length of major axis) the small scale effect decreases (the internal characteristic length is assumed to be unchanged). In fact, the size dependency in the nonlocal elasticity theory enters through the second term in the left side of the nonlocal constitutive equation (Eq. (3)). By normalizing Eq. (3) and using Eq. (21), the size-dependency will be better understood and dimensional parameters will appear in the term includes Laplacian of the stress tensor. This can also be seen in terms related to the nonlocal effect in Eq. (22) (see Appendix A). The dimensional parameters have a decreasing effect on the non-locality of nano-structures [25-31]. By further increasing the major length, the curves approach towards the local one. Approximately for $a \geq 25$ *nm* all curves converge to those obtained by classical assumption ( $\mu = 0$ ). With $\mu = 4$ $nm^2$, the relative error percents due to ignoring the nonlocal effect for a =10 *nm* and a =30 *nm* are % 23.36 and % 3.66, respectively. Here and afterwards, the relative error percent is defined as:

$$\text{Relative error percent} = \frac{|\text{Local Result - Nonlocal Result}|}{|\text{Local Result}|} \times 100$$

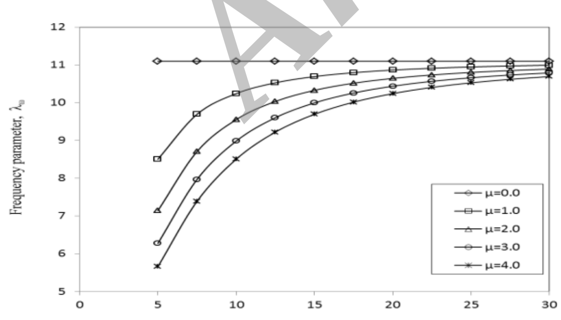


**Fig. 2**
Variations of the frequency parameter with semi-major axis length for different nonlocal parameters.

Fig. 3 shows the variations of the nano-plate frequency parameter against aspect ratio ( $a/b$ ) for various nonlocal parameter values. The semi-major axis of the elliptical nano-plate is taken as 5 *nm*. From this figure it is found that for greater aspect ratios, the nonlocal effect is more prominent. An interpretation for such behavior is that for a specific length of semi-major axis and nonlocal parameter, while the aspect ratio increases, the nano-plate becomes smaller and this leads to the increase of the small scale effect (size dependency). The relative error percents due to ignoring the nonlocal effect for $\mu = 4\ nm^2$ and aspect ratios $a/b = 1$ and $a/b = 4$ are % 31.35 and % 69.35, respectively. Obviously, it can be concluded that for higher aspect ratios the nonlocal theory should be used.

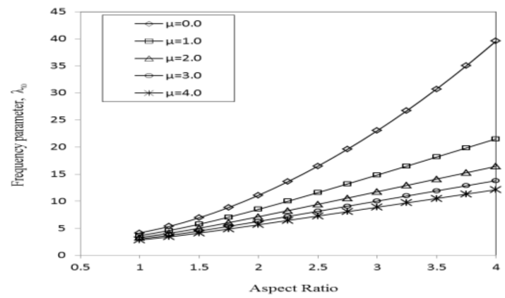


**Fig. 3**
Variations of the frequency parameter with aspect ratio for different nonlocal parameters.

The effect of small scale on the frequency parameter in different modes is shown in Fig. 4. It can be seen that in all of the modes, as the nonlocal parameter increases, the frequency parameter decreases. Further, the influence of the nonlocal effect becomes more pronounced at higher modes, as has been reported for square and circular nano-plates [26, 27, 31]. For the nonlocal parameter $\mu = 4\ nm^2$, the relative error percents due to ignoring the nonlocal effect for the first and fourth modes are % 48.98 and % 68.09, respectively.

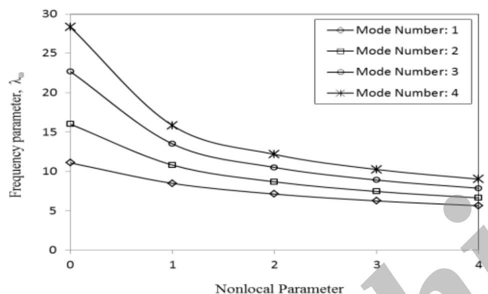


**Fig. 4**
Variations of the frequency parameter with small scale effect for different vibration modes.

4.3 *Effect of anisotropy*

An investigation is performed to account for the effect of anisotropy in the orthotropic case. For this purpose, variations of the non-dimensional frequencies with various anisotropy ratios i.e., $E_y/E_x$ are presented in Fig. 5 for different nonlocal parameters. This figure shows that degree of anisotropy has an increasing effect on the natural frequencies and the nonlocal effect becomes more prominent for greater degrees of anisotropy.

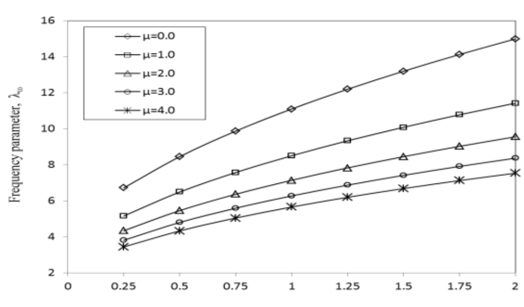


**Fig. 5**
Variations of the frequency parameter with degree of anisotropy for different nonlocal parameters.

4.4 *Effect of elastic foundation*

A two-parameter Winkler-Pasternak elastic medium is considered for proper presentation of the foundation effects. The foundation parameters can alter the total stiffness of the nano-plate (see Eq. (22) and Appendix A). Variations of frequency parameters with respect to the Winkler and Pasternak parameters are shown in Figs. 6 and 7 for various nonlocal parameter values. It can be observed that for all nonlocal parameters, by increasing the foundation stiffness either through springy (Winkler coefficient, $\bar{K}_W = 16K_W$) or shear effect (Pasternak coefficient, $\bar{K}_P = 4K_P$), the natural frequencies increase.

Variations of the relative error percent with the Winkler and Pasternak parameters are respectively shown in Figs. 8 and 9 for various nonlocal parameters. It may be seen that as the stiffness of the elastic foundation increases, the relative error percent decreases. Therefore, for stiffer foundations, the effect of considering nonlocal theory diminishes.

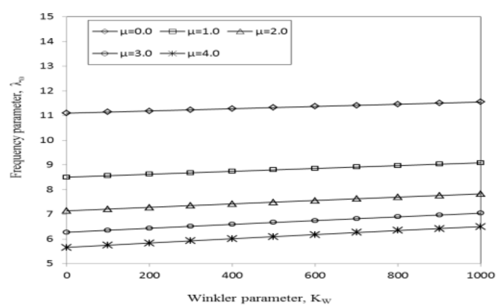


**Fig. 6**
Variations of the frequency parameter with Winkler parameter for different nonlocal parameters.

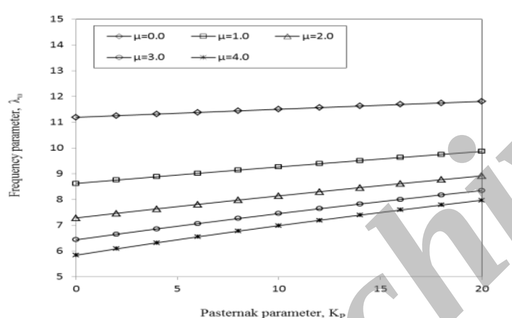


**Fig. 7**
Variations of the frequency parameter with Pasternak parameter for different nonlocal parameters.

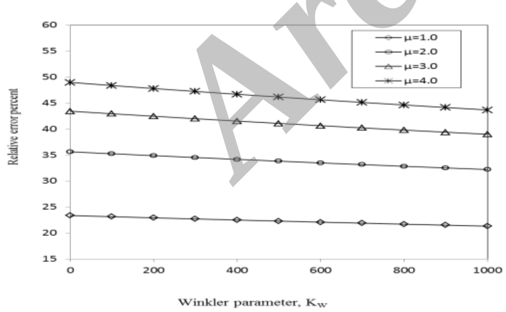


**Fig. 8**
Variations of the relative error percent with Winkler parameter for different nonlocal parameters.

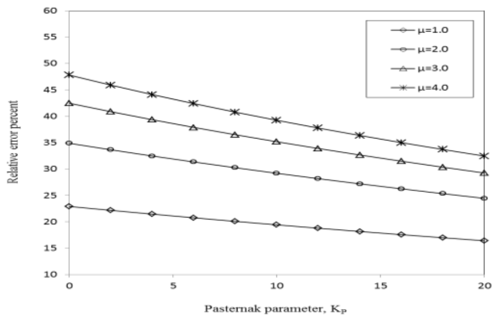


**Fig. 9**
Variations of the relative error percent with Pasternak parameter for different nonlocal parameters.

## 5 CONCLUSIONS

In the present study, transverse vibration analysis of the embedded orthotropic circular and elliptical micro/nano plates at the small scale is carried out using the NLPT model. The Winkler-Pasternak medium is considered for proper modeling of foundation effects. Based on the nonlocal Eringen's theory, the governing equation for transverse vibration of orthotropic nano-plates is derived. The variational formulation and the Galerkin method are applied to find the weak forms. The quadratic functional is obtained by considering elastic foundation and rotational vibration effects. The Ritz functions are employed to form an assumed expression for the displacement which satisfies the kinematic boundary conditions. The present method can be applied for vibration analysis of micro/nano-plates with any arbitrary shape.

The following conclusions can be drawn from this study:

- The small scale has a decreasing effect on the frequencies of orthotropic elliptical nano-plates.
- The nonlocal effect becomes more prominent when the length of the elliptical nano-plate decreases.
- The small scale effect increases when the nano-plate aspect ratio increases.
- In higher modes, the small scale effect is more pronounced.
- By increasing the degree of anisotropy, the non-dimensional frequencies of an orthotropic circular-elliptical nano-plate increase.
- The nonlocal effect becomes more prominent for greater degrees of anisotropy.
- The elastic foundation has an increasing effect on the relative error percent either through springy or shear effect.
- Stiffer elastic foundation decreases the nonlocal effects.

## APPENDIX A

$$\begin{aligned} K_{ij} &= \Upsilon_{ij}^{2020} + \nu_y \left(\frac{a}{b}\right)^2 \left(\Upsilon_{ij}^{2002} + \Upsilon_{ij}^{0220}\right) + \frac{E_y}{E_x}\left(\frac{a}{b}\right)^4 \Upsilon_{ij}^{0202} + 4\left(1-\nu_x\nu_y\right)\frac{G_{xy}}{E_x}\left(\frac{a}{b}\right)^2 \Upsilon_{ij}^{1111} \\ &+ K_W\left[\Upsilon_{ij}^{0000} + \mu\left(\frac{1}{a}\right)^2\left(\Upsilon_{ij}^{1010} + \left(\frac{a}{b}\right)^2 \Upsilon_{ij}^{0101}\right)\right] + K_P\left[\Upsilon_{ij}^{1010} + \left(\frac{a}{b}\right)^2 \Upsilon_{ij}^{0101} + \mu\left(\frac{1}{a}\right)^2\left(\Upsilon_{ij}^{2020} + \left(\frac{a}{b}\right)^2 \Upsilon_{ij}^{1111}\right)\right. \\ &\left. + \mu\left(\frac{1}{b}\right)^2\left(\Upsilon_{ij}^{1111} + \left(\frac{a}{b}\right)^2 \Upsilon_{ij}^{0202}\right)\right] \end{aligned} \tag{A.1}$$

$$\begin{aligned} M_{ij} &= \Upsilon_{ij}^{0000} + \mu\left(\frac{1}{a}\right)^2\left(\Upsilon_{ij}^{1010} + \left(\frac{a}{b}\right)^2 \Upsilon_{ij}^{0101}\right) \\ &+ \frac{1}{12}\left(\frac{h}{a}\right)^2\left[\Upsilon_{ij}^{1010} + \left(\frac{a}{b}\right)^2 \Upsilon_{ij}^{0101} + \mu\left(\frac{1}{a}\right)^2\left(\Upsilon_{ij}^{2020} + \left(\frac{a}{b}\right)^2 \Upsilon_{ij}^{1111}\right) + \mu\left(\frac{1}{b}\right)^2\left(\Upsilon_{ij}^{1111} + \left(\frac{a}{b}\right)^2 \Upsilon_{ij}^{0202}\right)\right] \end{aligned} \tag{A.2}$$

$$B_{ij} = \Upsilon_{ij}^{1010} + \mu\left(\frac{1}{a}\right)^2\left(\Upsilon_{ij}^{2020} + \left(\frac{a}{b}\right)^2\Upsilon_{ij}^{1111}\right) + \frac{N_{yy}}{N_{xx}}\left(\frac{a}{b}\right)^2\left[\Upsilon_{ij}^{0101} + \mu\left(\frac{1}{b}\right)^2\left(\Upsilon_{ij}^{1111} + \left(\frac{b}{a}\right)^2\Upsilon_{ij}^{0202}\right)\right]$$
$$+\frac{N_{xy}}{N_{xx}}\left(\frac{a}{b}\right)\left[\Upsilon_{ij}^{1001} + \Upsilon_{ij}^{0110} + \mu\left(\frac{1}{a}\right)^2\left(\Upsilon_{ij}^{2011} + \Upsilon_{ij}^{1120}\right) + \mu\left(\frac{1}{b}\right)^2\left(\Upsilon_{ij}^{0211} + \Upsilon_{ij}^{1102}\right)\right] \tag{A.3}$$

$$\Upsilon_{ij}^{klmn} = \iint_R \left[\frac{\partial^{k+l}\Phi_i(\xi,\eta)}{\partial\xi^k\partial\eta^l}\right]\left[\frac{\partial^{m+n}\Phi_j(\xi,\eta)}{\partial\xi^m\partial\eta^n}\right]d\xi d\eta \tag{A.4}$$

$$K_W = \frac{k_w a^4}{D_{11}}, \quad K_P = \frac{k_p a^2}{D_{11}} \tag{A.5}$$

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