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The Effect of Modified Couple Stress Theory on Buckling and Vibration Analysis of Functionally Graded Double-Layer Boron Nitride Piezoelectric Plate Based on CPT

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ABSTRACT

In this article, the effect of size-dependent on the buckling and vibration analysis of functionally graded (FG) double-layer boron nitride plate based on classical plate theory (CPT) under electro-thermo-mechanical loadings which is surrounded by elastic foundation is examined. This subject is developed using modified couple stress theory. Using Hamilton's principle, the governing equations of motion are obtained by applying a modified couple stress and von Karman nonlinear strain for piezoelectric material and Kirchhoff plate. These equations are coupled for the FG double-layer plate using Pasternak foundation and solved using Navier's type solution. Then, the dimensionless natural frequencies and critical buckling load for simply supported boundary condition are obtained. Also, the effects of material length scale parameter, elastic foundation coefficients and power law index on the dimensionless natural frequency and critical buckling load are investigated. The results demonstrate that the dimensionless natural frequency of the piezoelectric plate increases steadily by growing the power law index. Also, the effect of the power law index on the dimensionless critical buckling load of double layer boron nitride piezoelectric for higher dimensionless material length scale parameter is the most.

Keywords : Buckling and vibration analysis; Modified couple stress theory; Functionally graded double-layer Piezoelectric plate; CPT.

1 INTRODUCTION

IN the past years, the application of micro and nano scale structures has been expanded in many engineering devices such as atomic force, micro and nano probes and switches. Some theories are considered for the micro and nano structures including couple stress and strain gradient theories. The couple stress theory can be viewed as a special case of the strain gradient theory. The couple stress theory considers the rotations as a variable to describe curvature, while the strain gradient theory considers the strains as a variable to describe curvature. A number of investigations about these theories have been published in the scientific literatures [1-12].

Nowadays, many researchers are worked in the fields of bending, buckling, and vibration analysis of nanocomposite reinforced by single-walled carbon nanotube (SWCNT) (Ghorbanpour Arani et al. [13], Mohammadimehr et al. [14, 15]) or single-walled boron nitride nanotube (SWBNNT) (Mohammadimehr et al. [16])

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under electro-thermo-mechanical loadings. However, sudden changes in the layers of composite materials can cause to create crack and undesirable stress in the laminated composite material. Thus because of continuous variation of material properties in functionally graded materials (FGMs), there are not any interlaminar stresses for these materials unlike laminated composite material. Today, the FGMs due to their vast applications in many engineering fields are used in various systems such as micro/nano electromechanical systems (MEMS/NEMS). Size effect has a significant role on static and dynamic behavior of the micro and nano structures. For example, Asghari et al [17] studied the effect of size dependent on static and vibration analysis of FG micro beams based on modified couple stress theory. Also, the other studies are performed to investigate vibration, bending and buckling of the FGMs. Post-buckling of piezoelectric FGM rectangular plates under combined loadings is investigated by Liew et al [18]. Rao and Kuna [19] studied the fracture of functionally graded piezoelectric materials (FGPMs). They used interaction integrals based on three independent formulations to calculate stress and electric displacement intensity factors for cracks in the FGPMs. Comparison between the three formulations showed that if value of loading combination and non-homogeneity parameters be neglected, the three formulations have almost the same value of the normalized intensity factors. Golmakani and Kadkhodayan [20] analyzed the nonlinear bending of annular FGM plate based on first/third order shear deformation theories (FSDT/TSDT). They illustrated that the larger thickness becomes, the greater difference between FSDT and TSDT increases. Kim and Reddy [21] presented the analytical solution for bending, vibration and buckling of the FGM plates. They used TSDT and couple stress theory to obtain the equations of motion. Their results demonstrated that by decreasing power law index, the plate became stiff on the basis of mechanical properties of two constituents. Liu et al. [22] investigated the vibration analysis of piezoelectric simply supported rectangular nanoplates undergoing the electro-thermo-mechanical loadings based on the nonlocal and Kirchhoff theories. They found that the natural frequencies of piezoelectric nanoplates are very sensitive to the electro-mechanical loadings and insensitive to the thermal loading.

The problem of plates and beams resting on elastic foundations is one of the most important topics in engineering applications. For this problem, many studies have been performed; for instance, Ozgan and Daloglu [23] illustrated the effects of transverse shear strains on plates resting on elastic foundation using modified Vlasov model. They obtained the terms of vertical deflection and shear deformation stiffness matrices of the subsoil and evaluated using finite element method (FEM), and presented in explicit forms. Fallah and Aghdam [24] investigated large amplitude free vibration and buckling of an FGM Euler-Bernoulli beam resting on nonlinear elastic foundation. Their results indicated that by increasing nonlinear parameter of the elastic foundation, frequency and buckling load ratios increase. Ghorbanpour Arani et al. [25] investigated the dynamic stability of the double-walled carbon nanotube (DWCNTs) under axial loading embedded in an elastic medium by the energy method. Hygrothermal bending of the FGM plate resting on elastic foundation is analyzed by Zenkour [26]. Kiani et al [27] studied the static, dynamic and free vibration of an FGM doubly curved panel on elastic foundation. They obtained the governing equations of motion using the FSDT and modified Sanders shell theory. Khalili et al. [28] investigated buckling analysis of non-ideal laminated plate resting on elastic foundation. Their analytical results are compared with the obtained results of FEM and ANSYS program. They demonstrated that by increasing shear modulus of the elastic foundation, the buckling load of the plate increases. Rahmati and Mohammadimehr [29] presented the electro-thermo-mechanical vibration analysis of non-uniform and non-homogeneous boron nitride nanorod (BNNR) embedded in an elastic medium. They investigated the effects of attached mass, lower and higher vibrational mode, elastic medium, piezoelectric coefficient, dielectric coefficient, cross section coefficient, non-homogeneity parameter and small scale parameter on the natural frequency.

The governing equations of motion are obtained using Kirchhoff plate theory and Hamilton's principle in this article. Then, these equations can be solved using Navier's type solution. The effect of size dependent on the vibration and buckling of FG plate under electro-thermo-mechanical loadings surrounded by an elastic medium was investigated based on the modified couple stress theory.

2 THE EQUATION OF MOTION FOR SINGLE- LAYER PIEZOELECTRIC PLATE ON AN ELASTIC MEDIUM

The displacement fields of the classical plate theory (CPT) are given as [36]:

$$u_1(x,y,z,t)=u(x,y,t)-z\,\frac{\partial w(x,y,t)}{\partial x}$$
$$u_2(x,y,z,t)=v(x,y,t)-z\,\frac{\partial w(x,y,t)}{\partial y} \tag{1}$$
$$u_3(x,y,z,t)=w(x,y,t)$$

where u_1,u_2 and u_3 are the displacement fields. u , v and w denote the displacements in the x , y and z directions on the mid plane of CPT, respectively. In addition, t is time.

According to the displacement fields in Eq.(1) and the von-Karman nonlinear strains, the normal and shear strains can be obtained as the following form:

$$\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-z\,\frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_y=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2-z\,\frac{\partial^2 w}{\partial y^2}$$
$$\gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-2z\,\frac{\partial^2 w}{\partial x\,\partial y} \qquad \gamma_{xz}=\gamma_{yz}=\varepsilon_z=0 \tag{2}$$

Based on the modified couple stress theory, the components of the symmetric curvature tensor (χ_{ij}) are defined as [31]:

$$\chi_{ij}=\frac{1}{2}\left(\frac{\partial\theta_i}{\partial x_j}+\frac{\partial\theta_j}{\partial x_i}\right),\quad i,j=1,2,3 \tag{3}$$

where x_i denote the coordinate axes of a point on the mid plane of the plate that x_1,x_2, , and x_3 axes are corresponding to x , y and z axes, respectively. θ_i are the components of the rotation vector that can be expressed as follows [31]:

$$\theta_1=\frac{1}{2}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right) \qquad \theta_2=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right) \qquad \theta_3=\frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right) \tag{4}$$

Substituting Eq.(1) into Eq. (4) yields the following equations:

$$\theta_1=\frac{\partial w}{\partial y} \qquad \theta_2=-\frac{\partial w}{\partial x} \qquad \theta_3=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \tag{5}$$

Substituting Eq. (5) into Eq. (3) can be considered as:

$$\chi_x=\frac{\partial^2 w}{\partial x\,\partial y} \qquad \chi_y=-\frac{\partial^2 w}{\partial x\,\partial y} \qquad \chi_{xy}=\frac{1}{2}\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x^2}\right)$$
$$\chi_{xz}=\frac{1}{4}\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial x\,\partial y}\right) \qquad \chi_{yz}=\frac{1}{4}\left(\frac{\partial^2 v}{\partial x\,\partial y}-\frac{\partial^2 u}{\partial y^2}\right) \qquad \chi_z=0 \tag{6}$$

m_{ij} are the components of the deviatoric part of the symmetric couple stress tensor which are defined as follows [32]:

$$m_{ij}=\frac{E(z)}{1+\upsilon}l^2\chi_{ij} \tag{7}$$

where l is the material length scale parameter. Lagrangian function for FG double-layer boron nitride plate can be written as:

$$L = K - (U + W) \tag{8}$$

where L is the Lagrangian function and K , U and W denote the kinetic energy, strain energy, and virtual work done by the external forces, respectively. According to the Hamilton's principle, the variation form of Lagrangian function can be considered as [33]:

$$\int_o^T \delta L dt = \int_0^T (\delta U + \delta W - \delta K) dt = 0 \tag{9}$$

According to the modified couple stress theory, the variation form of strain energy is expressed as follows [31]:

$$\delta U = \int_V (\sigma_{ij}\delta\varepsilon_{ij} + m_{ij}\delta\chi_{ij})dV \tag{10}$$

where σ_{ij} and ε_{ij} are the components of stress and strain tensors, respectively. If i and j are equal to x , y , z then Eq. (10) can be rewritten as:

$$\begin{aligned}\delta U = \int_A \int_{-h/2}^{h/2} (&\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_{xy}\delta\gamma_{xy} + m_x \delta\chi_x + m_y \delta\chi_y \\ &+ 2m_{xy}\delta\chi_{xy} + 2m_{xz}\delta\chi_{xz} + 2m_{yz}\delta\chi_{yz})dA\,dz\end{aligned} \tag{11}$$

The resultant forces and moments may be defined as the following form [34]:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} dz \qquad \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} z\,dz \qquad \begin{Bmatrix} P_x \\ P_y \\ P_{xy} \\ P_{xz} \\ P_{yz} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} m_x \\ m_y \\ m_{xy} \\ m_{xz} \\ m_{yz} \end{Bmatrix} dz \tag{12}$$

Using Eqs. (12), substituting Eqs. (2) and (6) into Eq. (11) yields:

$$\begin{aligned}\delta U = \int_A \Bigg[& N_x \left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right) - M_x \frac{\partial^2 \delta w}{\partial x^2} + N_y \left(\frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\right) - M_y \frac{\partial^2 \delta w}{\partial y^2} \\ & + N_{xy}\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\right) - 2M_{xy}\frac{\partial^2 \delta w}{\partial x \partial y} + P_x \frac{\partial^2 \delta w}{\partial x \partial y} - P_y \frac{\partial^2 \delta w}{\partial x \partial y} \\ & + P_{xy}\left(\frac{\partial^2 \delta w}{\partial y^2} - \frac{\partial^2 \delta w}{\partial x^2}\right) + \frac{1}{2}P_{xz}\left(\frac{\partial^2 \delta v}{\partial x^2} - \frac{\partial^2 \delta u}{\partial x \partial y}\right) + \frac{1}{2}P_{yz}\left(\frac{\partial^2 \delta v}{\partial x \partial y} - \frac{\partial^2 \delta u}{\partial y^2}\right)\Bigg] dA\end{aligned} \tag{13}$$

The variation form of the kinetic energy is written as [33]:

$$\delta K = \int_A \int_{-h/2}^{h/2} \rho(z)\frac{\partial u_i}{\partial t}\delta\left(\frac{\partial u_i}{\partial t}\right) dA\,dz\ , \quad i = 1,2,3 \tag{14}$$

where $\rho(z)$ is the density of FG double-layer boron nitride plate. Substituting Eq. (1) into Eq. (14), one can obtain the following equation:

$$\delta K = \int_A \int_{-h/2}^{h/2} \rho(\dot{u}_1\delta\dot{u}_1 + \dot{u}_2\delta\dot{u}_2 + \dot{u}_3\delta\dot{u}_3)dA\,dz$$
$$= \int_A \left[I_0(\dot{u}\,\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\,\delta\dot{w}) + I_2\left(\frac{\partial\dot{w}}{\partial x}\frac{\partial\delta\dot{w}}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial\delta\dot{w}}{\partial y}\right) \right. \tag{15}$$
$$\left. -I_1\left(\dot{u}\,\frac{\partial\delta\dot{w}}{\partial x} + \frac{\partial\dot{w}}{\partial x} + \dot{v}\,\frac{\partial\delta\dot{w}}{\partial y} + \frac{\partial\dot{w}}{\partial y}\delta\dot{v}\right)\right] dA$$

where I_0, I_1, I_2 are the mass inertias which are defined as follows:

$$(I_0, I_1, I_2) = \int_{-h/2}^{h/2} \rho(1, z, z^2)dz \tag{16}$$

The variation form of the virtual work done by the external forces can be presented as follows [35, 36]:

$$\delta W = -\left[\int_\Omega (f_x\,\delta u + f_y\,\delta v + f_z\,\delta w + q_x\,\delta u + q_y\,\delta v + q_z\,\delta w + c_x\,\delta\theta_x + c_y\,\delta\theta_y \right. \tag{17a}$$
$$\left. + c_z\,\delta\theta_z + F_{elastic}\,\delta w)dxdy + \int_\Gamma (t_x\,\delta u + t_y\,\delta v + t_z\,\delta w + s_x\,\delta\theta_x + s_y\,\delta\theta_y + s_z\,\delta\theta_z)d\Gamma\right]$$

where Ω and Γ are the middle surface of the plate and the boundary of the middle surface, repectively. Also $(f_x, f_y, f_z), (c_x, c_y, c_z)$ and (q_x, q_y, q_z) denote the body forces, the body couples, and the tractions acting on Ω, respectively. Moreover, (t_x, t_y, t_z) and (s_x, s_y, s_z) are the Cauchy traction and surface couple acting on *s*, respectively. Then, the Eq. (17a) can be rewritten after simplifying as:

$$\delta W = -\left\{\int_\Omega \left[(f_x + q_x)\delta u + (f_y + q_y)\delta v + (f_z + q_z + F_{elastic})\delta w \right.\right.$$
$$\left. + c_x\,\frac{\partial\delta w}{\partial y} + c_y\,\frac{\partial\delta w}{\partial x} + \frac{c_z}{2}\left(\frac{\partial\delta v}{\partial x} - \frac{\partial\delta u}{\partial y}\right)\right] dxdy \tag{17b}$$
$$\left. + \int_\Gamma \left[t_x\,\delta u + t_y\,\delta v + t_z\,\delta w + s_x\,\frac{\partial\delta w}{\partial y} - s_y\,\frac{\partial\delta w}{\partial x} + \frac{s_z}{2}\left(\frac{\partial\delta v}{\partial x} - \frac{\partial\delta u}{\partial y}\right)\right] d\Gamma\right\}$$

where

$$F_{elastic} = K_w w(x, y) - K_G\,\nabla^2 w(x, y) \tag{18}$$

where K_w and K_G are the spring constant of the Winkler type and the shear constant of the Pasternak type, respectively. Substituting Eqs. (13), (15) and (17) into Eq. (9), the equations of motion and boundary conditions can be derived as follows:

$$\delta u: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \frac{1}{2}\left(\frac{\partial^2 P_{xz}}{\partial x\,\partial y} + \frac{\partial^2 P_{yz}}{\partial y^2}\right) + f_x + q_x + \frac{1}{2}\frac{\partial c_z}{\partial y} = I_0\ddot{u} - I_1\frac{\partial\ddot{w}}{\partial x} \tag{19a}$$

$$\delta v: \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} - \frac{1}{2}\left(\frac{\partial^2 P_{xz}}{\partial x^2} + \frac{\partial^2 P_{yz}}{\partial x\,\partial y}\right) + f_y + q_y - \frac{1}{2}\frac{\partial c_z}{\partial x} = I_0\ddot{v} - I_1\frac{\partial\ddot{w}}{\partial y} \tag{19b}$$

$$\delta w: \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial^2 P_{xy}}{\partial x^2} + \frac{\partial^2 P_y}{\partial x\,\partial y} - \frac{\partial^2 P_{xy}}{\partial y^2} - \frac{\partial^2 P_x}{\partial x\,\partial y} + N(w)$$
$$+ f_z + q_z + F_{elastic} + \frac{\partial c_y}{\partial x} - \frac{\partial c_x}{\partial y} = I_0\ddot{w} + I_1\left(\frac{\partial\ddot{u}}{\partial x} + \frac{\partial\ddot{v}}{\partial y}\right) - I_2\nabla^2\ddot{w} \tag{19c}$$

$$\delta u: N_x n_x + N_{xy} n_y + \frac{1}{2}\left(\frac{\partial P_{xz}}{\partial x} + \frac{\partial P_{yz}}{\partial y}\right) n_y + \frac{1}{2} c_z n_y - t_x - \frac{1}{2}\frac{\partial s_z}{\partial y} = 0 \tag{20a}$$

$$\delta v: N_{xy} n_x + N_y n_y - \frac{1}{2}\left(\frac{\partial P_{xz}}{\partial x} + \frac{\partial P_{yz}}{\partial y}\right) n_x - \frac{1}{2} c_z n_x - t_y + \frac{1}{2}\frac{\partial s_z}{\partial x} = 0 \tag{20b}$$

$$\begin{aligned}\delta w: &\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \frac{\partial P_{xz}}{\partial x} + \frac{\partial P_y}{\partial y} + I_1 \ddot{u} - I_2 \frac{\partial \ddot{w}}{\partial x}\right) n_x + c_y n_x - c_x n_y - t_z - \frac{\partial s_y}{\partial x} \\ &+ \frac{\partial s_x}{\partial y} + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} + I_1 \ddot{v} - I_2 \frac{\partial \ddot{w}}{\partial y} - \frac{\partial P_x}{\partial x} + \frac{\partial P_{xy}}{\partial y}\right) n_y + \frac{\partial M_{ns}}{\partial s} + \bar{N}(w) = 0\end{aligned} \tag{20c}$$

$$\frac{\partial \delta w}{\partial n}: M_{nn} = 0 \tag{20d}$$

where

$$N(w) = \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y}\right) \tag{21a}$$

$$\bar{N}(w) = \left(N_x \frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) n_x + \left(N_{xy}\frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y}\right) n_y \tag{21b}$$

$$M_{ns} = (M_y - M_x) n_x n_y + M_{xy}(n_x^2 - n_y^2) \qquad M_{nn} = M_x n_x^2 + M_y n_y^2 + 2M_{xy} n_x n_y \tag{21c}$$

3 CONSTITUTIVE EQUATIONS OF FG PIEZOELECTRIC PLATE

In a piezoelectric material, the constitutive equations of the FG double-layer boron nitride plate under electro-thermo-mechanical loadings are considered as follows [37]:

$$\begin{aligned}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{xz}\end{Bmatrix} = \frac{E(z)}{1-\upsilon^2}\begin{bmatrix}1 & \upsilon & 0 & 0 & 0\\ \nu & 1 & 0 & 0 & 0\\ 0 & 0 & \frac{1-\upsilon}{2} & 0 & 0\\ 0 & 0 & 0 & \frac{1-\upsilon}{2} & 0\\ 0 & 0 & 0 & 0 & \frac{1-\upsilon}{2}\end{bmatrix}\left(\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{bmatrix} - \begin{bmatrix}\alpha_x(z)\\ \alpha_y(z)\\ 0\\ 0\\ 0\end{bmatrix}\Delta T\right) \\ -\frac{E(z)}{1-\upsilon^2}\begin{bmatrix}e_{11}(z) & 0 & 0\\ e_{12}(z) & 0 & 0\\ 0 & e_{32}(z) & 0\\ 0 & 0 & e_{43}(z)\end{bmatrix}\begin{Bmatrix}E_x\\ E_y\\ E_z\end{Bmatrix}\end{aligned} \tag{22}$$

The constitutive relations of Eq. (22) are valid under plane stress assumption. Where $\upsilon, E(z)$ and ΔT are the Poisson's ratio, Young modulus, and temperature rise, respectively. $\alpha_x(z)$ and $\alpha_y(z)$ denote the thermal expansion coefficient in *x* and *y* directions, respectively. $e_{ij}(z)$ and E_i are the piezoelectric constants and electric fields, respectively. The electro-thermo-mechanical properties for the FG boron nitride piezoelectric plate are illustrated as the following form [11]:

$$
\begin{aligned}
&\alpha_x(z)=\alpha_{x2}+(\alpha_{x1}-\alpha_{x2})\left(\frac{1}{2}+\frac{z}{h}\right)^p \qquad \alpha_y(z)=\alpha_{y2}+(\alpha_{y1}-\alpha_{y2})\left(\frac{1}{2}+\frac{z}{h}\right)^p\\
&E(z)=E_2+(E_1-E_2)\left(\frac{1}{2}+\frac{z}{h}\right)^p \qquad \rho(z)=\rho_2+(\rho_1-\rho_2)\left(\frac{1}{2}+\frac{z}{h}\right)^p\\
&e_{11}(z)=e_{112}+(e_{111}-e_{112})\left(\frac{1}{2}+\frac{z}{h}\right)^p \qquad e_{12}(z)=e_{122}+(e_{121}-e_{122})\left(\frac{1}{2}+\frac{z}{h}\right)^p
\end{aligned}
\tag{23}
$$

where the subscripts 1 and 2 present the two materials used, and p is the power law index indicating the volume fraction of material. Substituting Eqs. (22) and (23) into Eq. (12) and integrating through the thickness of the plate, the results force and moments are obtained as follows:

$$
\begin{aligned}
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} &= A\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix}\begin{Bmatrix} \frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{Bmatrix} + B\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix}\begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x\,\partial y} \end{Bmatrix}\\
&+R_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} -E_x \\ 0 \\ 0 \end{Bmatrix}+F_1\begin{bmatrix} 1 & 0 & 0 \\ \upsilon & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} -\Delta T \\ 0 \\ 0 \end{Bmatrix}+G_1\begin{bmatrix} \upsilon & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} -\Delta T \\ 0 \\ 0 \end{Bmatrix}+S_1\begin{bmatrix} 0 & 0 & 0 \\ \upsilon & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} -E_x \\ 0 \\ 0 \end{Bmatrix}
\end{aligned}
\tag{24a}
$$

$$
\begin{aligned}
\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} &= B\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix}\begin{Bmatrix} \frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{Bmatrix} + D\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix}\begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x\,\partial y} \end{Bmatrix}+R_2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} -E_x \\ 0 \\ 0 \end{Bmatrix}\\
&+F_2\begin{bmatrix} 1 & 0 & 0 \\ \upsilon & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} -\Delta T \\ 0 \\ 0 \end{Bmatrix}+G_2\begin{bmatrix} \upsilon & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} -\Delta T \\ 0 \\ 0 \end{Bmatrix}+S_2\begin{bmatrix} 0 & 0 & 0 \\ \upsilon & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} -E_x \\ 0 \\ 0 \end{Bmatrix}
\end{aligned}
\tag{24b}
$$

$$
\begin{Bmatrix} P_x \\ P_y \\ P_{xy} \\ P_{xz} \\ P_{yz} \end{Bmatrix} = A_n\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \frac{\partial_2 w}{\partial w\partial y} \\ -\frac{\partial_2 w}{\partial x\partial y} \\ \frac{\partial_2 w}{\partial y^2}-\frac{\partial_2 w}{\partial x^2} \\ \frac{1}{2}\left(\frac{\partial_2 v}{\partial x^2}-\frac{\partial_2 u}{\partial x\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial_2 v}{\partial x\partial y}-\frac{\partial_2 u}{\partial y^2}\right) \end{Bmatrix}
\tag{24c}
$$

where

$$(A,B,D)=\int_{-h/2}^{h/2}\frac{E(z)}{1-\upsilon^2}(1,z,z^2)dz$$
$$(A_n,B_n,D_n)=l^2\frac{1-\upsilon}{2}(A,B,D)$$
$$(R_1,S_1,F_1,G_1)=\int_{-h/2}^{h/2}\frac{E(z)}{1-\upsilon^2}(e_{11}(z),e_{12}(z),\alpha_x(z),\alpha_y(z))dz \tag{25}$$
$$(R_2,S_2,F_2,G_2)=\int_{-h/2}^{h/2}(z)\frac{E(z)}{1-\upsilon^2}(e_{11}(z),e_{12}(z),\alpha_x(z),\alpha_y(z))dz$$

4 THE GOVERNING EQATION OF MOTION FOR DOUBLE- LAYER BORON NITRIDE PIEZOELECTRIC PLATE

In this section, based on the modified couple stress theory, the governing equations of motion for the FG single-layer piezoelectric plate resting on elastic foundation can be derived. Substituting Eqs. (24a), (24b), and (24c) into Eqs. (19a), (19b), and (19c) yields the following form:

$$\begin{aligned}&A\left(\frac{\partial^2 u}{\partial x^2}+\frac{1-\upsilon}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\upsilon}{2}\frac{\partial^2 v}{\partial x\partial y}\right)-G_1\upsilon\frac{\partial\Delta T(x,y,t)}{\partial x}-F_1\frac{\partial\Delta T(x,y,t)}{\partial x}-R_1\frac{\partial E_x(x,y,t)}{\partial x}\\&-B\nabla^2\frac{\partial w}{\partial x}+\frac{A_n}{4}\nabla^2\left(\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}\right)+f_x+q_x+\frac{1}{2}\frac{\partial^2 c_z}{\partial y}\\&+A\left[\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\upsilon\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+\left(\frac{1-\upsilon}{2}\right)\left(\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right)\right]=I_0\ddot{u}-I_1\frac{\partial\ddot{w}}{\partial x}\end{aligned}\tag{26a}$$

$$\begin{aligned}&A\left(\frac{\partial^2 v}{\partial y^2}+\frac{1-\upsilon}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\upsilon}{2}\frac{\partial^2 u}{\partial x\partial y}\right)-F_1\upsilon\frac{\partial\Delta T(x,y,t)}{\partial y}-G_1\frac{\partial\Delta T(x,y,t)}{\partial y}-S_1\upsilon\frac{\partial E_x(x,y,t)}{\partial y}\\&-B\nabla^2\frac{\partial w}{\partial y}+\frac{A_n}{4}\nabla^2\left(\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 v}{\partial x^2}\right)+f_y+q_y-\frac{1}{2}\frac{\partial c_z}{\partial y}\\&+A\left[\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}+\upsilon\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+\left(\frac{1-\upsilon}{2}\right)\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}\right)\right]=I_0\ddot{v}-I_1\frac{\partial\ddot{w}}{\partial y}\end{aligned}\tag{26b}$$

So far, the governing equations of motion are obtained for an FGM piezoelectric plate embedded on elastic foundation undergoing electro-thermo-mechanical load using the modified couple stress theory. In Fig .1 is shown an FGM rectangular plate that a , b , h are length, width and total thickness of the plate, respectively. In this section, according to Fig .2 two plates are coupled by elastic foundation and both of them are fixed to wall. The following equations are stated without considering f body force, c body couple and small rotations:

$$\begin{aligned}&A\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{1-\upsilon}{2}\frac{\partial^2 u_1}{\partial y^2}+\frac{1+\upsilon}{2}\frac{\partial^2 v_1}{\partial x\partial y}\right)-R_1\frac{\partial E_x(x,y,t)}{\partial x}-F_1\frac{\partial\Delta T(x,y,t)}{\partial x}\\&-\upsilon G_1\frac{\partial\Delta T(x,y,t)}{\partial x}-B\nabla^2\frac{\partial w_1}{\partial x}+\frac{A_n}{4}\nabla^2\left(\frac{\partial^2 v_1}{\partial x\partial y}-\frac{\partial^2 u_1}{\partial y^2}\right)+q_x=I_0\ddot{u}_1-I_1\frac{\partial\ddot{w}_1}{\partial x}\end{aligned}\tag{27a}$$

$$A\left(\frac{\partial^2 u_2}{\partial x^2}+\frac{1-\upsilon}{2}\frac{\partial^2 u_2}{\partial y^2}+\frac{1+\upsilon}{2}\frac{\partial^2 v_2}{\partial x\,\partial y}\right)-R_1\frac{\partial E_x(x,y,t)}{\partial x}-F_1\frac{\partial \Delta T(x,y,t)}{\partial x}$$
$$-\upsilon G_1\frac{\partial \Delta T(x,y,t)}{\partial x}-B\nabla^2\frac{\partial w_2}{\partial x}+\frac{A_n}{4}\nabla^2\left(\frac{\partial^2 v_2}{\partial x\,\partial y}-\frac{\partial^2 u_2}{\partial y^2}\right)+q_x=I_0\ddot{u}_2-I_1\frac{\partial \ddot{w}_2}{\partial x} \tag{27b}$$

$$A\left(\frac{\partial^2 v_1}{\partial y^2}+\frac{1-\upsilon}{2}\frac{\partial^2 v_1}{\partial x^2}+\frac{1+\upsilon}{2}\frac{\partial^2 u_1}{\partial x\,\partial y}\right)-\upsilon F_1\frac{\partial \Delta T(x,y,t)}{\partial y}-\upsilon S_1\frac{\partial E_x(x,y,t)}{\partial y}$$
$$-G_1\frac{\partial \Delta T(x,y,t)}{\partial y}-B\nabla^2\frac{\partial w_1}{\partial x}+\frac{A_n}{4}\nabla^2\left(\frac{\partial^2 u_1}{\partial x\,\partial y}-\frac{\partial^2 v_1}{\partial x^2}\right)+q_y=I_0\ddot{v}_1-I_1\frac{\partial \ddot{w}_1}{\partial y} \tag{27c}$$

$$A\left(\frac{\partial^2 v_2}{\partial y^2}+\frac{1-\upsilon}{2}\frac{\partial^2 v_2}{\partial x^2}+\frac{1+\upsilon}{2}\frac{\partial^2 u_2}{\partial x\,\partial y}\right)-\upsilon F_1\frac{\partial \Delta T(x,y,t)}{\partial y}-\upsilon S_1\frac{\partial E_x(x,y,t)}{\partial y}$$
$$-G_1\frac{\partial \Delta T(x,y,t)}{\partial y}-B\nabla^2\frac{\partial w_2}{\partial x}+\frac{A_n}{4}\nabla^2\left(\frac{\partial^2 u_2}{\partial x\,\partial y}-\frac{\partial^2 v_2}{\partial x^2}\right)+q_y=I_0\ddot{v}_2-I_1\frac{\partial \ddot{w}_2}{\partial y} \tag{27d}$$

$$B\nabla^2\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)-(D+A_n)\nabla^4 w_1-R_2\frac{\partial^2 E_x(x,y,t)}{\partial x^2}-(F_2+\upsilon G_2)\frac{\partial^2 \Delta T(x,y,t)}{\partial x^2}$$
$$-K_w w_1+K_G\nabla^2 w_1+K_w(w_2-w_1)-K_G\nabla^2(w_2-w_1)-(\upsilon F_2+G_2)\frac{\partial^2 \Delta T(x,y,t)}{\partial y^2}$$
$$-\upsilon S_2\frac{\partial^2 E_x(x,y,t)}{\partial y^2}-(R_1 E_x+F_1\Delta T+\upsilon G_1\Delta T)\frac{\partial^2 w_1}{\partial x^2} \tag{27e}$$
$$-(\upsilon F_1\Delta T+\upsilon S_1 E_x+G_1\Delta T)\frac{\partial^2 w_1}{\partial y^2}+q_z=I_0\ddot{w}_1+I_1\left(\frac{\partial \ddot{u}_1}{\partial x}-\frac{\partial \ddot{v}_1}{\partial y}\right)-I_2\nabla^2\ddot{w}_1$$

$$B\nabla^2\left(\frac{\partial u^2}{\partial x}+\frac{\partial v_2}{\partial y}\right)-(D+A_n)\nabla^4 w_2-R_2\frac{\partial^2 E_x(x,y,t)}{\partial x^2}-(F_2+\upsilon G_2)\frac{\partial^2 \Delta T(x,y,t)}{\partial x^2}$$
$$-K_w w_2+K_G\nabla^2 w_2-K_w(w_2-w_1)+K_G\nabla^2(w_2-w_1)-(\upsilon F_2+G_2)\frac{\partial^2 \Delta T(x,y,t)}{\partial y^2}$$
$$-\upsilon S_2\frac{\partial^2 E_x(x,y,t)}{\partial y^2}-(R_1 E_x+F_1\Delta T+\upsilon G_1\Delta T)\frac{\partial^2 w_2}{\partial x^2} \tag{27f}$$
$$-(\upsilon F_1\Delta T+\upsilon S_1 E_x+G_1\Delta T)\frac{\partial^2 w_2}{\partial y^2}+q_z=I_0\ddot{w}_2+I_1\left(\frac{\partial \ddot{u}_2}{\partial x}-\frac{\partial \ddot{v}_2}{\partial y}\right)-I_2\nabla^2\ddot{w}_2$$

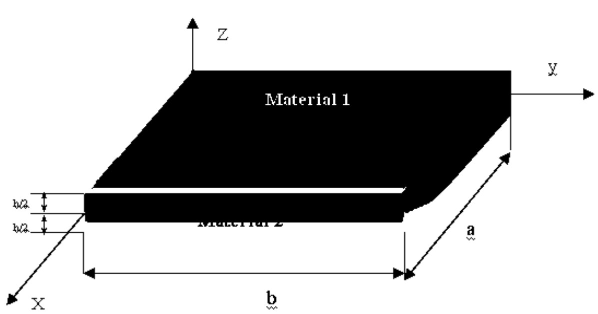

Fig.1
An schematic of an FGM Plate.

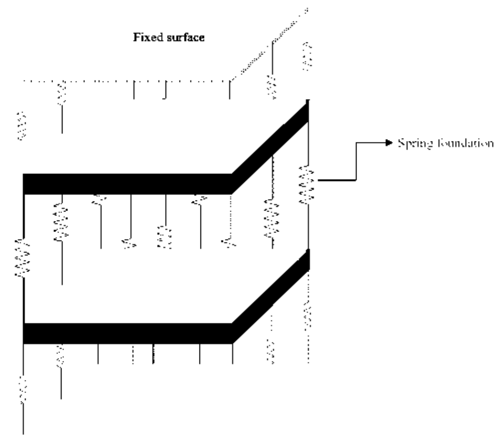

Fig. 2
Coupled system of double-layer boron nitride piezoelectric plate by a Pasternak foundation.

5 ANALYTICAL SOLUTIONS

The Navier's type solution for four edges simply supported boundary conditions of rectangular plate is used to obtain the natural frequency and critical buckling load for the FG double-layer piezoelectric plate resting on elastic foundation based on the modified couple stress theory. Then, the displacement fields to satisfy the simply supported boundary conditions and the governing equations of motion can be defined as follows:

$$\begin{aligned}
&u_1(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn1}\sin\alpha_m x\,\sin\beta_n y\;e^{i\omega t} \qquad v_1(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn1}\sin\alpha_m x\,\sin\beta_n y\;e^{i\omega t}\\
&w_1(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn1}\sin\alpha_m x\,\sin\beta_n y\;e^{i\omega t} \qquad u_2(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn2}\sin\alpha_m x\,\sin\beta_n y\;e^{i\omega t}\\
&v_2(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn2}\sin\alpha_m x\,\sin\beta_n y\;e^{i\omega t} \qquad w_2(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn2}\sin\alpha_m x\,\sin\beta_n y\;e^{i\omega t}
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\Delta T(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}T_{mn}\sin\alpha_m x\,\sin\beta_n y\;e^{i\omega t}\\
&E_x(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}E_{mn}\sin\alpha_m x\,\sin\beta_n y\;e^{i\omega t}
\end{aligned} \tag{29}$$

where $\alpha_m=\frac{m\pi}{a},\beta_n=\frac{n\pi}{b},i=\sqrt{-1}$ and ω is the natural frequency of the FG piezoelectric plate. The transverse load $q(x,y)$ using the double-Fourier series is defined as:

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Q_{mn}\sin\alpha_m x\,\sin\beta_n y \tag{30}$$

where

$$Q_{mn}=\frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\alpha_m x\,\sin\beta_n y\,dx\,dy \tag{31}$$

Substituting Eqs.(28) and (29) into Eq.(27), the equations of motion as matrix form for the FG double-layer piezoelectric plate are considered as:

$$\begin{bmatrix} c_{11} & 0 & c_{13} & 0 & c_{15} & 0 \\ 0 & c_{22} & 0 & c_{24} & 0 & c_{26} \\ c_{31} & 0 & c_{33} & 0 & c_{35} & 0 \\ 0 & c_{42} & 0 & c_{44} & 0 & c_{46} \\ c_{51} & 0 & c_{53} & 0 & c_{55}+d_{55} & d_{56} \\ 0 & c_{62} & 0 & c_{64} & d_{65} & c_{66}+d_{66} \end{bmatrix} \begin{Bmatrix} U_{mn1} \\ U_{mn2} \\ V_{mn1} \\ V_{mn2} \\ W_{mn1} \\ W_{mn2} \end{Bmatrix} - \omega^2 \begin{bmatrix} m_{11} & 0 & 0 & 0 & m_{15} & 0 \\ 0 & m_{22} & 0 & 0 & 0 & m_{26} \\ 0 & 0 & m_{33} & 0 & m_{35} & 0 \\ 0 & 0 & 0 & m_{44} & 0 & m_{46} \\ m_{51} & 0 & m_{53} & 0 & m_{55} & 0 \\ 0 & m_{62} & 0 & m_{64} & 0 & m_{66} \end{bmatrix} \begin{Bmatrix} U_{mn1} \\ U_{mn2} \\ V_{mn1} \\ V_{mn2} \\ W_{mn1} \\ W_{mn2} \end{Bmatrix} = \begin{Bmatrix} f \\ f \\ g \\ g \\ Q_{mn}+h \\ Q_{mn}+h \end{Bmatrix} \tag{32}$$

where

$$\begin{aligned}
&c_{11} = A\alpha_m{}^2 + \frac{1-v}{2}A\beta_n{}^2 + \frac{A_n}{4}\beta_n{}^2(\alpha_m{}^2+\beta_n{}^2) \qquad c_{13} = \frac{1+v}{2}A\alpha_m\beta_n - \frac{A_n}{4}\alpha_m\beta_n(\alpha_m{}^2+\beta_n{}^2) \\
&c_{33} = A\beta_n{}^2 + \frac{1-v}{2}A\alpha_m{}^2 + \frac{A_n}{4}\alpha_m{}^2(\alpha_m{}^2+\beta_n{}^2) \qquad c_{55} = (D+A_n)\left(\alpha_m{}^2+\beta_n{}^2\right)^2 \\
&c_{15} = -B\alpha_m(\alpha_m{}^2+\beta_n{}^2) \qquad c_{35} = -B\beta_n(\alpha_m{}^2+\beta_n{}^2) \qquad c_{46} = -B\beta_n(\alpha_m{}^2+\beta_n{}^2) \\
&c_{26} = -B\alpha_m(\alpha_m{}^2+\beta_n{}^2) \qquad c_{31} = \frac{1+v}{2}A\alpha_m\beta_n - \frac{A_n}{4}\alpha_m\beta_n(\alpha_m{}^2+\beta_n{}^2) \\
&c_{22} = A\alpha_m{}^2 + \frac{1-v}{2}A\beta_n{}^2 + \frac{A_n}{4}\beta_n{}^2(\alpha_m{}^2+\beta_n{}^2) \qquad c_{24} = \frac{1+v}{2}A\alpha_m\beta_n - \frac{A_n}{4}\alpha_m\beta_n(\alpha_m{}^2+\beta_n{}^2) \\
&k = N_{Cr}(\gamma_1\alpha_m{}^2+\gamma_2\beta_n{}^2) \qquad c_{51} = -B\alpha_m(\alpha_m{}^2+\beta_n{}^2) \qquad c_{53} = -B\beta_n(\alpha_m{}^2+\beta_n{}^2) \\
&c_{42} = \frac{1+v}{2}A\alpha_m\beta_n - \frac{A_n}{4}\alpha_m\beta_n(\alpha_m{}^2+\beta_n{}^2) \qquad c_{44} = A\beta_n{}^2 + \frac{1-v}{2}A\alpha_m{}^2 + \frac{A_n}{4}\alpha_m{}^2(\alpha_m{}^2+\beta_n{}^2) \\
&c_{62} = -B\alpha_m(\alpha_m{}^2+\beta_n{}^2) \qquad c_{64} = -B\beta_n(\alpha_m{}^2+\beta_n{}^2) \qquad c_{66} = (D+A_n)\left(\alpha_m{}^2+\beta_n{}^2\right)^2 \\
&m_{13} = m_{24} = m_{61} = m_{51} = -\alpha_m I_1 \qquad m_{11} = m_{42} = m_{21} = m_{32} = I_0 \qquad m_{33} = m_{44} = m_{62} = m_{52} = -\beta_n I_1 \\
&m_{33} = I_0 + I_2(\alpha_m{}^2+\beta_n{}^2) \qquad m_{53} = m_{64} = I_0 + I_2(\alpha_m{}^2+\beta_n{}^2) \qquad m_{11} = m_{22} = I_0 \qquad m_{13} = -\alpha_m I_1 \\
&m_{23} = -\beta_n I_1 \qquad d_{56} = d_{65} = -K_w - K_G(\alpha_m{}^2+\beta_n{}^2) \\
&d_{55} = d_{66} = 2K_w + 2K_G(\alpha_m{}^2+\beta_n{}^2) - (R_1E_x + F_1\Delta T + \upsilon G_1\Delta T)\alpha_m{}^2 - (\upsilon F_1\Delta T + \upsilon S_1E_x + G_1\Delta T)\beta_n{}^2 \\
&f = (R_1\alpha_m)\left[\frac{4E_0}{ab}\frac{1}{\alpha_m\beta_n}(-\cos\alpha_m a+1)(\cos\beta_n b-1)\right] \\
&\quad + (F_1\alpha_m + \upsilon G_1\alpha_m)\left[\frac{4T_0}{ab}\frac{1}{\alpha_m\beta_n}(-\cos\alpha_m a+1)(\cos\beta_n b-1)\right] \\
&g = (\upsilon F_1 + G_1)\left[\frac{4T_0}{ab}\frac{1}{\alpha_m\beta_n}(-\cos\alpha_m a+1)(\cos\beta_n b-1)\right] \\
&\quad + (\upsilon S_1)e_{12}\left[\frac{4E_0}{ab}\frac{1}{\alpha_m\beta_n}(-\cos\alpha_m a+1)(\cos\beta_n b-1)\right] \\
&h = [(F_2+\upsilon G_2)\alpha_m{}^2 + (\upsilon F_2+G_2)\beta_n{}^2]\left[\frac{4T_0}{ab}\frac{1}{\alpha_m\beta_n}(\cos\alpha_m a-1)(\cos\beta_n b-1)\right] \\
&\quad + \left[R_2\alpha_m{}^2 + S_2\upsilon\beta^2\right]\left[\frac{4E_0}{ab}\frac{1}{\alpha_m\beta_n}(\cos\alpha_m a-1)(\cos\beta_n b-1)\right]
\end{aligned} \tag{33}$$

6 NUMERICAL RESULTS

In this section, by utilizing the Navier's type solution, the dimensionless natural frequency and critical buckling load of the FG double-layer boron nitride piezoelectric plate based on the modified couple stress theory can be obtained using geometric and electro-thermo-mechanical properties as follows [36, 38, 39]:

$$\begin{array}{lll} E_1 = 18(Tpa) & E_2 = 1.8(Tpa) & q_0 = 1(N/m) \\ \alpha_{x1} = 1.2\times10^{-6}(1/K) & \alpha_{x2} = 0.12\times10^{-6}(1/K) & \alpha_{y1} = 0.6\times10^{-6}(1/K) \\ \alpha_{y2} = 0.06\times10^{-6}(1/K) & e_{111} = 0.95(C/m^2) & e_{112} = 0.095(C/m^2) \\ e_{121} = 0.95(C/m^2) & e_{122} = 0.095(C/m^2) & h = 0.075(nm) \\ \rho_1 = 23.3\times10^3(kg/m^3) & \rho_2 = 2.3\times10^3(kg/m^3) & \\ k_G = 2.071273(N/m) & k_w = 8.9995035\times10^{17}(N/m^3) & \end{array} \tag{34}$$

The dimensionless buckling load N^*, temperature T^* and frequency λ are defined as:

$$\begin{array}{lll} N^* = \dfrac{Na^2}{E_2h^3} & T^* = T\times(\alpha_{x2} + \upsilon\alpha_{y2}) & \lambda = \dfrac{\omega a^2}{h}\sqrt{\dfrac{\rho_2}{h_2}} \\ \dfrac{a}{h} = 10 & \dfrac{a}{b} = 1 & p = 10 \end{array} \tag{35}$$

The dimensionless natural frequencies of the piezoelectric plate subjected to the electro-thermo-mechanical loadings are calculated and compared with the obtained results by Thai and Choi [35] for mechanical properties (Eq. (34)).

The dimensionless frequencies for $a/h = 5$, 10 and $l/h = 0, 0.2, 0.4, 0.6, 0.8, 1$ and $p = 10$ have been shown in Table 1.

Table1
The dimensionless frequency of a simply support piezoelectric square plate.

Present work			Thai and Choi, 2013 [35]		
a/h	l/h	$p=10$	a/h	l/h	$p=10$
5	0	4.3221	5	0	4.2535
5	0.2	4.6562	5	0.2	4.5925
5	0.4	5.5377	5	0.4	5.4843
5	0.6	6.7529	5	0.6	6.7091
5	0.8	8.1492	5	0.8	8.1129
5	1	10.1776	5	1	10.0046
10	0	4.7883	10	0	4.0861
10	0.2	5.1143	10	0.2	4.7606
10	0.4	5.9865	10	0.4	5.6874
10	0.6	7.2093	10	0.6	6.9629
10	0.8	8.6347	10	0.8	8.4301
10	1	10.1776	10	1	10.0046

The influences of aspect ratio a/b and a/h on the dimensionless natural frequency for different dimensionless material length scale parameters l/h based on the modified couple stress are shown in Figs. 3 and 4, respectively. According to Figs. 3 and 4, the dimensionless natural frequency increases with an increase of aspect ratio. Also, by increasing the dimensionless material length scale parameter, the dimensionless natural frequency rises.

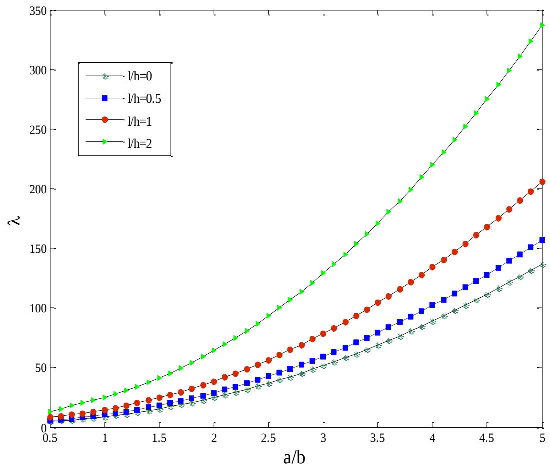

Fig.3
Influence of dimensionless material length scale parameter on the dimensionless natural frequency and aspect ratio (*a/b*).

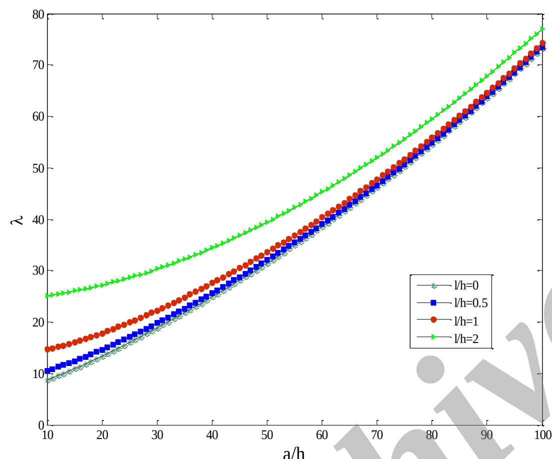

Fig.4
Influence of dimensionless material length scale parameter on the dimensionless natural frequency and aspect ratio (*a/h*).

Fig. 5 illustrates the effects of aspect ratio a/h on the natural frequency for different dimensionless material length scale parameters l/h based on the modified couple stress. According to Fig. 5, the natural frequency has a dramatic decrease with an increase of the aspect ratio. Also, by growing the dimensionless material length scale parameter, the natural frequency rises.

The effect of transverse wave number on the dimensionless frequency of double layer boron nitride piezoelectric plate is shown in Fig. 6. As depicted in this figure, by increasing transverse wave number, the dimensionless natural frequency rises gradually. Moreover, it can be seen that the difference between curves for higher dimensionless material length scale parameter l/h rises.

The effect of electric field on the dimensionless frequency of the piezoelectric plate for various l/h is investigated in Fig. 7. The influence of the electric field on the dimensionless frequency is reduced by increasing the dimensionless material length scale parameter.

Fig. 8 illustrated the influence of the power law index (volume fraction of material) for various dimensionless material length scale parameter on the dimensionless natural frequency. It is shown from the results that the dimensionless natural frequency of the piezoelectric plate rises steadily by increasing the power law index but the slop of these curves becomes approximately constant for $p\rangle 10$. Also, by growing the dimensionless material length scale parameter, the effect of the power law index on the dimensionless frequency decreases.

The influence of temperature change on the dimensionless natural frequency for various l/h is shown in Fig.9. The effect of temperature change on the dimensionless frequency in the specified l/h is approximately negligible.

As depicted in Fig. 10, the dimensionless critical buckling load of the double layer plate increases by growing the power law index. Although the buckling load rises dramatically for $l/h = 2$, the increase of it for $l/h = 0$, 0.5 and 1 is gradually.

The influence of the power law index on the critical buckling load due to electric field and temperature change of the piezoelectric plate are investigated in Figs. 11 and 12, respectively. In Figs. 11 and 12, it is depicted that the more the FGM power law index grows, the greater the electric and thermal fields grow especially by increasing ratio of l/h .

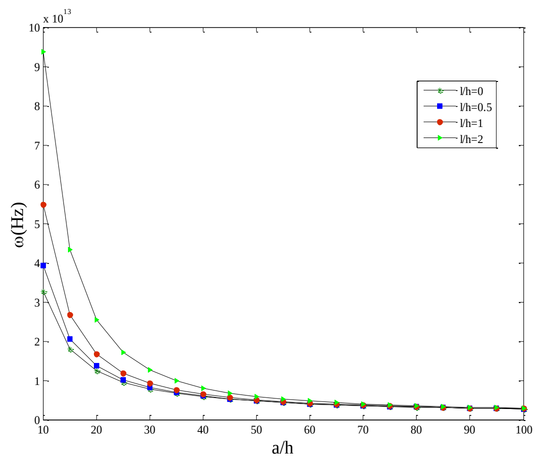

Fig.5
Effect of dimensionless material length scale parameter on the natural frequency and aspect ratio (*a/h*).

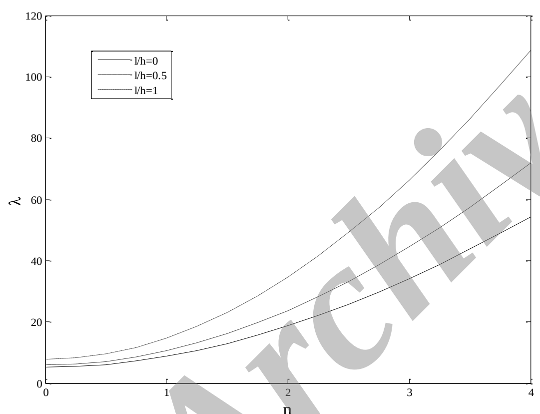

Fig.6
Effect of transverse wave number on the dimensionless natural frequency for various *l/h*.

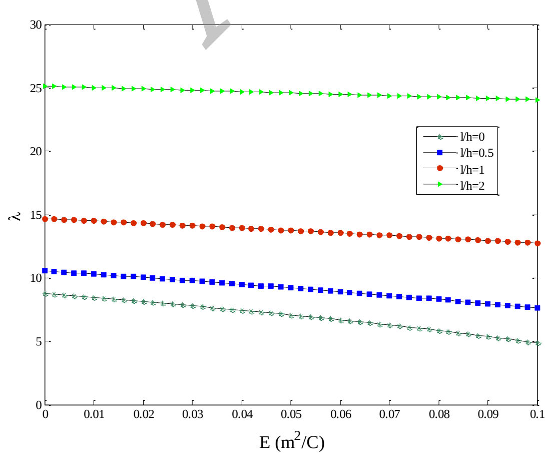

Fig.7
Effect of electric field on dimensionless natural frequency of piezoelectric plate for various *l/h*.

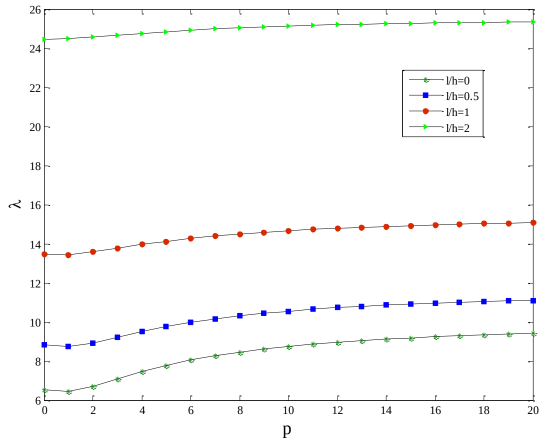

Fig.8
Effect of power law index on the dimensionless natural frequency of piezoelectric plate for various *l/h*.

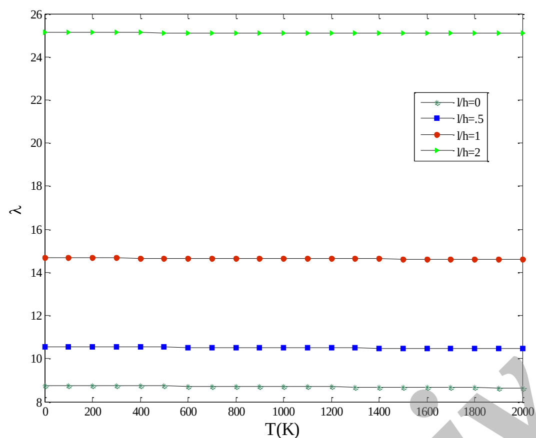

Fig.9
Effect of temperature change on the dimensionless natural frequency for various *l/h*.

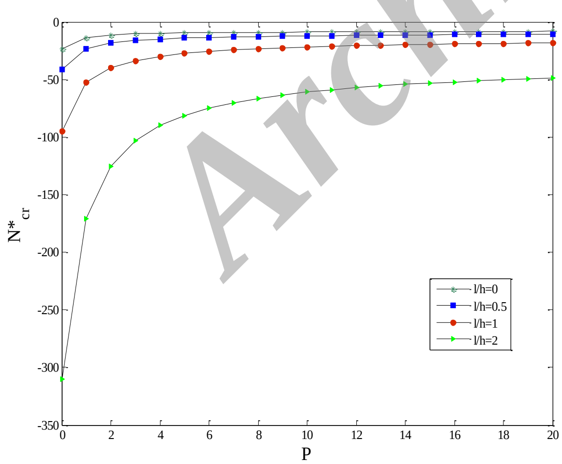

Fig.10
Effect of power lax index on dimensionless buckling load N of FG double-layer boron nitride piezoelectric plate.

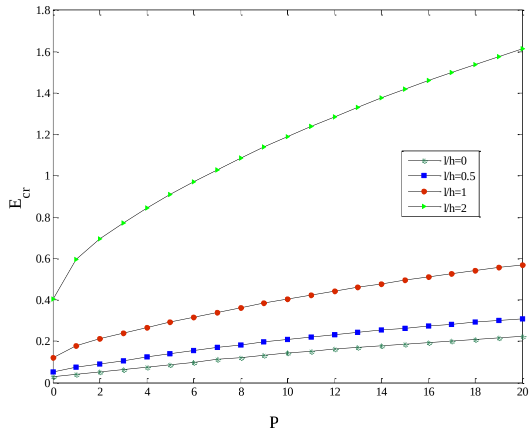

Fig.11
Effect of power lax index on electric field E of FG double-layer boron nitride piezoelectric plate.

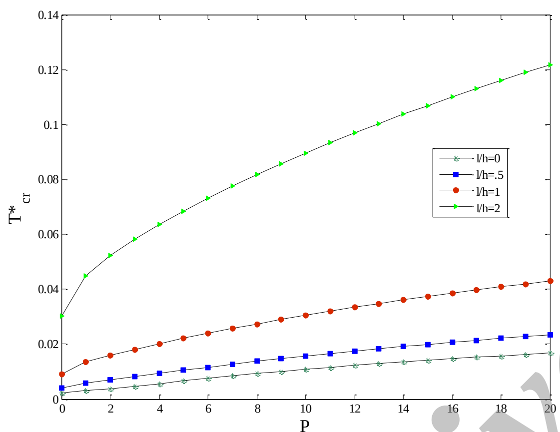

Fig.12
Effect of power lax index on the critical thermal buckling for simply supported boundary conditions.

7 CONCLUSIONS

In this article, the effect of size-dependent on the critical buckling load and natural frequency of the FG double-layer boron nitride plate under electro-thermo-mechanical loadings surrounded by an elastic foundation was developed. Using Hamilton's principle, the governing equations of motion was obtained by applying a modified couple stress and von Karman nonlinear strain for piezoelectric material and Kirchhoff plate. These equations were coupled for the FG double-layer plate using Pasternak foundation and solved using Navier's type solution for simply supported boundary conditions. The results of the research can be illustrated as:

1. As aspect ratios a/b and a/h increases, the dimensionless natural frequency rises. Also, by growing the dimensionless material length scale parameter l/h , the dimensionless frequency increases.
2. By growing the transverse wave number, the dimensionless natural frequency increases.
3. The effect of the electric field on the dimensionless frequency is declined gradually by increasing the dimensionless material length scale parameter.
4. The effect of the power law index on the dimensionless frequency decreases by increasing the dimensionless material length scale parameter.
5. The effect of the temperature change on the dimensionless frequency in the specified l/h is approximately negligible.
6. The power law index has a significant influence on the dimensionless critical buckling load of the double layer plate for $l/h = 2$, but it has a gradual effect on the buckling load for the others.
7. The electric field and temperature change rises with an increase in the FGM power law index.

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