

# A Modified Couple Stress Theory for Postbuckling Analysis of Timoshenko and Reddy-Levinson Single-Walled Carbon Nanobeams

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## ABSTRACT

The novelty of this study is presentation of an exact solution for prediction of postbuckling behavior of shear deformable micro- and nano-scale beams based on modified couple stress theory and using principle of minimum potential energy. Timoshenko and Reddy-Levinson beam theories are applied to consider the shear deformation effect and Von Karman nonlinear kinematics is used to describe the nonlinear behavior of the postbuckling, and the Poisson's effect is also considered in stress-strain relation. Also, the size effect is exposed by introducing a material length scale parameter. Finally, the influences of shear deformation, Poisson's ratio and variations of length and thickness are investigated. The results indicate that the classical theory exaggerates the postbuckling amplitude of the nanobeam and overstates the effect of shear deformation on the postbuckling response of the nanobeam.

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**Keywords :** Postbuckling; Single-walled carbon nanobeam; Timoshenko and Reddy-Levinson beam theories.

## 1 INTRODUCTION

**B**UCKLING and postbuckling of solid structures such as beams, plates, and shells under thermal or mechanical loadings have been studied by many researchers [1-6]. In many of these studies, the shear deformation effect has been considered as a significant influence on the buckling and postbuckling behaviors. Emam [7] investigated the nonlinear response of the composite beams modeled on the basis of higher order shear deformation theories in postbuckling. Also, Daneshmehr et al. [8] presented an exact solution for postbuckling of shear deformable beam made of functionally graded material (FGM) under axial loading. Both of the studies expressed that the shear deformation has a considerable effect on buckling and postbuckling behaviors of beams. Therefore, it is necessary to consider the shear deformation effect in postbuckling analysis.

There are many high order beam theories, but one of the most common of these theories is Reddy-Levinson beam theory (RLBT). The RLBT was developed by Levinson [9] and Reddy [10] for beams with rectangular cross sections using a third-order displacement field. So, in the present study, RLBT is employed to predict the postbuckling response of small scale beams. The Timoshenko beam theory (TBT) is also used, which this theory utilizes first order variations in displacement field.

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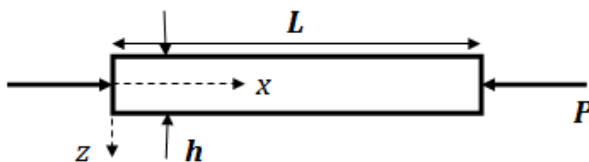
Since the classical continuum theories do not satisfy the behavior of very thin structures (for example micro- and nanostructures), some non-classical theories must be used for these structures. One of the non-classical theories is the modified couple stress theory (MCST) which has only one material constant to capture the size effect of small scale structures. The modified couple stress theory is developed from classical couple stress theory by Yang et al. [11]. This theory has two considerable advantages over the classical couple stress theory; the first one is involvement of only one material length scale parameter to satisfy the size-dependent behavior of micro and nanostructures, the second one is inclusion of a symmetric couple stress tensor. Many studies on static and dynamic analysis of beams are accomplished based on modified couple stress theories. A microstructure-dependent Timoshenko beam has been proposed based on modified couple stress theory and Hamilton's principle by Ma et al. [12]. Asghari et al. [13] developed the nonlinear behavior of the new model due to the present of induced midplane stretching, so, derived nonlinear Timoshenko beam formulation based on the modified couple stress theory and free vibration and static bending analyses have been presented by them. Static bending and free vibration of a simply supported Reddy-Levinson beam based on modified couple stress theory have been presented by Ma et al. [14]. Ke and Wang [15] investigated the dynamic stability of an FG microbeam using the modified couple stress and Timoshenko beam theories. Akgoz and Civalek [16] presented a buckling analysis of Euler-Bernoulli microbeams using Strain gradient and modified couple stress theories. Salamat-talab et al. [17] developed static and dynamic analysis of third-order FG microbeam based on modified couple stress theory. Bending analysis of a composite Timoshenko beam has been studied based on modified couple stress theory by Roque et al. [18]. Mohammad-Abadi and Daneshmehr [19] developed buckling analysis of microbeams with higher order beam theories based on modified couple stress theory. As can be observed, a couple stress based postbuckling analysis of micro or nanobeams is not published yet. Therefore, study on postbuckling of nanobeam based on the modified couple stress theory can be useful for researchers and scientists.

Moreover some researchers presented the postbuckling analysis of different nanostructures. Shen and Zhang [20] presented buckling and postbuckling behaviors of double-walled carbon nanotubes under an axial loading and thermal environment using a continuum mechanics model with the presence of van der Waals interaction forces. Eringen's nonlocal beam theory has been used for postbuckling analysis of a cantilever nanorod by Wang et al. [21]. Setoodeh et al. [22] investigated an exact solution for postbuckling of single-walled carbon nanotubes based on nonlocal elasticity model and Euler-Bernoulli beam theory. Li et al. [23] presented postbuckling analysis of piezoelectric nanobeam with surface effects including surface elasticity, surface piezoelectricity and residual surface stress. Prediction of the postbuckling deflection for Timoshenko nanobeams incorporating the surface stress effect has been investigated by Ansari et al. [24]. Additionally, there are some related studies that can be useful for readers [25-29].

The significance of this study is to investigate the static postbuckling response of the shear deformable micro and nanobeams based on modified couple stress theory with first and higher order shear deformation theories, as in the present study, is concentrated on beam deflections and the relation between the postbuckling load and maximum deflection of beam is presented. The Poisson's effect and Von Karman nonlinear kinematics are applied in the study, and the nonlinear governing equations are derived using principle of minimum potential energy and variational method. The amplitude of the static postbuckling is obtained by solving the nonlinear governing equations. Finally, influences of size, Poisson's ratio and shear deformation on the postbuckling response of nanobeams are studied.

## 2 FORMULATIONS

In this section, the principle of minimum strain energy is employed to derive the governing equations. Consider a beam of length  $L$  with rectangular cross section of width  $b$  and thickness  $h$  which is subjected to an axial load  $P$  in both ends of the beam (see Fig. 1).



**Fig.1** Geometry of the beam subjected to axial loading and coordinate system.

Using the modified couple stress theory [11], the strain energy  $U$  occupying region  $\Omega$  is given by:

$$\delta U = \int_{\Omega} (\sigma_{ij} \delta \varepsilon_{ij} + m_{ij} \delta \chi_{ij}) dV - \frac{1}{2} \int_0^L P \left( \frac{\partial w}{\partial x} \right) dx \quad i, j = 1, 2, 3 \quad (1)$$

where  $\varepsilon, \sigma, \chi$  and  $m$  are the strain tensor, Cauchy stress tensor, symmetric curvature tensor and the deviatoric part of the couple stress tensor, respectively. The symmetric curvature and the deviatoric part of the couple stress tensors are defined as follow

$$\chi_{ij} = \frac{1}{2} (\nabla \theta + (\nabla \theta)^T) = \frac{1}{2} (\theta_{i,j} + \theta_{j,i}) \quad (2)$$

$$m_{ij} = 2\xi^2 \mu \chi_{ij} \quad (3)$$

where  $u$  and  $\theta$  are the displacement and rotation vectors, respectively.  $\xi$  is a material length scale parameter which expresses the effect of couple stress [30] and can be determined from torsion tests of slim cylinders of different diameters [31] or bending tests of thin beams of different thickness [32]. Also,  $\lambda$  and  $\mu$  are Lamé's constants that are given as:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \quad (4)$$

The rotation vector is given as:

$$\theta = \frac{1}{2} \text{curl}(u) \quad (5)$$

where  $\nu$  is Poisson's ratio.

Using the Cartesian coordinate system shown in Fig. 1, the displacement field in the shear deformable beam is depicted as [14]

$$u_1 = u(x) - z \phi(x) - c_1 z^3 \left[ -\phi(x) + \frac{\partial w(x)}{\partial x} \right], \quad u_2 = 0, \quad u_3 = w(x) \quad (6)$$

where  $u_1, u_2$  and  $u_3$  are  $x$ -,  $y$ - and  $z$ - components of displacement vector at each point respectively, also  $u$  and  $w$  are respectively  $x$ - and  $z$ -components of displacement vector at the points corresponding to middle surface of beam, and  $\phi$  is the angle of rotation of the cross section. It is noted that, the  $y$ -axis is the neutral axis and the  $z$ -axis is the symmetry axis. Also,  $c_1$  is a constant given by

$$c_1 = 0 \quad \text{Timoshenko beam theory}$$

$$c_1 = \frac{4}{3h^2} \quad \text{Reddy beam theory}$$

By assuming very small slopes in the beam after deformation but possible finite transverse deflection  $w$ , the axial strain can be approximately expressed by the Von Karman nonlinear relation as [13]

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2} \left( \frac{\partial u_3}{\partial x} \right)^2 = \frac{\partial u}{\partial x} - (z - c_1 z^3) \frac{\partial \phi}{\partial x} - c_1 z^3 \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \quad (7)$$

And other non-zero component of the strain tensor can be obtained as:

$$\varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} \right) = \frac{1}{2} (1 - 3c_1 z^2) \left( \frac{\partial w}{\partial x} - \phi \right) \quad (8)$$

Using Eqs. (5) and (6) the non-zero component of the rotation vector is obtained as:

$$\theta_y = -\frac{1}{2} \left( (1 + 3c_1 z^2) \frac{\partial w}{\partial x} + (1 - 3c_1 z^2) \phi \right) \quad (9)$$

and from Eqs. (2) and (9) the non-zero components of the symmetric curvature tensor are expressed as:

$$\chi_{xy} = -\frac{1}{4} \left( (1 + 3c_1 z^2) \frac{\partial^2 w}{\partial x^2} + (1 - 3c_1 z^2) \frac{\partial \phi}{\partial x} \right) = \chi_{yx} \quad \chi_{yz} = -\frac{3}{2} c_1 z \left( -\phi + \frac{\partial w}{\partial x} \right) = \chi_{zy} \quad (10)$$

By considering the Poisson's effect, the non-zero components of stress tensor are obtained as:

$$\begin{aligned} \sigma_{xx} &= (\lambda + 2\mu)\varepsilon_{xx} = (\lambda + 2\mu) \left( \frac{\partial u}{\partial x} - (z - c_1 z^3) \frac{\partial \phi}{\partial x} - c_1 z^3 \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) \\ \sigma_{xz} &= 2\mu\varepsilon_{xz} = \mu(1 - 3c_1 z^2) \left( \frac{\partial w}{\partial x} - \phi \right) \end{aligned} \quad (11)$$

Substituting Eq. (10) into Eq. (3), the non-zero component of the deviatoric part of couple stress tensor is obtained as:

$$m_{xy} = -\frac{1}{2} \mu \xi^2 \left( (1 + 3c_1 z^2) \frac{\partial^2 w}{\partial x^2} + (1 - 3c_1 z^2) \frac{\partial \phi}{\partial x} \right) \quad m_{yz} = -3\mu \xi^2 c_1 z \left( -\phi + \frac{\partial w}{\partial x} \right) \quad (12)$$

Using Eq. (1) and applying the principle of minimum potential energy as  $\delta U = 0$ , also by neglecting the non-conservative forces, the governing equations obtain as follow

$$\delta u : \int_{-h/2}^{h/2} (\lambda + 2\mu) \left[ \frac{\partial^2 u}{\partial x^2} - (z - c_1 z^3) \frac{\partial^2 \phi}{\partial x^2} - c_1 z^3 \frac{\partial^3 w}{\partial x^3} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right] dz = 0 \quad (13a)$$

$$\begin{aligned} \delta \phi : \int_{-h/2}^{h/2} & \left[ (\lambda + 2\mu)(z - c_1 z^3) \left[ \frac{\partial^2 u}{\partial x^2} - (z - c_1 z^3) \frac{\partial^2 \phi}{\partial x^2} - c_1 z^3 \frac{\partial^3 w}{\partial x^3} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right] \right. \\ & \left. + (\mu(1 - 3c_1 z^2)^2 + 9\mu \xi^2 c_1^2 z^2) \left[ \phi - \frac{\partial w}{\partial x} \right] - \frac{1}{4} \mu \xi^2 \left[ (1 - 9c_1^2 z^4) \frac{\partial^3 w}{\partial x^3} + (1 - 3c_1 z^2)^2 \frac{\partial^2 \phi}{\partial x^2} \right] \right] dz = 0 \end{aligned} \quad (13b)$$

$$\begin{aligned} \delta w : \int_{-h/2}^{h/2} & \left[ (\lambda + 2\mu) \left[ c_1 z^3 \left( -\frac{\partial^3 u}{\partial x^3} + (z - c_1 z^3) \frac{\partial^3 \phi}{\partial x^3} + c_1 z^3 \frac{\partial^4 w}{\partial x^4} - \frac{\partial w}{\partial x} \frac{\partial^3 w}{\partial x^3} - \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right) \right. \right. \\ & \left. \left. + \left( -\frac{\partial^2 u}{\partial x^2} \frac{\partial w}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} + (z - c_1 z^3) \left( \frac{\partial^2 \phi}{\partial x^2} \frac{\partial w}{\partial x} + \frac{\partial \phi}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) + c_1 z^3 \left( \frac{\partial w}{\partial x} \frac{\partial^3 w}{\partial x^3} + \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \right) - \frac{3}{2} \frac{\partial^2 w}{\partial x^2} \left( \frac{\partial w}{\partial x} \right)^2 \right) \right. \right. \\ & \left. \left. + \frac{1}{4} \mu \xi^2 \left[ (1 - 9c_1^2 z^4) \frac{\partial^3 \phi}{\partial x^3} + (1 + 3c_1 z^2)^2 \frac{\partial^4 w}{\partial x^4} \right] \right] dz - P_b \frac{\partial^2 w}{\partial x^2} = 0 \end{aligned} \quad (13c)$$

### 3 EXACT SOLUTIONS

In this section, the critical buckling load and static postbuckling response are investigated for simply supported Timoshenko and Reddy-Levinson nanobeams. By integrating Eq. (13a), we have;

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} = 0 \rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \tag{14}$$

Integrating Eq. (14) with respect to the spatial coordinate  $x$  yields

$$\frac{\partial u}{\partial x} = c - \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \tag{15}$$

where  $c$  is integral constant. By substituting Eqs. (14) and (15) into Eqs. (13b) and (13c), and then integration with respect to  $z$  yield

$$(A_1 - A_5) \frac{\partial^2 \phi}{\partial x^2} + (A_2 - A_4) \frac{\partial^3 w}{\partial x^3} + A_3 \left( \phi - \frac{\partial w}{\partial x} \right) = 0 \tag{16}$$

$$(A_6 + A_7) \frac{\partial^4 w}{\partial x^4} - (A_2 - A_4) \frac{\partial^3 \phi}{\partial x^3} + (P_b - A_8 c) \frac{\partial^2 w}{\partial x^2} = 0 \tag{17}$$

where  $P_b = P/b$  and coefficients  $A_i, i = 1, 2, \dots, 8$  are defined as:

$$\begin{aligned} A_1 &= (\lambda + 2\mu) \left[ \frac{h^3}{12} - \frac{c_1 h^5}{40} + \frac{c_1^2 h^7}{448} \right], & A_2 &= (\lambda + 2\mu) \left[ \frac{c_1 h^5}{80} - \frac{c_1^2 h^7}{448} \right], & A_3 &= (\lambda + 2\mu) \frac{c_1^2 h^7}{448}, \\ A_4 &= (\lambda + 2\mu) h, & A_5 &= \mu \left[ h - \frac{c_1 h^3}{2} + \frac{9c_1^2 h^5}{80} \right], & A_6 &= \frac{1}{4} \xi^2 \mu \left[ h - \frac{9c_1^2 h^5}{80} \right], \\ A_7 &= \frac{1}{4} \xi^2 \mu \left[ h - \frac{c_1 h^3}{2} + \frac{9c_1^2 h^5}{80} \right], & A_8 &= \frac{1}{4} \xi^2 \mu \left[ h + \frac{c_1 h^3}{2} + \frac{9c_1^2 h^5}{80} \right], & A_9 &= \frac{3}{4} \xi^2 \mu c_1 h^3 \end{aligned} \tag{18}$$

Integrating Eq. (15) once more, we obtain

$$u(x) = cx + c' - \frac{1}{2} \int_0^x \left( \frac{\partial w}{\partial \eta} \right)^2 d\eta \tag{19}$$

where  $c'$  is integral constant, too. For the midplane stretching to be significant, the beam ends must be restrained [7]. The boundary conditions for the axial displacement are assumed as follows

$$u(0) = 0, \quad u(L) = 0$$

The constants are now given by

$$c' = 0, \quad c = \frac{1}{2L} \int_0^L \left( \frac{\partial w}{\partial \eta} \right)^2 d\eta \tag{20}$$

For simply supported boundary conditions and the first buckling mode, the following displacement field is assumed [7]

$$w(x) = \alpha \sin \frac{\pi x}{L} \quad \phi(x) = \beta \cos \frac{\pi x}{L} \quad (21)$$

where  $\alpha$  and  $\beta$  are unknowns to be determined. Substituting Eqs. (21) into Eqs. (16), (17) and (20), The postbuckling response can be obtained as follows

$$P_b = \left( A_9 + A_5 + \left( \frac{1}{4} \frac{\pi^2 \alpha^2}{L^2} \right) A_4 \right) + (A_3 + A_8) \left( \frac{\pi}{L} \right)^2 + R \quad (22)$$

where coefficient  $R$  is defined as:

$$R = \frac{-(A_2 + A_6)^2 \left( \frac{\pi}{L} \right)^4 + 2(A_5 + A_9)(A_2 + A_6) \left( \frac{\pi}{L} \right)^2 - (A_5 + A_9)^2}{(A_1 + A_7) \left( \frac{\pi}{L} \right)^2 + (A_5 + A_9)} \quad (23)$$

It is noted that the buckling amplitude  $\alpha$  corresponds to the maximum buckling level that occurs at the midspan of the beam where  $x = L/2$ . Also, by replacing  $\alpha = 0$  the critical buckling load per unit width is obtained as follow

$$P_{cr} = (A_9 + A_5) + (A_3 + A_8) \left( \frac{\pi}{L} \right)^2 + R \quad (24)$$

#### 4 NUMERICAL RESULTS

To study on static postbuckling response, we consider a small-scale beam made of epoxy with the following properties [12]

$$E = 1.44 \text{ GPa}, \quad \xi = 17.6 \times 10^{-6} \text{ m}, \\ \nu = 0.38, \quad L = 20h, \quad b = 2h$$

At first for validation, the non-dimensional critical buckling loads are compared with obtained results in Ref [19]. This comparison shows very well agreement for present solution (see Table 1).

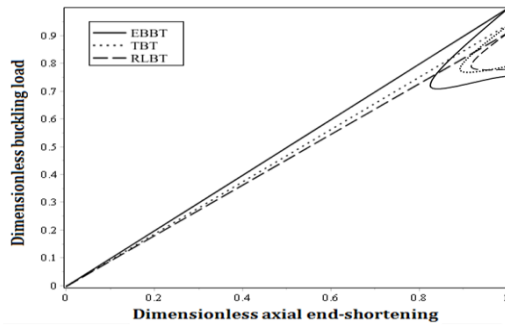
**Table 1**

Comparison of the non-dimensional critical buckling load ( $12P_{cr}L^2 / Eh^3$ ) obtained by present solution and obtained results in Ref [19] for TBT.

h(μm)	Present	Ref [19]
17.6	60.48	60.35
52.8	22.99	22.96
88	19.97	19.94
123.2	19.14	19.12

Fig. 2. shows the postbuckling behavior of a carbon nanobeam under  $h / \xi = 1, \nu = 0.38, L / h = 20$  and different beam theories. This figure illustrates the nonlinear behavior of the buckling load-end shortening curves, and this behavior is verified by obtained result in Ref [26]. In this figure, the dimensionless end shortening and

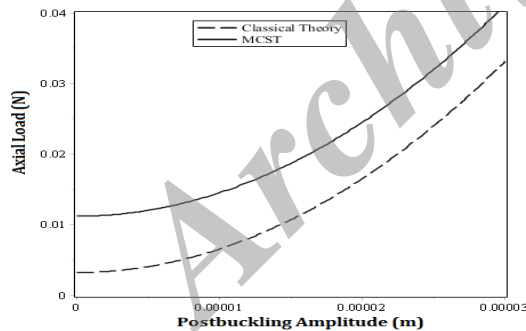
dimensionless buckling load are defined as  $(E/\sigma_{cr})e$  and  $\sigma_x/\sigma_{cr}$ , respectively; where  $e$  is value of beam end shortening.



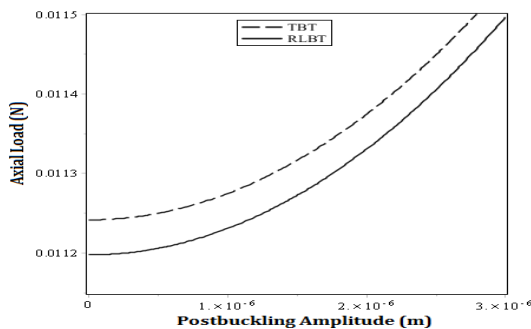
**Fig.2**  
Postbuckling behavior of carbon nanobeam for different beam theories with  $h/\xi=1, \nu=0.38, L/h=20$ .

Fig. 3 compares the static postbuckling response of the classical and modified couple stress theories. This figure shows that the classical theory exaggerates the postbuckling amplitude of the small scale beams and is not a convenient theory for postbuckling analysis of micro and nanobeams. Fig. 4 expresses the effect of different shear deformation theories on static postbuckling response. The postbuckling behaviors of the beam based on Timoshenko beam theory (TBT) and Reddy-Levinson beam theory (RLBT) are compared in this figure. It is found that TBT underestimates the postbuckling amplitude than RLBT at a given axial load. Therefore, TBT has errors to predict the postbuckling behavior, and these errors are shown in Fig. 5 at a given postbuckling amplitude ( $\alpha = 2 \times 10^{-6} m$ ). Also, this figure compares the classical theory and the modified couple stress theory according to values of difference between the TBT and RLBT. Fig. 5 displays that the classical theory estimates more difference between the TBT and RLBT at  $L/h < 50$ . Moreover, Fig. 5 shows that by increasing the slenderness ratio ( $L/h$ ) the effect of shear deformation becomes less and difference of TBT and RLBT is near to zero at very high slenderness ratios.

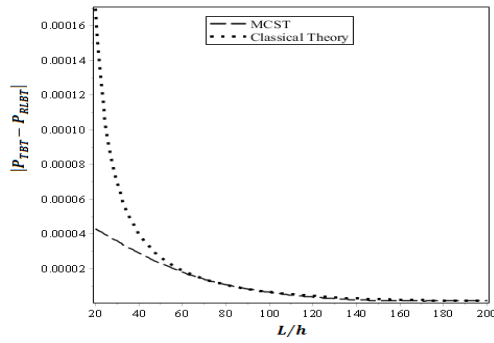
Fig. 6 presents the variation of the midspan postbuckling amplitude with the applied axial load for different thickness-to-material parameter ratios ( $h/\xi$ ). This figure indicates that by increasing the thickness-to-material parameter ratio at a given slenderness ratio the beam becomes stiffer, so, the critical buckling load and the postbuckling load are increased. Therefore, the buckling and the postbuckling behaviors of the beam are improved.



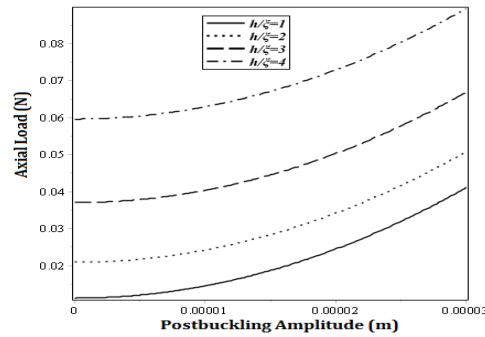
**Fig.3**  
Comparison of the static postbuckling response of the classical theory and the modified couple stress theory (MCST) with RLBT and  $h/\xi=1$ .



**Fig.4**  
Comparison of the static postbuckling response of the TBT and RLBT with  $h/\xi=1$ .

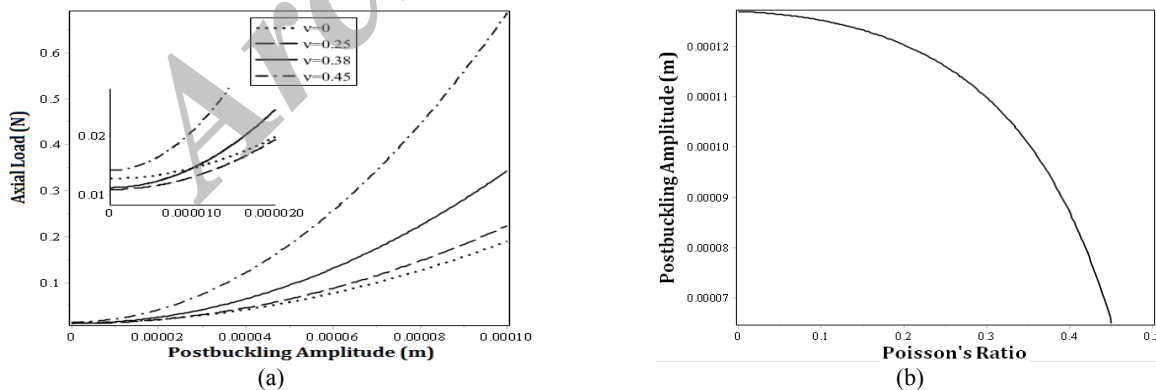


**Fig.5**  
Differences of the postbuckling load between the TBT and RLBT for both the classical theory and the modified couple stress theory (MCST) versus variations of the length-to-thickness ratios with  $h / \xi = 1$ .



**Fig.6**  
Effect of thickness-to-material parameter ratio on the static postbuckling response of the RLBT beam.

Effect of Poisson's ratio on postbuckling response of small scale beam is shown in Figs. 7(a) and 7(b). Fig. 7(a) presents the variations of the midspan postbuckling amplitude with the applied axial load for different Poisson's ratios. As shown in Figs. 7, the postbuckling amplitude decreases by increasing the Poisson's ratio at a given axial load, because the beam becomes stiffer by increasing the Poisson's ratio. Also, neglecting of Poisson's ratio ( $\nu=0$ ) makes considerable errors in the postbuckling behavior and predicts unexpected behavior in small amplitudes. Fig. 7(b) shows the variations of the maximum deflection of the beam versus the Poisson's ratio, at a given applied load ( $P=0.3 \text{ N}$ ). As stated in Fig. 7(a), this figure shows that by increasing Poisson's ratio, the maximum deflection of beam (postbuckling amplitude of beam) is decreased. The lateral displacement leads to absorb a part of the applied energy via the external load. Therefore, the beam receives the less deflection.

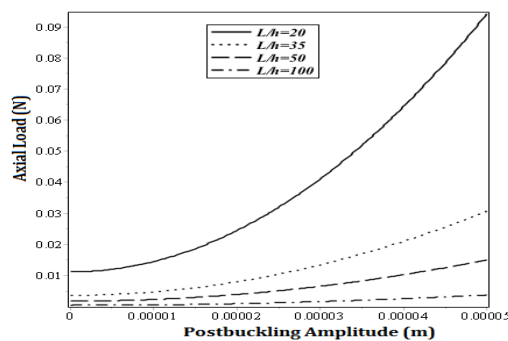


**Fig.7**  
a) Effect of Poisson's ratio on the static postbuckling response of the RLBT beam with  $h / \xi = 1$ .  
b) Variations of the postbuckling amplitude versus Poisson's ratio with the RLBT beam with  $h / \xi = 1$ .

Fig. 8 presents the static postbuckling response for different length-to-thickness ratios (slenderness ratio,  $L/h$ ). It is noted that at low length-to-thickness ratios, the beam is stiffer than slender beams, because the small amplitudes are estimates for them at a given axial load. In fact, by applying the axial loads larger than critical buckling load, the



slender beams make larger amplitudes, very fast and by increasing the length-to-thickness ratio the sensitivity of the postbuckling amplitude than applied load is increased. All figures (Figs. 2-8) show the nonlinear behavior of the postbuckling.



**Fig.8**  
Effect of length-to-thickness ratio on the static postbuckling response of the RLBT beam with  $h / \xi = 1$ .

## 5 CONCLUSIONS

The postbuckling analysis of the Timoshenko and Reddy-Levinson simply supported micro/nanobeams is presented based on a modified couple stress theory, which investigates behavior of the postbuckling of the beams with different geometries and properties by an exact solution. A material length scale parameter is used to capture the size effect. Also, Poisson's effect and Von Karman nonlinear kinematics are applied. The governing differential equations are derived using principle of minimum potential energy and are solved by closed-form method. Studying on effects of shear deformation, Poisson's ratio and variations of length and thickness the following points are concluded:

- TBT and RLBT predict the different behaviors for postbuckling response of beam and this difference reduces by increasing the slenderness ratio. Namely, for high slender beams the effect of shear deformation becomes negligible.
- The classical theory exaggerates the postbuckling amplitude of the small scale beam in comparison of the modified couple stress theory. Also, the effect of shear deformation is more significant for classical theory than modified couple stress theory.
- The thickness-to-material parameter ratio has a considerable effect on the buckling and postbuckling behaviors. Increasing this ratio at a given slenderness ratio, the buckling and postbuckling behaviors of the beam are improved.
- Poisson's effect on the postbuckling response of the beam is large, especially when the applied axial load is increased. Accordingly, this assumption that Poisson's effect is negligible ( $\nu=0$ ) leads to unexpected postbuckling behavior. In fact, at a given applied energy, the lateral deformation leads to reduce the required energy for the deflection of the beam after the happening of the buckling.
- Increase of length-to-thickness ratio leads to more sensitivity of the beam against increase of the applied axial load in the postbuckling behavior and makes high postbuckling amplitude.

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