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A Simple Finite Element Procedure for Free Vibration and Buckling Analysis of Cracked Beam-Like Structures

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ABSTRACT

In this study, a novel and very simple finite element procedure is presented for free vibration and buckling analysis of slim beam-like structures damaged by edge cracks. A cracked region of a beam is modeled using a very short element with reduced second moment of area (*I*). For computing reduced *I* in a cracked region, the elementary theory of bending of beams and local flexibility approach are used. The method is able to model cracked beam-columns by using ordinary beam elements. Therefore, it is possible to solve these problems with much less computational costs compared to 2D and 3D standard FE models. Numerical examples are offered to demonstrate the efficiency and effectiveness of the presented method.

Keywords : Cracked beam ; Modal analysis ; Buckling load; F.E.M.

1 INTRODUCTION

BEAM-Like structures are widely used in machines and structures, especially in mechanical, aerospace and civil engineering. The study of dynamic and stability behavior of these components is one of the main steps in the design process that must be accomplished because it is necessary to ensure the safety of structures against catastrophic collapses caused by resonance or buckling phenomena.

The occurrence of crack in slim elements, which may be developed due to applied dynamic loads, mechanical vibrations, etc. can change total behavior of the structure and reduce its safety. Therefore, many researchers have focused on developing reliable models to study dynamic and buckling behavior of cracked beam-like structures. These attempts could be classified into three main groups; an equivalent reduced section method [3], a local flexibility or rotational spring method, and especial finite element models.

Krishmar [1] and Thomson [2] introduced the equivalent reduced section method to simulate the effect of crack on natural frequencies of beams. The behind philosophy of the method is simulating the effects of the cracked region using a local bending moment and a reduced section. The experimental method used to evaluate the stiffness reduction due to a crack was so time-consuming and tedious. Zheng and Ji [3] have recently developed an approximate approach for analysis of cracked beam utilizing the concept of the equivalent reduced section and improved Rayleigh method. A general expression of the natural frequencies of the beam was established in a close

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form within their study, but the unknown coefficients revealed in the statement must be specified separately for any case of boundary conditions.

The local flexibility approach was first introduced by Okamura et al [4]. In this method it is considered that the crack divides a beam into two separated parts, which are connected to each other with a massless rotational spring. The local flexibility of spring is a function of the cross-sectional geometry and crack size, and can be evaluated by the fundamental theory of fracture mechanics. Many researchers attempted to introduce local flexibility functions for different cross sections [5-8]. This method has been widely implemented to study the free and forced vibrations, the buckling capacity and the identification of crack depth and its location in inverse problems [9-14]. It must be noted that, the analytical solution method has been used in these works, so this method can only be utilized for the structures with simple geometry and boundary conditions.

To remove the mentioned deficiency, numerous computational methods have been considered by researchers. For example, Standard 2D Finite Element Method (FEM) was implemented by chati et al to study natural frequencies of edge-cracked beams [15]. The 2 and 3D standard FEM can reliably model discontinuities in the structure using very fine girds and singular elements though due to high computational efforts they are not appropriate for primary design and solving inverse problems.

To overcome this challenge, researchers focused on developing especial FE methods, which are able to model structural behavior of cracked slim components, with minimum computational costs. For example, a zero length element for representing a beam with rectangular cross section and a single edge crack was presented by Tharp [16]. In his work, the stiffness matrix of the cracked beam element is obtained using the compliance coefficients by partial derivative of the total strain energy. Gounaris and Dimarogonas [17] introduced a special element to evaluate the dynamic response of a cracked cantilevered beam. Ostachowicz and Krawczuk[18] presented a point finite element to model an open and closed non-propagating edge crack. Skrinar and Plibersek [19] introduced a new cracked beam finite element considering the effects of shear force for analyzing moderately thick beam structures. Bouboulas and Anifantis [20] introduced a cracked beam element for modeling of fractured skeletal structures based on the direct stiffness method.

Most of these FE models are very simple to deal with, and can predict the behavior of fractured beam-columns with reasonable accuracy. But unfortunately they are not commonly addressed by commercial FEM packages. Thus, applying these models in practical engineering could become very tedious.

In sum, the main objective of this investigation is to introduce a novel and very simple procedure for analyzing cracked beam-like structures using ordinary beam elements that are available within all commercial FEM packages. For this purpose, the basic concept of the equivalent reduced section method is combined with the local flexibility approach. This procedure has four main steps. In the first step, by using the basic theory of bending beams, the local flexibility of a beam in the crack position is computed in terms of mechanical and geometrical properties of it. In second step, the local flexural flexibility of cracked segment is computed using the available flexural stiffness functions in the literature. Afterward, the equivalent reduced cross sectional properties are computed by comparing two estimated local flexibilities in the two first steps. Finally, the cracked region of the beam is modeled using a very short ordinary beam element with reduced cross sectional properties. Numerical verifications show the effectiveness and the accuracy of the proposed procedure.

2 LOCAL FLIXIBILITY DUE TO AN EDGE CRACK

Fig. 1 shows a cracked beam under general loading condition. It is accepted that an open crack on an elastic structure is a source of local flexibility, so the cracked beam is modeled as two separated parts that interconnected by a massless flexible joint. In general form, the flexibility of cracked region is modeled by 6×6 compliance matrix [18]. In this study, the in-plane bending of beams is only considered. Therefore, a crack can be modeled as a massless rotational spring, whose flexibility is oriented in the *z* direction. To evaluate the flexibility of a rotational spring the relation between the strain energy concentration and the applied load is established according to the theory of linear elastic fracture mechanics. Then, it is expressed in terms of stress intensity factors by utilizing Castigliano's theorem.

Numerous rotational spring stiffness expressions for different geometries have been established by researchers. Some of the most important ones of these functions have been illustrated in Table.1. Although they are algebraically different, the predicted numerical values by them are in good agreement. These stiffness functions will be used to evaluate the equivalent reduced section of a cracked beam in the next section. In this way, it will be possible to use ordinary beam finite elements available in all FE commercial packages for structural analysis of cracked beams.

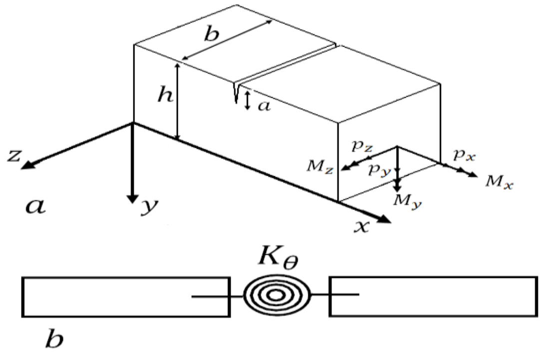
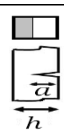


Fig.1
a) A cracked beam under general loading. b) Mathematical model.

Table1
Different expressions for local stiffness of a cracked beam.

Reference	Geometry	Stiffness function
Okamura et al.[4]		$f(\lambda)=1.98\lambda^2-3.277\lambda^3+14.43\lambda^4-31.26\lambda^5+63.56\lambda^6-103.36\lambda^7+147.52\lambda^8-127.69\lambda^9+61.5\lambda^{10}$ $k=\dfrac{E\,b.h^2}{72(1-v^2)f(\lambda)}$
Yazdchi and Gowhari.[8]		$f(\lambda)=-0.0038+1.0459\lambda-24.7331\lambda^2+450.54\lambda^3-2800.2\lambda^4+8918.17\lambda^5-13907.2\lambda^6+8481.12\lambda^7$ $k=\dfrac{E\,b.h^2}{12f(\lambda)}$
Zheng and Fan.[7]		$c=\dfrac{72\pi}{E\,b.h^2}\int_0^{\lambda}x\left(\dfrac{1}{\cos\left(\dfrac{\pi x}{2}\right)}\left(\left(\dfrac{2}{\pi x}\tan\left(\dfrac{\pi x}{2}\right)\right)^{0.5}\left(0.932+0.199\left(1-\sin\left(\dfrac{\pi x}{2}\right)\right)^4\right)\right)\right)^2dx$ $k=\dfrac{1}{c}$
Ostachovicz and Krauczuk.[6]		$f(\lambda)=0.6384-1.035\lambda+3.7201\lambda^2-5.1773\lambda^3+7.553\lambda^4-7.332\lambda^5+2.4909\lambda^6$ $k=\dfrac{E}{72\pi.\lambda^2.f(\lambda)}$
Yazdchi and Gowhari.[8]		$f(\lambda)=-0.0017+0.5506\lambda-20.819\lambda^2+334.396\lambda^3-1833.32\lambda^4+5993.55\lambda^5-9600.42\lambda^6+6030.7\lambda^7$ $k=\dfrac{E\,b.h^2}{12f(\lambda)}$
Ostachovicz and Krauczuk.[6]		$f(\lambda)=0.5335-0.929\lambda+3.5\lambda^2-3.181\lambda^3+5.793\lambda^4$ $k=\dfrac{b.h^2.E}{36.\pi.\lambda^2.f(\lambda)}$
Yazdchi and Gowhari.[8]		$f(\lambda)=0.0007+0.3255\lambda-8.4253\lambda^2+167.486\lambda^3-831.418\lambda^4+2266.89\lambda^5-3154.06\lambda^6+1852.87\lambda^7$ $k=\dfrac{E.\pi.h^3}{64f(\lambda)}$

Zheng and Fan.[7]		$f=\frac{1}{\cos\left(\frac{\pi x}{2}\right)}\left(\left(\frac{2}{\pi x}\tan\left(\frac{\pi x}{2}\right)\right)^{0.5}\left(0.932+0.199\left(1-\sin\left(\frac{\pi x}{2}\right)\right)^{4}\right)\right);x=\frac{\left(2x+\sqrt{1-4y^{2}}-1\right)}{\left(2\sqrt{1-4y^{2}}\right)};$ $c=\left(\frac{1024}{\pi . E . h^{3}}\right)\int_{0}^{\lambda}\int_{(-\sqrt{(x-x^{2})})}^{(\sqrt{x-x^{2}})}(1-4y^{2})(2x+\sqrt{(1-4y^{2})}-1)f^{2}dydx$ $k=\frac{1}{c}$

In the above equations k is the local stiffness and λ is a/h .

3 THE CONCEPT OF EQUIVALENT REDUCED FLEXURAL RIGIDITY

The presence of edge cracks can significantly reduce flexural rigidity of beam structures. Therefore it is possible to model a cracked beam as an intact structure with reduced cross-sectional properties. There are two possible approaches to use this concept.

In the first approach, it is considered that an edge crack has a uniform effect throughout the beam. Therefore the cracked beam can be replaced by an intact beam with a uniform equivalent reduced second moment of area (see Fig. 2 a). This approach has been utilized by Zheng and Ji [3] to establish an approximate method for determining the natural frequency and the static deflection of cracked beams. However, the format of equivalent I computation is highly dependent on the number and location of edge cracks as well as loading and boundary conditions.

In the second approach, which is used in this research, it is assumed that an edge crack only decreases a flexural rigidity of a very small segment of the beam and does not effect on the other parts (see Fig. 2 b). To verify this concept, 2D Finite Elements Analysis is conducted on the cracked beam of Fig. 3 a. The achieved results for lateral displacements are plotted in Fig. 4a.

As evident the results for the intact beam is nearly equal to the analytical results. But the existence of crack reduced the rigidity of the beam and caused additional deflection in the cracked beam. According to the fundamental theory of bending of beams, a curvature of the beam can be written as Eq. (1).

$$\kappa=\frac{M}{EI}=y^{"} \tag{1}$$

The bending moment and the elastic modules are constant throughout the beam. Therefore it can be concluded that:

$$\kappa\propto\frac{1}{I} \tag{2}$$

So, to compute the equivalent second moment of area, the amount of the curvature of the beam should be determined. For this purpose cubic polynomials are fitted to the finite elements results by using least square technique. To capture the true behavior of the cracked beam, the results are divided in to three parts (one short part in the region of the crack and two intact parts) and curve fitting is done for each section individually. The achieved results for the curvature of the cracked and intact beam are depicted in the Fig. 4 b. As evident the result for intact parts are very close to analytical value of the curvature. But a jumped increase in the region of the crack is detected. According to the Eq. (2), this jumping increase can be considered as a jumped decrease in the second moment of area. Therefore the cracked beam can be modeled as an intact beam with a reduced second moment of area in a very short region around the crack. In the next section a very simple method to evaluate the reduced I is presented using the fundamental theory of bending of beams and the concept of local flexibility.

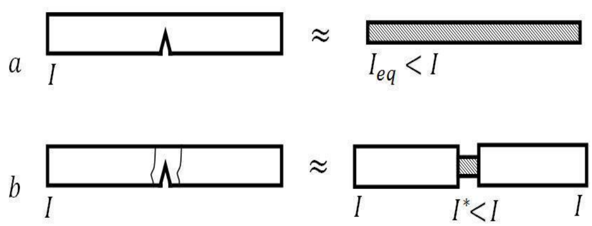


Fig.2
a) Uniform effect approach. b) Local effect approach.

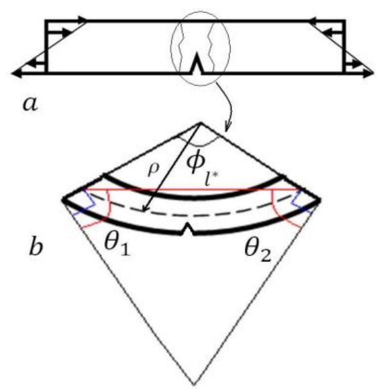


Fig.3
a) Cracked beam under pure bending. b) Deformation in cracked segment.

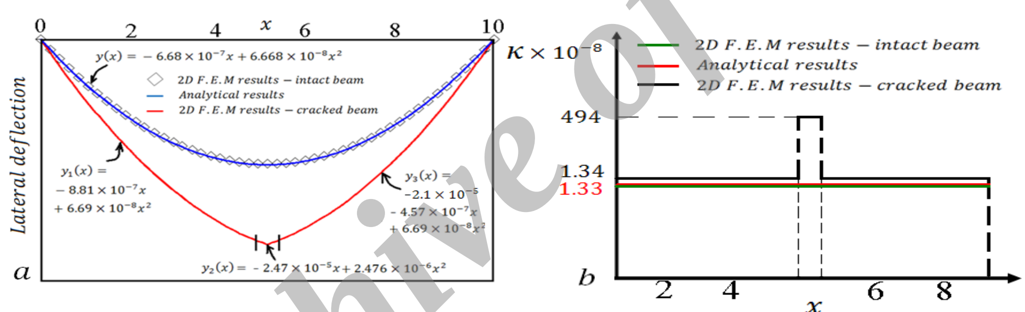


Fig.4
Results for a beam under pure bending. a) Lateral deflection. b) Curvature.

4 THE EVALUATION OF REDUCED SECOND MOMENT OF AREA

Consider Fig. 3 b. The flexural stiffness of a small segment of the beam that is affected by the edge crack can be written as:

$$K = \frac{M}{\Delta\theta} \tag{3}$$

where, M is the bending moment and $\Delta\theta = \theta_1 + \theta_2 = \phi$.

The bending deflection of the beam is assumed very small. Therefore it can be concluded that:

$$\phi = \rho \cdot l^* \qquad or \qquad \phi = \frac{l^* M}{EI^*} \tag{4}$$

Substituting Eq. (4) into Eq. (3), the flexural stiffness of the cracked region is obtained as:

$$K = \frac{EI^*}{l^*} \tag{5}$$

As noted before, the local flexural stiffness of a cracked beam can be obtained by available especial stiffness functions in the literature. Hence, the reduced second moment of area in the edge crack region can be easily determined as:

$$I^* = \frac{K \cdot l^*}{E} \tag{6}$$

The above formulations lead us to introduce a very simple procedure for structural analysis of cracked beam-columns, which is summarized as follows:

Step 1: Determining the flexural stiffness of the cracked region by one of the stiffness functions introduced in Table.1.

Step 2: Selecting the suitable value of I^* for the cracked region.

Step 3: Determining the reduced second moment of area I^* for the cracked region using Eq. (6).

Step 4: Establishing the appropriate mesh using ordinary beam elements to model the problem.

In this step a cracked region of the beam is being modeled by only one element having reduced second moment of area I^* and length I^*.

Step 5: Solving the problem.

In the next section, the above procedure is utilized to solve some problems related to free vibration and buckling analysis of cracked beams. The effect of some parameters like I^* and mesh density in convergence of the problem will be illustrated as well.

5 NUMERICAL RESULTS AND DISCUSSION

5.1 Natural frequencies of a prismatic cracked beam

In this example, the first three natural frequencies of a prismatic cracked beam with different boundary conditions have been studied. The geometry and mechanical characteristics of the beam is depicted in Fig. 5. To study the effect of mesh density, each case is studied using three different mesh patterns, which are illustrated in Fig.6.

As it is evident, the first mesh pattern is the very rough one with the minimum computational costs and equal-length elements in intact region. Also, a very short reduced beam element is used to model the cracked region of the beam. In the second mesh pattern, variable-length beam elements are utilized to discrete the model. The lengths of elements have become smaller near to the cracked region in this pattern. And finally, in the third pattern a very fine grid with equal-length elements in both cracked and non-cracked region are used.

Also, three different I^*, 0.01, 0.001, 0.0001 m, is used to model each case.The local stiffness expression introduced by Yazdchi and Gowhari [8] is used to compute the reduced second moment of area, I^*. To validate the presented results 2D FE model has been developed using ANSYS FEA commercial package [21]. Four- node quadratic plane stress element has been used to establish the model with total of 2154 DOFs. The results are summarized in Table. 2 . These have been non-dimensionalized based on the corresponding natural frequency of intact beam.

Evidently, the presented procedure is definitely mesh and I^* independent. Obviously, in most cases the presented results are reasonably close to those achieved by 2D FE model, and the differences are less than 3%, even thought the obtained differences for third natural frequency in a fixed-free case exceed 8%. To explain the reason of this disagreement, all extracted natural mode shapes are shown in Fig.7. As evident, for the lateral or bending vibration mode shapes, the results are close and the presented procedure can handle the problem with much less computational efforts compared to 2D FE model. But one of the nine extracted natural frequencies, where the maximum difference between two methods is happened, is related to the axial vibration of the beam.

Table2
The natural frequencies obtained for the first example.

				Cracked/Intact								
				mode 1			mode 2			mode 3		
case	boundary conditions	mesh pattern	$I^*(m)$	present	2D FEM	Diff%	present	2D FEM	Diff%	present	2D FEM	Diff%
1	Free-Free	1	0.01	0.7540		2.237	1.000		0.532	0.8182		2.819
2			0.001	0.7512		1.854	1.000		0.510	0.8196		2.998
3			0.0001	0.7488		1.529	1.000		0.503	0.8187		2.877
4		2	0.01	0.7559		2.489	1.000		0.524	0.8180		2.790
5			0.001	0.7489	0.7375	1.539	1.000	0.9949	0.503	0.8194	0.7958	2.974
6			0.0001	0.7488		1.529	1.000		0.503	0.8192		2.940
7		3	0.01	0.7527		2.055	1.000		0.510	0.8178		2.761
8			0.001	0.7485		1.487	1.000		0.489	0.8192		2.940
9			0.0001	0.7155		-	0.999		0.395	0.8187		2.877
10	Fixed-Free	1	0.01	0.9489		1.987	0.797		0.062	0.9997		8.561
11			0.001	0.9484		1.927	0.794		-	0.9998		8.569
12			0.0001	0.9470		1.779	0.794		-	0.9998		8.569
13		2	0.01	0.9453		1.600	0.798		0.191	0.9994		8.527
14			0.001	0.9424	0.9304	1.285	0.794	0.7962	-	0.9994	0.9209	8.527
15			0.0001	0.9386		0.877	0.794		-	0.9994		8.527
16		3	0.01	0.9441		1.471	0.798		0.191	0.9987		8.452
17			0.001	0.9434		1.395	0.794		-	0.9987		8.452
18			0.0001	0.9344		0.426	0.786		-	0.9987		8.452
19	Fixed-Fixed	1	0.01	0.8634		-	1.001		0.584	0.8998		2.824
20			0.001	0.8630		-	1.000		0.507	0.8972		2.524
21			0.0001	0.8607		-	1.000		0.507	0.8970		2.494
22		2	0.01	0.8649		-	1.000		0.521	0.8995		2.788
23			0.001	0.8666	0.8845	-	1.000	0.9949	0.493	0.9007	0.8751	2.925
24			0.0001	0.8625		-	1.000		0.486	0.8966		2.454
25		3	0.01	0.8644		-	1.000		0.507	0.8993		2.762
26			0.001	0.8628		-	1.000		0.479	0.8966		2.459
27			0.0001	0.8653		-	0.999		0.403	0.8961		2.397

Natural Frequencies for intact beam (Hz) : Free-Free & Fixed-Fixed: $\omega_1 = 53.132, \omega_1 = 146.45, \omega_2 = 287.34$
Fixed-Free: $\omega_1 = 8.3487, \omega_1 = 52.247, \omega_2 = 129.261$

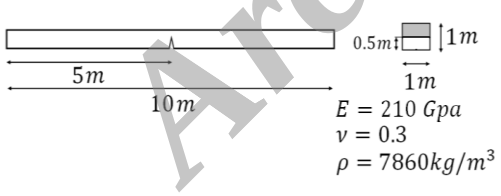


Fig.5
Setup of the first example.

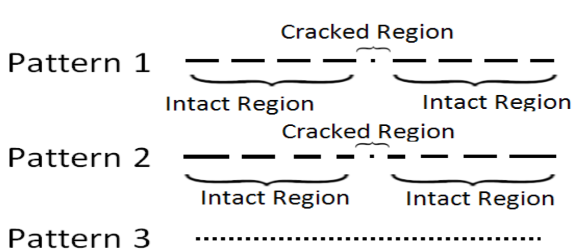


Fig.6
Different mesh patterns used to model cracked beam.

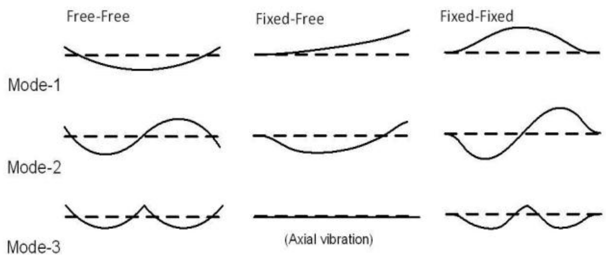


Fig.7
Extracted mode shapes for a cracked beam.

As noted earlier, the case of in-plane bending of beams is only considered in this study, so the presented procedure cannot estimate the effects of crack existence in the other natural frequencies of the beam. However this approach can be easily generalized to study all possible natural modes by rewriting the local flexibility function in a general form. It is also worth to note that when a crack is located close to the stationary point of the vibration mode, the second mode for free-free and fixed –fixed cases, it does not affect natural frequency of the beam. Another parameter that can affect the results of presented procedure is the selected local flexibility expression to compute the reduced second moment of area in the cracked region of the beam. To study the effect of this parameter in the convergence of problem, a sensitivity analysis is done on the case No.11 of Table. 2 , and the problem is solved using different local stiffness functions. The achieved results are summarized in Table. 3. As it is evident, the differences between results are negligible. This shows that the presented procedure is independent of the selected local flexibility function. In fact, predicted numerical values for the local stiffness of cracked region by the use of these functions are in good agreement. Therefore, all of them compute nearly same reduced second moment of area for the same cases.

Table3
The effect of different selected stiffness functions on results.

Local Stiffness Function	Predicted $I^*(m^4)$	Cracked/Intact	
		mode1	mode2
Yazdchi & Gowhari.[8]	0.00002531	0.9484	0.7940
Okamura et al.[4]	0.00002596	0.9591	0.8012
Zheng & Fan.[7]	0.00002512	0.9476	0.7917
Ostachovicz and Krauczuk.[6]	0.00002579	0.9537	0.8003

5.2 Buckling analysis of a prismatic cracked column

In the second example, buckling capacity of a prismatic fixed-pinned column with single and double edge cracks under concentric axial load has been studied (Fig.8). Thus, the effect of crack depth and its location on the buckling load of the column is investigated. The cracked region of the column is modeled using one reduced beam element with $I^* = 0.001m$. To compute the reduced second moment of area , I^*, the local stiffness expressions introduced by Ostachowicz and Krawczuk [6] for single and double edge crack are used. Again, to validate the presented results a 2D FE model is conducted using ANSYS FEA package. As shown in Table. 4, the results of both methods are in good agreement, and the maximum difference is less than 6%. In addition, two important points could be concluded from the results. As expected, the carrying capacity of the column decreases as the crack depth increases. In addition, for a constant crack depth, the maximum reduction of the buckling capacity occurs when the crack is located at the position of maximum curvature of the column buckling mode shape.

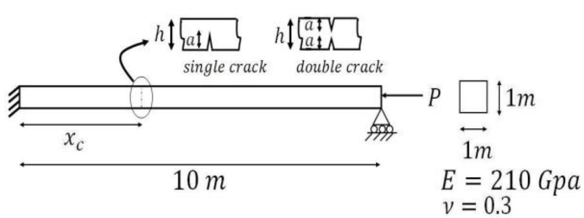


Fig.8
Setup of the second example.

Table 4
The buckling load obtained for the second example.

case	a/h	x_c		Buckling Load Cracked/Intact present	2D FEM ANSYS	Diff%
1	0.1	3	single crack	0.9938	0.9382	5.927
2			double crack	0.9929	0.9379	5.868
3		5	single crack	0.9734	0.9266	5.052
4			double crack	0.9651	0.9129	5.718
5		7	single crack	0.9558	0.9186	4.046
6			double crack	0.9365	0.8897	5.261
7	0.2	3	single crack	0.9938	0.9379	5.959
8			double crack	0.9907	0.9373	5.690
9		5	single crack	0.9422	0.8928	5.529
10			double crack	0.8656	0.8173	5.901
11		7	single crack	0.9098	0.8644	5.251
12			double crack	0.8054	0.7646	5.343
13	0.3	3	single crack	0.9932	0.9373	5.962
14			double crack	0.8860	0.8505	4.169
15		5	single crack	0.8795	0.8380	4.942
16			double crack	0.6502	0.6275	3.617
17		7	single crack	0.8227	0.7836	4.996
18			double crack	0.5090	0.5061	0.561

Buckling load for intact column; $P_{cr} = \pi^2 EI / (0.7L)^2 = 35.249 \times 10^8$

5.3 Free lateral vibration of a cracked conical shaft

As noted before, the main advantage of the presented method is the ability of modeling cracked slim structures with ordinary available beam elements within all commercial FEM packages. Therefore, the method can model problems with much less computational efforts compared to 2D and 3D FEMs. This advantage is more obvious when complex practical engineering problems are considered. To validate this claim, as a third example, free bending vibration of simply supported conical shaft of Fig.9 is studied. The details of the geometry and finite element model are depicted in that figure. Again, the mechanical properties of example 1 are used. The shaft is modeled using 118 Euler beam elements with total 354 DOFs. A cracked region of the shaft is modeled using one reduced element with $I^* = 0.005m$. The reduced second moment of area, I^*, is computed using the function introduced by Zheng and Fan [7]. To validate the presented results, 3D FE model with total 5437 DOFs is conducted using ANSYS. Table.5 shows the results achieved by both methods. As it is shown, despite the vast differences in computational efforts, there is a good agreement between two methods.

Table5
The Natural Frequencies obtained for the third example.

Natural Frequencies	Cracked/Intact Presented Method	3D FEM	Diff%
First Mode	0.975	0.951	2.5
Second Mode	0.978	0.961	1.5
Third Mode	0.979	0.948	3

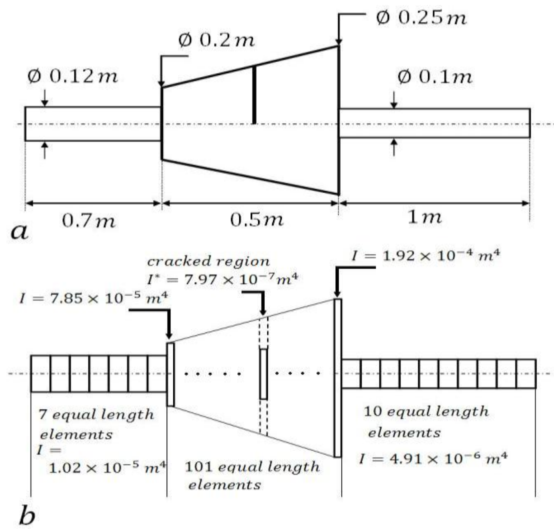


Fig.9
Setup of the third example a) Geometry b) Proposed FE model.

6 CONCLUSIONS

A very simple Finite Element procedure for modal and buckling analysis of cracked beam-like structures has been presented. Using this approach, it is possible to model fractured beam-like structures by ordinary beam elements rather than 2 & 3D FE models, and he computational efforts would be reduced significantly. This approach is introduced by the mixture of the local flexibility method and the concept of equivalent reduced section. The presented procedure is able to model complicated engineering cases of cracked slim structures. Also, this approach is validated by the comparison of achieved results with those obtained by 2 & 3D FEA. Although only in plane bending of structures is considered in this work, the approach can easily be generalized to calculate all possible vibration and instability modes. It is feasible by rewriting the local stiffness function of cracked region in the general form (6×6 matrix) and then computing the flexibility variation of the beam in all possible degrees of freedom.

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